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A CHARACTERIZATION OF EXTRINSIC SPHERES IN A RIEMANNIAN MANIFOLD

Dedicated to the memory of Professor Takeo Ohkubo

By

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Abstract. We give a characterization of a totally umbilic submanifold M^n with parallel mean curvature vector of a Riemannian manifold $\tilde{M}^{n+p}$, that is an extrinsic sphere M^n of $\tilde{M}^{n+p}$, in terms of the extrinsic shape of circles on M^n in the ambient manifold $\tilde{M}^{n+p}$. This characterization is an improvement of Nomizu and Yano's result ([2]).

1. Introduction

To what extent can we determine the properties of a submanifold by observing the extrinsic shape of geodesics or circles of a submanifold? It is well-known that a submanifold is totally geodesic (resp. totally umbilic with parallel mean curvature vector) if and only if all geodesics (resp. circles) of the submanifold are geodesics (resp. circles) in the ambient space.

From this point of view we here recall the following two surfaces. Let f_1 be a totally umbilic imbedding of a 2-dimensional standard sphere $S^2(c)$ of curvature c into Euclidean space $\boldsymbol{R}^5$, and let $f_2 = \iota \circ f$ be an isometric parallel immersion of $S^2(c)$ into $\boldsymbol{R}^5$. Here f is the second standard minimal immersion of $S^2(c)$ into $S^4(3c)$ and ι is a totally umbilic imbedding of $S^4(3c)$ into $\boldsymbol{R}^5$. We know that for each great circle γ on $S^2(c)$, both of the curves $f_1 \circ \gamma$ and $f_2 \circ \gamma$ are circles in the ambient space $\boldsymbol{R}^5$. This implies that we cannot distinguish f_1 from f_2 by the extrinsic shape of *geodesics* of $S^2(c)$ in $\boldsymbol{R}^5$. However we emphasize that we can distinguish these two isometric immersions f_1 and f_2 by the extrinsic shape of (small) *circles* of $S^2(c)$ in $\boldsymbol{R}^5$. In fact, for each small circle γ on $S^2(c)$, the curve

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$f_1 \circ \gamma$ is also a circle in $\boldsymbol{R}^5$ but the curve $f_2 \circ \gamma$ is a helix of proper order 4 in the ambient space $\boldsymbol{R}^5$ (for details, see [1, 4]).

In this context we are interested in the extrinsic shape of *circles* of the submanifold. Let M^n be an n-dimensional connected Riemannian manifold with $n \geq 2$, and let $\tilde{M}^N(c)$ be an N-dimensional complete simply conneted space form of constant curvature c. Namely, $\tilde{M}^N(c)$ is isometric to $\boldsymbol{R}^N$, $S^N(c)$ or $H^N(c)$. Our study is motivated by the following well-known result.

Theorem A ([2]). *Let M^n be a Riemannian submanifold of $\tilde{M}^{n+p}$ through an isometric immersion f. Then M^n is an extrinsic sphere of $\tilde{M}^{n+p}$ if and only if, for some positive constant k and for every circle $\gamma = \gamma(s)$ of curvature k in M^n, the curve $f \circ \gamma$ is a circle in $\tilde{M}^{n+p}$.*

The purpose of this paper is to improve Theorem A.

Theorem 1. *Let M^n be a Riemannian submanifold of $\tilde{M}^{n+p}$ through an isometric immersion f. Then M^n is an extrinsic sphere of $\tilde{M}^{n+p}$ if and only if, for some positive constant k and for every circle $\gamma = \gamma(s)$ of curvature k in M^n, the curve $f \circ \gamma$ is a Frenet curve of order 2 in $\tilde{M}^{n+p}$.*

As an immediate consequence of Theorem 1, we obtain the following.

Theorem 2. *Let M^n be a Riemannian submanifold of a space form $\tilde{M}^{n+p}(c)$ through an isometric immersion f. Then M^n is totally umbilic in $\tilde{M}^{n+p}(c)$ if and only if, for some positive constant k and for every circle $\gamma = \gamma(s)$ of curvature k in M^n, the curve $f \circ \gamma$ is a plane curve in $\tilde{M}^{n+p}(c)$.*

Theorem 2 is related to the following well-known classification theorem of planar geodesic submanifolds in a space form.

Theorem B ([4]). *Let M^n be a Riemannian submanifold of a space form $\tilde{M}^{n+p}(c)$ through an isometric immersion f. Suppose that for every geodesic $\gamma = \gamma(s)$ in M^n the curve $f \circ \gamma$ is a plane curve in $\tilde{M}^{n+p}(c)$ (that is, f is a planar geodesic immersion). Then M^n is totally umbilic in $\tilde{M}^{n+p}(c)$ or M^n is locally congruent to a compact symmetric space of rank one immersed into some totally umbilic submanifold of $\tilde{M}^{n+p}(c)$ through the parallel minimal immersion.*

2. Preliminaires

First of all we recall the notion of isotropic immersions. Let M and $\tilde{M}$ be Riemannian manifolds and $f : M \to \tilde{M}$ be an isometric immersion. We denote by σ the second fundamental form of f. Then the immersion f is said to be *isotropic* at $x \in M$ if $\|\sigma(X,X)\|/\|X\|^2$ is constant for each $X \neq 0$ on the tangent space $T_x(M)$ of M at x. If the isometric immersion is isotropic at every point, then the immersion is said to be isotropic. Note that a totally umbilic immersion is isotropic, but not *vice versa*.

The following is well-known ([3]).

Lemma 2.1. *Let f be an isometric immersion of M into $(\tilde{M}, \langle\, ,\rangle)$. Then f is isotropic at $x \in M$ if and only if the second fundamental form σ satisfies $\langle \sigma(X,X), \sigma(X,Y)\rangle = 0$ for an arbitrary orthogonal pair $X, Y \in T_x(M)$.*

We next recall the Frenet formula for a smooth Frenet curve in a Riemannian manifold M with Riemannian metric $\langle\, ,\rangle$. A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s is called a *Frenet curve of proper order d* if there exist orthonormal frame fields $\{V_1 = \dot{\gamma}, \dots, V_d\}$ along γ and positive functions $\kappa_1(s), \dots, \kappa_{d-1}(s)$ satisfying the following system of ordinary equations

$$\nabla_{\dot{\gamma}} V_j(s) = -\kappa_{j-1}(s) V_{j-1}(s) + \kappa_j(s) V_{j+1}(s), \quad j = 1, \dots, d, \tag{2.1}$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ. We call Equation (2.1) the Frenet formula for the Frenet curve γ. The functions $\kappa_j(s)$ $(j = 1, \dots, d-1)$ and the orthonormal frame $\{V_1, \dots, V_d\}$ are called the curvatures and the Frenet frame of γ, respectively.

A Frenet curve is called a *Frenet curve of order d* if it is a Frenet curve of proper order r $(\leqq d)$. For a Frenet curve of order d which is of proper order r $(\leqq d)$, we use the convention in (2.1) that $\kappa_j \equiv 0$ $(r \leqq j \leqq d-1)$ and $V_j \equiv 0$ $(r+1 \leqq j \leqq d)$. In this paper a curve means a smooth Frenet curve. We call a smooth Frenet curve a *helix* when all its curvatures are constant. A helix of order 1 is nothing but a geodesic and a helix of order 2, that is a curve which satisfies $\nabla_{\dot{\gamma}} V_1(s) = \kappa V_2(s)$, $\nabla_{\dot{\gamma}} V_2(s) = -\kappa V_1(s)$, $V_1(s) = \dot{\gamma}(s)$, is called a *circle* of curvature κ.

For later use, we prepare the following lemmas.

Lemma 2.2 ([2]). *A circle $\gamma = \gamma(s)$ satisfies the following differential equation*

$$\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}\dot{\gamma}) + \langle \nabla_{\dot{\gamma}}\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}\rangle \dot{\gamma} = 0, \tag{2.2}$$

where the curvature $\kappa = \|\nabla_{\dot{\gamma}}\dot{\gamma}\|$ is constant along γ. Conversely, if a curve $\gamma = \gamma(s)$ satisfies (2.2), *then it is a helix of order* 2, *that is, it is a circle or a geodesic.*

LEMMA 2.3. *A Frenet curve $\gamma = \gamma(s)$ of order* 2 *satisfies the following differential equation*

$$\kappa(s)(\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) + \langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle\dot\gamma) = \dot\kappa(s)\nabla_{\dot\gamma}\dot\gamma, \tag{2.3}$$

where $\kappa(s) = \|\nabla_{\dot\gamma}\dot\gamma\|$. Conversely, if a Frenet curve $\gamma = \gamma(s)$ satisfies (2.3), *then it is of order* 2.

PROOF. We first suppose that γ is a Frenet curve of order 2. Then by definition we have

$$\nabla_{\dot\gamma}\dot\gamma = \kappa(s)V_2(s), \tag{2.4}$$

$$\nabla_{\dot\gamma}V_2(s) = -\kappa(s)\dot\gamma. \tag{2.5}$$

We get from (2.4) and (2.5)

$$\kappa(s)(\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma)) = \kappa(s)\nabla_{\dot\gamma}(\kappa(s)V_2(s)) = \dot\kappa(s)\nabla_{\dot\gamma}\dot\gamma - \kappa(s)^3\dot\gamma$$

as well as

$$\langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle = \kappa(s)^2,$$

so that we obtain Equation (2.3).

Conversely, assume that (2.3) holds. We set $\kappa(s) = \|\nabla_{\dot\gamma}\dot\gamma\|$. If $\kappa(s) \equiv 0$, then γ is a geodesic. When $\kappa(s) > 0$, we can set $\nabla_{\dot\gamma}\dot\gamma = \kappa(s)V_2(s)$. Hence,

$$\nabla_{\dot\gamma}V_2(s) = \nabla_{\dot\gamma}\left(\frac{1}{\kappa(s)}\nabla_{\dot\gamma}\dot\gamma\right) = -\kappa(s)\dot\gamma,$$

by virtue of (2.3). Thus γ is a Frenet curve of order 2. □

Finally we review fundamental equations in submanifold theory. Let M be an n-dimensional Riemannian submanifold of $\tilde M^{n+p}$ with metric $\langle\ ,\ \rangle$. We denote by ∇ and $\tilde\nabla$ the covariant differentiations of M and $\tilde M$, respectively. Then the second fundamental form σ of the immersion is defined by

$$\sigma(X, Y) = \tilde\nabla_X Y - \nabla_X Y, \tag{2.6}$$

where X and Y are vector fields tangent to M. For a vector field ξ normal to M, we write

$$\tilde\nabla_X\xi = -A_\xi X + D_X\xi, \tag{2.7}$$

where $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. the normal) component of

$\tilde{\nabla}_X \xi$. We define the covariant differentiation $\overline{\nabla}$ of the second fundamental form σ with respect to the connection in (tangent bundle) + (normal bundle) as follows:

$$(2.8) \qquad (\overline{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

The second fundamental form σ is said to be *parallel* if $(\overline{\nabla}_X \sigma)(Y, Z) = 0$ for all tangent vector fields X, Y and Z on M.

3. Proof of Theorems

Let x be any point of M^n. In the following, we take and fix an orthonormal pair of vectors $X, Y \in T_x M$.

Let $\gamma = \gamma(s)$ be a circle of curvature k on M^n satisfying the equations $\nabla_{\dot\gamma}\dot\gamma = kY_s$ and $\nabla_{\dot\gamma} Y_s = -k\dot\gamma$, $|s| < \varepsilon$ for some $\varepsilon > 0$ with initial condition that $\gamma(0) = x$, $\dot\gamma(0) = X$ and $Y_0 = Y$. Needless to say, the curve γ satisfies (2.2). By assumption the curve $f \circ \gamma$ is a Frenet curve of order 2, thus from (2.3) it satisfies the differential equation

$$(3.1) \qquad \kappa(s)(\tilde{\nabla}_{\dot\gamma}(\tilde{\nabla}_{\dot\gamma}\dot\gamma) + \langle \tilde{\nabla}_{\dot\gamma}\dot\gamma, \tilde{\nabla}_{\dot\gamma}\dot\gamma \rangle \dot\gamma) = \dot\kappa(s)\tilde{\nabla}_{\dot\gamma}\dot\gamma,$$

where $\kappa(s) = \|\tilde{\nabla}_{\dot\gamma}\dot\gamma\|$ and $\tilde{\nabla}$ is the covariant differentiation of $\tilde{M}^{n+p}$. We here note that $\kappa(s) > 0$ for any s, that is, the curve $f \circ \gamma$ is of proper order 2. Indeed, suppose that the Frenet curve $f \circ \gamma$ satisfies $\kappa \equiv 0$. This implies that the curve $f \circ \gamma$ is a geodesic in the ambient space $\tilde{M}^{n+p}$, so that the curve $\gamma = \gamma(s)$ is a geodesic in M^n, which is a contradiction. It follows from (2.6), (2.7) and (2.8) that

$$(3.2) \qquad \tilde{\nabla}_{\dot\gamma}(\tilde{\nabla}_{\dot\gamma}\dot\gamma) = \nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) + 3\sigma(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\overline{\nabla}_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma).$$

We find from (2.2), (2.6), (2.7), (3.1) and (3.2) that

$$(3.3) \qquad \begin{aligned} &\kappa(s)(3\sigma(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) - A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + (\overline{\nabla}_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma) + \|\sigma(\dot\gamma, \dot\gamma)\|^2\dot\gamma) \\ &\quad = \dot\kappa(s)(\nabla_{\dot\gamma}\dot\gamma + \sigma(\dot\gamma, \dot\gamma)). \end{aligned}$$

Considering the tangential component and the normal component for the submanifold M in Equation (3.3), we obtain the following:

$$(3.4) \qquad \kappa(s)(-A_{\sigma(\dot\gamma,\dot\gamma)}\dot\gamma + \|\sigma(\dot\gamma, \dot\gamma)\|^2\dot\gamma) = \dot\kappa(s)\nabla_{\dot\gamma}\dot\gamma.$$

$$(3.5) \qquad \kappa(s)(3\sigma(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) + (\overline{\nabla}_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma)) = \dot\kappa(s)\sigma(\dot\gamma, \dot\gamma).$$

Note that

$$\kappa(s) = \|\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\| = \sqrt{k^2 + \|\sigma(\dot{\gamma}, \dot{\gamma})\|^2} > 0.$$

Hence

$$\begin{aligned}\kappa(s)\dot{\kappa}(s) &= \frac{1}{2}\frac{d}{ds}\kappa(s)^2 \\ &= \frac{1}{2}\frac{d}{ds}\langle\sigma(\dot{\gamma}, \dot{\gamma}), \sigma(\dot{\gamma}, \dot{\gamma})\rangle \\ &= \langle D_{\dot{\gamma}}(\sigma(\dot{\gamma}, \dot{\gamma})), \sigma(\dot{\gamma}, \dot{\gamma})\rangle \\ &= \langle(\overline{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma}, \dot{\gamma}) + 2\sigma(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}), \sigma(\dot{\gamma}, \dot{\gamma})\rangle.\end{aligned}$$

Thus, at $s = 0$ we get the following:

$$\kappa(0) = \sqrt{k^2 + \|\sigma(X, X)\|^2}. \tag{3.6}$$

$$\kappa(0)\dot{\kappa}(0) = \langle(\overline{\nabla}_X\sigma)(X, X) + 2k\sigma(X, Y), \sigma(X, X)\rangle. \tag{3.7}$$

Evaluating (3.5) at $s = 0$, we find

$$\kappa(0)(3k\sigma(X, Y) + (\overline{\nabla}_X\sigma)(X, X)) = \dot{\kappa}(0)\sigma(X, X). \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}&3k\kappa(0)^2\sigma(X, Y) - 2k\langle\sigma(X, Y), \sigma(X, X)\rangle\sigma(X, X) \\ &\quad = \langle(\overline{\nabla}_X\sigma)(X, X), \sigma(X, X)\rangle\sigma(X, X) - \kappa(0)^2(\overline{\nabla}_X\sigma)(X, X).\end{aligned} \tag{3.9}$$

We here consider another circle $\tau = \tau(s)$ of the same curvature k on M^n satisfying the equations $\nabla_{\dot{\tau}}\dot{\tau} = kZ_s$ and $\nabla_{\dot{\tau}}Z_s = -k\dot{\tau}$, $|s| < \varepsilon_1$ for some $\varepsilon_1 > 0$ with initial condition that $\tau(0) = x$, $\dot{\tau}(0) = X$ and $Z_0 = -Y$. By assumption the curve $f \circ \tau$ is a Frenet curve of order 2 in the ambient manifold $\tilde{M}$. We set $\kappa_1(s) = \|\tilde{\nabla}_{\dot{\tau}}\dot{\tau}\| > 0$. Applying the above discussion to the circle τ, we find

$$\begin{aligned}&-3k\kappa_1(0)^2\sigma(X, Y) + 2k\langle\sigma(X, Y), \sigma(X, X)\rangle\sigma(X, X) \\ &\quad = \langle(\overline{\nabla}_X\sigma)(X, X), \sigma(X, X)\rangle\sigma(X, X) - \kappa_1(0)^2(\overline{\nabla}_X\sigma)(X, X).\end{aligned} \tag{3.9'}$$

Equation (3.6) guarantees $\kappa(0) = \kappa_1(0)$. Then, from (3.9) and (3.9′) we can see that

$$3k\kappa(0)^2\sigma(X, Y) - 2k\langle\sigma(X, Y), \sigma(X, X)\rangle\sigma(X, X) = 0.$$

This, together with (3.6), yields

$$(3k^2 + \|\sigma(X,X)\|^2)\langle\sigma(X,X),\sigma(X,Y)\rangle = 0,$$

so that

$$\langle\sigma(X,X),\sigma(X,Y)\rangle = 0.$$

Since x is arbitrary, thanks to Lemma 2.1, we find that our immersion f is isotropic. So, again by using Lemma 2.1, we get $A_{\sigma(\dot{\gamma},\dot{\gamma})}\dot{\gamma} = \|\sigma(\dot{\gamma},\dot{\gamma})\|^2\dot{\gamma}$. Hence, from (3.4) we obtain $0 = \dot{\kappa}(s)\nabla_{\dot{\gamma}}\dot{\gamma} = \dot{\kappa}(s)kY_s$, so that $\kappa = \kappa(s)$ is constant along the curve $f \circ \gamma$. Therefore we can see that the curve $f \circ \gamma$ is a circle in the ambient space $\tilde{M}^{n+p}$. Thus we get the statement of Theorem 1 (see Theorem A). □

Next, let the ambient space $\tilde{M}^{n+p}$ be a space form $\tilde{M}^{n+p}(c)$. We can easily find that in the manifold $\tilde{M}^{n+p}(c)$, a Frenet curve $\gamma = \gamma(s)$ is of order 2 if and only if the curve γ is a plane curve, (that is, the curve γ is locally contained in some 2-dimensional totally geodesic submanifold of $\tilde{M}^{n+p}(c)$). Therefore we establish the statement of Theorem 2. □

We finally remark that in any Riemannian manifold, a plane curve means a Frenet curve of order 2. However, in general the converse does not hold.

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