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DISCRETELY STAR-LINDELÖF SPACES

By

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Abstract. A space X is called (discretely) star-Lindelöf if for every open cover $\mathcal{U}$ of X, there exists a (discrete closed) countable subset B of X such that $St(B, \mathcal{U}) = X$. We investigate the relationship between these spaces and ω_1-compact spaces, and also study topological properties of discretely star-Lindelöf spaces.

1. Introduction

By a space we mean a topological space. Fleischman [4] defined a space X to be *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset B of X such that $St(B, \mathcal{U}) = X$, where $St(B, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap B \neq \varnothing\}$. He proved that every countably compact space is starcompact, and conversely, van Douwen-Reed-Roscoe-Tree [2] proved that every starcompact T_2-space is countably compact. As a generalization of starcompactness, the following class of spaces is also studied by several authors under different names (see [9]):

DEFINITION 1.1. A space X is *star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset B of X such that $St(B, \mathcal{U}) = X$.

Further, Yasui-Gao [13] defined *a space in countable discrete web* by replacing the word 'countable' by 'countable discrete closed' in the preceding definition. In this paper, we rename a space in countable discrete web as the following definition, which seems to be more natural in the context of the history of star-covering properties:

DEFINITION 1.2. A space X is *discretely star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable discrete closed subset B of X and $St(B, \mathcal{U}) = X$.

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Recall that a space X is *ω_1-compact* if there is no uncountable discrete closed subset of X. The following diagram illustrates the relationship among spaces we shall consider and more familiar ones:

$$\begin{array}{ccccc}
\text{Lindelöf} & \longrightarrow & \omega_1\text{-compact} & \longleftarrow & \text{countably compact} \\
 & & \Big\downarrow T_1 & & \\
 & & \text{discretely star-Lindelöf} & & \\
 & & \Big\downarrow & & \\
\text{separable} & \longrightarrow & \text{star-Lindelöf} & \longleftarrow & \text{starcompact}
\end{array}$$

The purpose of this paper is to investigate the relationship among spaces on the vertical centerline in the above diagram and to study topological properties of discretely star-Lindelöf spaces. In particular, we give various examples showing the difference between discretely star-Lindelöf spaces and ω_1-compact spaces, and improve some results due to Yasui-Gao [13].

Throughout the paper, the cardinality of a set A is denoted by $|A|$. For a cardinal κ, κ^+ denotes the smallest cardinal greater than κ. In particular, let ω denote the first infinite cardinal, $\omega_1 = \omega^+$ and $\mathfrak{c}$ the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. Other terms and symbols will be used as in [3].

2. Discretely Star-Lindelöf Spaces and Their Subspaces

The square of the Sorgenfrey line is star-Lindelöf since it is separable, while Yasui-Gao [13] proved that the square is not discretely star-Lindelöf. The following theorem gives an alternative proof of the latter fact.

THEOREM 2.1. *Let κ be an infinite cardinal with $\kappa^\omega = \kappa$ and let X be a discretely star-Lindelöf space with $|X| = \kappa$. Then, the cardinality of a discrete closed subset of X is less than κ.*

PROOF. The proof is based on the idea of that of van Douwen-Reed-Roscoe-Tree [2, Lemma 2.2.4]. Suppose on the contrary that there exists a discrete closed subset H of X with $|H| = \kappa$. Let $\mathscr{F}$ be the set of all countable discrete closed subsets of X. Then, $|\mathscr{F}| = \kappa$ since $|X| = \kappa = \kappa^\omega$, and thus, we can enumerate $\mathscr{F}$ as $\mathscr{F} = \{F_\alpha : \alpha < \kappa\}$. By transfinite induction, we can define a subset $H_0 = \{x_\alpha : \alpha < \kappa\}$ of H satisfying that $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$ and $x_\alpha \notin \bigcup_{\beta \leq \alpha} F_\beta$ for each $\alpha < \kappa$. Define

$U_\alpha = X \backslash (F_\alpha \cup (H_0 \backslash \{x_\alpha\}))$ for each $\alpha < \kappa$. Then, U_α is an open neighborhood of x_α in X. Let us consider the open cover

$$\mathscr{U} = \{U_\alpha : \alpha < \kappa\} \cup \{X \backslash H_0\}$$

of X. For each $\alpha < \kappa$, $x_\alpha \notin St(F_\alpha, \mathscr{U})$, because U_α is the only element of $\mathscr{U}$ containing the point x_α and $U_\alpha \cap F_\alpha = \varnothing$ by the definition. This shows that X is not discretely star-Lindelöf, which is a contradiction. $\square$

Corollary 2.2. *Let X be a discretely star-Lindelöf space with $|X| = \mathfrak{c}$. Then, the cardinality of a discrete closed subset of X is less than $\mathfrak{c}$.*

This is a special case of Theorem 2.1. The following corollaries are immediate consequences of Corollary 2.2.

Corollary 2.3. *The square of the Sorgenfrey line, the Niemytzki plane and every Isbell-Mrówka space Ψ with $|\Psi| = \mathfrak{c}$ are not discretely star-Lindelöf.*

It is worth noting that all of the spaces stated in Corollary 2.3 are star-Lindelöf since they are separable.

Corollary 2.4. *Under assuming the continuum hypothesis, every discretely star-Lindelöf space with cardinality $\mathfrak{c}$ is ω_1-compact.*

Next, we give a machine which produces discretely star-Lindelöf spaces. For a separable space X and its countable dense subset D, we define

$$S(X, D) = X \cup (D \times \kappa^+), \quad \text{where } \kappa = |X|,$$

and topologize $S(X, D)$ as follows: A basic neighborhood of $x \in X$ in $S(X, D)$ is a set of the form

$$G_{U,\alpha}(x) = U \cup ((U \cap D) \times \{\beta : \alpha < \beta < \kappa^+\}),$$

for a neighborhood U of x in X and for $\alpha < \kappa^+$, and a basic neighborhood of $\langle x, \alpha \rangle \in D \times \kappa^+$ in $S(X, D)$ is a set of the form

$$G_V(\langle x, \alpha \rangle) = \{x\} \times V$$

for a neighborhood V of α in κ^+. When it is not necessary to specify D, we simply write $S(X)$ instead of $S(X, D)$. By a *Tychonoff space* we mean a completely regular T_1-space.

Theorem 2.5. *Let X be a separable space with a countable dense set D. Then, the space $S(X,D)$ is discretely star-Lindelöf. Moreover,*

(1) *if X is a Tychonoff space, so is $S(X,D)$;*

(2) *if X is a normal space, so is $S(X,D)$.*

Proof. Put $S = S(X,D)$ and let $\mathscr{U}$ be an open cover of S. For every $x \in X$, there exist a neighborhood U of x in X and $\alpha(x) < \kappa^+$ such that $G_{U,\alpha(x)}(x)$ is included in some member of $\mathscr{U}$. Since $|X| = \kappa$, we can find $\alpha < \kappa^+$ such that $\alpha > \alpha(x)$ for each $x \in X$. Then, the set $B_1 = D \times \{\alpha\}$ is countable, discrete closed in S and $St(B_1,\mathscr{U}) \supseteq X$. For each $x \in D$, there exists a finite set $F_x \subseteq \{x\} \times \kappa^+$ such that $St(F_x,\mathscr{U}) \supseteq \{x\} \times \kappa^+$, because $\{x\} \times \kappa^+$ is countably compact. Then, the set $B_2 = \bigcup\{F_x : x \in D\}$ is countable, discrete closed in S and $St(B_2,\mathscr{U}) \supseteq D \times \kappa^+$. If we put $B = B_1 \cup B_2$, then B is a countable discrete closed set in X such that $St(B,\mathscr{U}) = S$, which proves that S is discretely star-Lindelöf. The proof of the statement (1) is left to the reader since it is not difficult.

Finally, to prove the statement (2), assume that X is normal. Let A_0 and A_1 be disjoint closed subsets of $S(X,D)$. Since X is normal and $\kappa^+ > |X|$, we can find disjoint open subsets U_0, U_1 of X and $\alpha < \kappa^+$ such that $A_i \cap X \subseteq U_i$ and

$$(U_i \cup ((U_i \cap D) \times (\alpha,\kappa^+))) \cap A_{1-i} = \varnothing$$

for each $i = 0,1$. Let $X_0 = D \times \kappa^+$ and put

$$B_i = ((U_i \cap D) \times (\alpha,\kappa^+)) \cup (A_i \cap X_0) \quad \text{for } i = 0,1.$$

Then, B_0 and B_1 are disjoint closed in X_0. Since X_0 is normal, there exist disjoint open sets V_0 and V_1 in X_0 such that $B_i \subseteq V_i$ for each $i = 0,1$. Let $G_i = U_i \cup V_i$ for $i = 0,1$. Then, G_0 and G_1 are disjoint open sets in $S(X,D)$ such that $A_i \subseteq G_i$ for each $i = 0,1$. The proof is complete. □

Corollary 2.6. *Every Tychonoff space X with $w(X) \leq \mathfrak{c}$ can be embedded in a discretely star-Lindelöf Tychonoff space as a closed subspace.*

Proof. Let X be a Tychonoff space X with $w(X) \leq \mathfrak{c}$. Then, it is known that X can be embedded in a separable Tychonoff space Y as a closed subspace. Indeed, embed X into $[0,1]^{\mathfrak{c}}$ and take a countable dense subset D of $[0,1]^{\mathfrak{c}}$. Then, the space Y is obtained from the subspace $X \cup D$ by making each point of $D \setminus X$ isolated. Next, consider the space $S(Y)$ defined above. Then, $S(Y)$ is discretely star-Lindelöf by Theorem 2.5 and X is closed in $S(Y)$. □

REMARK 1. If X is one of the spaces stated in Corollary 2.3, then $S(X)$ is discretely star-Lindelöf but not ω_1-compact. Examples of discretely star-Lindelöf spaces with richer properties but not ω_1-compact were also given by Matveev [10].

It is quite interesting to find an example of a normal (discretely) star-Lindelöf space which is not ω_1-compact. Now, we give a consistency example:

COROLLARY 2.7. *Assume Martin's axiom and the negation of the continuum hypothesis and let* $\omega_1 \leq \kappa < \mathfrak{c}$. *Then, there exists a normal, discretely star-Lindelöf space* X *containing a closed discrete subset* B *with* $|B| = \kappa$.

PROOF. Under the assumption, it is known ([12]) that there exists a separable normal space Y with a closed discrete subset B with $|B| = \kappa$. Then, the space $X = S(Y)$ is a required one by Theorem 2.5. □

REMARK 2. Matveev [10] also showed, independently, the existence of a normal discretely star-Lindelöf space which is not ω_1-compact under certain set-theoretic assumption weaker than ours. He also asked if there exists an example within ZFC.

If X is a discretely star-Lindelöf space which is not ω_1-compact, then X contains an uncountable discrete closed subset B. Since B is not star-Lindelöf, this shows that a closed subspace of a discretely star-Lindelöf space need not be star-Lindelöf. Yasui-Gao [13] also gave an example showing that a closed subspace of a discretely star-Lindelöf space need not be discretely star-Lindelöf; however, their space is not Hausdorff. Now, we give another stronger example.

EXAMPLE 2.8. There exists a discretely star-Lindelöf, Tychonoff space X having a regular-closed subspace which is not discretely star-Lindelöf.

PROOF. Let $\mathscr{R}$ be a maximal almost disjoint family of infinite subsets of ω with $|\mathscr{R}| = \mathfrak{c}$, and consider the Isbell-Mrówka space $\Psi = \omega \cup \mathscr{R}$ (see [5, 5I, p. 79]). Let X be the space obtained from the space $S(\Psi, \omega) = \Psi \cup (\omega \times \mathfrak{c}^+)$ by making each point of ω in Ψ isolated. Then, Ψ is a regular-closed subspace of X and is not discretely star-Lindelöf by Corollary 2.3. Hence, it remains to show that X is discretely star-Lindelöf. Let $\mathscr{U}$ be an open cover of X. Then, by a similar argument to the proof of Theorem 2.5, we can find countable discrete closed subsets B_1 and B_2 of X such that $\Psi \setminus \omega \subseteq St(B_1, \mathscr{U})$ and $\omega \times \mathfrak{c}^+ \subseteq St(B_2, \mathscr{U})$. Since no infinite subset of ω is closed in Ψ, the set $B_3 = \omega \setminus St(B_1, \mathscr{U})$ is finite. Hence, if we put $B = B_1 \cup B_2 \cup B_3$, then B is a countable discrete closed set in X such that $St(B, \mathscr{U}) = X$. This proves that X is discretely star-Lindelöf. $\square$

3. Mappings

In [2, Theorem 2.4.1], van Douwen-Reed-Roscoe-Tree proved that a continuous image of a star-Lindelöf space is star-Lindelöf. First, we give examples showing that a parallel result does not hold for discretely star-Lindelöf spaces.

EXAMPLE 3.1. There exists a continuous bijection $f : X \to Y$ from a discretely star-Lindelöf, Tychonoff space X to a Tychonoff space Y which is not discretely star-Lindelöf.

PROOF. Let $\Psi = \omega \cup \mathscr{R}$ be the same Isbell-Mrówka space as in the proof of Example 2.8. Then, the space $S(\Psi, \omega) = \Psi \cup (\omega \times \mathfrak{c}^+)$ is discretely star-Lindelöf by Theorem 2.5. Now, we change the topology of $S(\Psi, \omega)$ by declaring that a basic neighborhood of $r \in \mathscr{R}$ is a set of the form

$$G_U(r) = U \cup ((U \cap \omega) \times \mathfrak{c}^+)$$

for a neighborhood U of r in Ψ, and that basic neighborhoods of other points are the same as those in $S(\Psi, \omega)$. We show that the resulting space Y is not discretely star-Lindelöf. For this end, we enumerate the set of all finite subsets of ω as $\{K_n : n \in \omega\}$. Since $|\mathscr{R}| = \mathfrak{c}$, we can write $\mathscr{R} = \{r_{n,\alpha} : \langle n, \alpha \rangle \in \omega \times \mathfrak{c}\}$, where $r_{n,\alpha} \neq r_{n',\alpha'}$ if $\langle n, \alpha \rangle \neq \langle n', \alpha' \rangle$. For each $\langle n, \alpha \rangle \in \omega \times \mathfrak{c}$, define

$$U_{n,\alpha} = (\{r_{n,\alpha}\} \cup (r_{n,\alpha} \setminus K_n)) \cup ((r_{n,\alpha} \setminus K_n) \times \mathfrak{c}^+).$$

Then, $U_{n,\alpha}$ is an open neighborhood of $r_{n,\alpha}$ in Y. Let us consider the open cover

$$\mathscr{U} = \{U_{n,\alpha} : \langle n, \alpha \rangle \in \omega \times \mathfrak{c}\} \cup \{\omega \cup (\omega \times \mathfrak{c}^+)\}.$$

It remains to show that $St(B, \mathscr{U}) \neq Y$ for every countable discrete closed set B in Y. To show this, let B be a countable discrete closed set in Y. Since $B \cap \mathscr{R}$ is countable, there exists $\beta < \mathfrak{c}$ such that

$$B \cap \{r_{n,\beta} : n \in \omega\} = \emptyset. \tag{1}$$

On the other hand, $B \setminus \mathscr{R}$ is finite since every infinite subset of $\omega \cup (\omega \times \mathfrak{c}^+)$ has an accumulation point in Y. Thus, there exists $m \in \omega$ such that

$$B \setminus \mathscr{R} \subseteq K_m \cup (K_m \times \mathfrak{c}^+). \tag{2}$$

Now, $U_{m,\beta}$ is the only element of $\mathscr{U}$ containing the point $r_{m,\beta}$ and $B \cap U_{m,\beta} = \varnothing$ by (1) and (2). Hence, $r_{m,\beta} \notin St(B, \mathscr{U})$, which proves that Y is not discretely star-Lindelöf. Finally, let $X = S(\Psi, \omega)$ and let $f : X \to Y$ be the identity map. Since f is continuous, the proof is complete. □

Yasui-Gao [13] proved that the image of a discretely star-Lindelöf space under a closed continuous map is discretely star-Lindelöf. The following example shows that 'closed map' cannot be replaced by 'open map' in their result.

EXAMPLE 3.2. There exists an open continuous map $f : X \to Y$ from a discretely star-Lindelöf, Tychonoff space X onto a Tychonoff space Y which is not discretely star-Lindelöf.

PROOF. Let Ψ be the same space as in the proof of Example 2.8 and consider the space $S(\Psi, \omega) = \Psi \cup (\omega \times \mathfrak{c}^+)$. Define a retraction $f : S(\Psi, \omega) \to \Psi$ by $f(p) = p$ for $p \in \Psi$ and $f(\langle n, \alpha \rangle) = n$ for $\langle n, \alpha \rangle \in \omega \times \mathfrak{c}^+$. Then, it is easily checked that f is an open continuous map. The space $S(\Psi, \omega)$ is discretely star-Lindelöf by Theorem 2.5, but the space Ψ is not discretely star-Lindelöf by Corollary 2.3. □

Next, we turn to consider preimages. To show that the preimage of a discretely star-Lindelöf space under a closed 2-to-1 continuous map need not be discretely star-Lindelöf, we use the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighbourhood of a point $\langle x, 0 \rangle \in X \times \{0\}$ is a set of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1 \rangle\})$, where U is a neighborhood of x in X.

THEOREM 3.3. *For a space X, the following conditions are equivalent*:

(1) *X is ω_1-compact*;

(2) *$A(X)$ is ω_1-compact*;

(3) *$A(X)$ is discretely star-Lindelöf*;

(4) *$A(X)$ is star-Lindelöf.*

PROOF. The implication (1) $\Rightarrow$ (2) follows from the fact that a perfect preimage of an ω_1-compact space is ω_1-compact. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. To show that (4) $\Rightarrow$ (1), suppose that X is not ω_1-compact. Then, there exists an uncountable discrete closed set D in X. Since $D \times \{1\}$ is an uncountable, discrete, open and closed set in $A(X)$, $A(X)$ is not star-Lindelöf. □

Let X be a discretely star-Lindelöf space which is not ω_1-compact (see Remark 1 above). Then, the space $A(X)$ is not star-Lindelöf by Theorem 3.3. Since the projection $A(X) \to X$ is a closed continuous map, this shows that the preimage of a discretely star-Lindelöf space under a closed 2-to-1 continuous map need not be star-Lindelöf. Now, we give a positive result:

THEOREM 3.4. *Assume that there exists an open and closed, finite-to-one, continuous map f from a space X to a discretely star-Lindelöf space Y. Then, X is discretely star-Lindelöf.*

PROOF. Since $f[X]$ is open and closed in Y, we may assume that $f[X] = Y$. Let $\mathscr{U}$ be an open cover of X and let $y \in Y$. Since $f^{-1}(y)$ is finite, there exists a finite subcollection $\mathscr{U}_y$ of $\mathscr{U}$ such that $f^{-1}(y) \subseteq \bigcup\{U : U \in \mathscr{U}_y\}$ and $U \cap f^{-1}(y) \neq \emptyset$ for each $U \in \mathscr{U}_y$. Since f is closed, there exists an open neighborhood V_y of y in Y such that $f^{-1}[V_y] \subseteq \bigcup\{U : U \in \mathscr{U}_y\}$. Since f is open, we can assume that

$$V_y \subseteq \bigcap\{f[U] : U \in \mathscr{U}_y\}. \tag{3}$$

Taking such open set V_y for each $y \in Y$, we have an open cover $\mathscr{V} = \{V_y : y \in Y\}$ of Y. Since Y is discretely star-Lindelöf, there exists a countable discrete closed subset D of Y such that $St(D, \mathscr{V}) = Y$. Since f is finite-to-one and continuous, the set $E = f^{-1}[D]$ is also a countable discrete closed set in X. To show that $St(E, \mathscr{U}) = X$, let $x \in X$. Then, there exist $y \in Y$ such that $f(x) \in V_y$ and $V_y \cap D \neq \emptyset$. Since

$$x \in f^{-1}[V_y] \subseteq \bigcup\{U : U \in \mathscr{U}_y\},$$

we can choose $U \in \mathscr{U}_y$ with $x \in U$. Then, $U \cap E \neq \emptyset$, because $V_y \subseteq f[U]$ by (3). Hence, $x \in St(E, \mathscr{U})$, and consequently, we have that $St(E, \mathscr{U}) = X$. □

As we shall show in Remark 3 below, Theorem 3.4 fails to be true if 'open and closed, finite-to-one' is replaced by 'open perfect'.

4. Products

In [2], van Douwen-Reed-Roscoe-Tree showed that the product of a star-Lindelöf Tychonoff space and a compact Hausdorff space need not be star-Lindelöf. We begin by showing a similar example for discretely star-Lindelöf spaces:

Example 4.1. There exist a discretely star-Lindelöf Tychonoff space X and a compact Hausdorff space Y such that $X \times Y$ is not star-Lindelöf.

Proof. The proof is essentially same as that of [2, Example 3.3.4]. Let $\Psi = \omega \cup \mathscr{R}$ be the same as in the proof of Example 2.8 and define the space $X = S(\Psi, \omega)$ $(= \Psi \cup (\omega \times \mathfrak{c}^+))$. Then, X is discretely star-Lindelöf by Theorem 2.5. Since $|\mathscr{R}| = \mathfrak{c}$, we can enumerate it as $\mathscr{R} = \{r_\alpha : \alpha < \mathfrak{c}\}$. On the other hand, let $D = \{y_\alpha : \alpha < \mathfrak{c}\}$ be the discrete space of cardinality $\mathfrak{c}$ and let $Y = D \cup \{y_\infty\}$ be the one-point compactification of D. To show that $X \times Y$ is not star-Lindelöf, we consider the open cover

$$\mathscr{U} = \{(\{r_\alpha\} \cup (\omega \times \mathfrak{c}^+)) \times (Y \setminus \{y_\alpha\}) : \alpha < \mathfrak{c}\}$$
$$\cup \{X \times \{y_\alpha\} : \alpha < \mathfrak{c}\} \cup \{(\omega \cup (\omega \times \mathfrak{c}^+)) \times Y\}$$

of $X \times Y$. For every countable subset B of $X \times Y$, there exists $\alpha < \mathfrak{c}$ such that $B \cap (X \times \{y_\alpha\}) = \varnothing$. Then, $\langle r_\alpha, y_\alpha \rangle \notin St(B, \mathscr{U})$ since $X \times \{y_\alpha\}$ is the only element of $\mathscr{U}$ containing $\langle r_\alpha, y_\alpha \rangle$. Hence, $X \times Y$ is not star-Lindelöf. □

Remark 3. By Example 4.1, we can see that the preimage of a discretely star-Lindelöf space under an open perfect map need not be star-Lindelöf.

A map $f : X \to Y$ is called an *s-map* if $f^{-1}(y)$ is separable for each $y \in Y$. Ikenaga [6] proved that the preimage of a star-Lindelöf space under an open perfect continuous s-map is star-Lindelöf. Hence, the product of a star-Lindelöf space and a separable compact space is star-Lindelöf. Moreover, Ikenaga [6] showed that the product of a Lindelöf space and a separable metric space need not be star-Lindelöf. For products of star-Lindelöf spaces, the reader is also referred to [1]. By contrast, little seems to be known about discretely star-Lindelöf spaces. In fact, the following problem is open:

Problem 4.2. Is the product of a discretely star-Lindelöf Tychonoff space and a separable compact Hausdorff space discretely star-Lindelöf? In particular, is the product of a discretely star-Lindelöf Tychonoff space and a compact metric space discretely star-Lindelöf?

The following theorem is a partial answer to Problem 4.2.

Theorem 4.3. *Let X be a discretely star-Lindelöf, countably metacompact space and Y a compact metric space. Then, $X \times Y$ is discretely star-Lindelöf.*

Proof. Let $\mathscr{U}$ be an open cover of $X \times Y$. Fix a countable base $\mathscr{B}$ of Y and let $\{\mathscr{B}_n : n \in \omega\}$ be the set of all finite covers of Y by members of $\mathscr{B}$. For each $n \in \omega$, we choose a finite set $C_n \subseteq Y$ such that $B \cap C_n \neq \emptyset$ for each $B \in \mathscr{B}_n$. For each $x \in X$, Y being compact, we can find an open neighborhood G_x of x in X and $n(x) \in \omega$ such that $G_x \times B$ is included in some member of $\mathscr{U}$ for each $B \in \mathscr{B}_{n(x)}$. For each $n \in \omega$, let $U_n = \bigcup\{G_x : n(x) = n\}$. Then, $\{U_n : n \in \omega\}$ is a countable open cover of X. Since X is countably metacompact, there exists a point-finite open cover $\mathscr{V} = \{V_n : n \in \omega\}$ of X such that $V_n \subseteq U_n$ for each $n \in \omega$. For each $n \in \omega$, let

$$\mathscr{W}_n = \{G_x \cap V_n : x \in X \text{ and } n(x) = n\}.$$

Then, $\mathscr{W} = \bigcup_{n\in\omega} \mathscr{W}_n$ is an open cover of X. Since X is discretely star-Lindelöf, there exists a countable discrete closed set $D = \{x_k : k \in \omega\}$ in X such that $St(D, \mathscr{W}) = X$. Since $\mathscr{V}$ is point-finite, the set $M_k = \{n \in \omega : x_k \in V_n\}$ is finite for each $k \in \omega$. Thus, if we put

$$E = \bigcup_{k\in\omega}\left(\{x_k\} \times \bigcup_{n\in M_k} C_n\right),$$

then E is a countable discrete closed subset of $X \times Y$. To show that $St(E, \mathscr{U}) = X \times Y$, let $\langle s,t\rangle \in X \times Y$ be fixed. Then, there exists $W \in \mathscr{W}$ such that $s \in W$ and $W \cap D \neq \emptyset$, because $St(D, \mathscr{W}) = X$. By the definitions of D and $\mathscr{W}$, there exist $k, n \in \omega$ and $x \in X$ such that $x_k \in D \cap W$, $W = G_x \cap V_n$ and $n(x) = n$. Choose $B \in \mathscr{B}_n$ with $t \in B$. Then, $\langle s,t\rangle \in G_x \times B$ and $(G_x \times B) \cap (\{x_k\} \times C_n) \neq \emptyset$. This implies that $\langle s,t\rangle \in St(E, \mathscr{U})$, since $G_x \times B$ is included in some member of $\mathscr{U}$. Hence, $St(E, \mathscr{U}) = X \times Y$. □

The author does not know if the assumption that X is countably metacompact can be removed from Theorem 4.3 in case X is Hausdorff. The following example shows that the assumption is necessary at least in the realm of T_1-spaces.

Example 4.4. There exists a discretely star-Lindelöf T_1-space X such that the product $X \times (\omega + 1)$ is not discretely star-Lindelöf.

Proof. Let Y be a set with $|Y| = \omega_1$. Define $X = Y \cup \omega_2$, where $\omega_2 = \omega_1^+$, and topologize X as follows: A basic neighborhood of $y \in Y$ is a set of the form

$$G_\alpha(y) = \{y\} \cup \{\beta : \alpha < \beta < \omega_2\}$$

for $\alpha < \omega_2$, and ω_2 is an open subspace of X with the usual order topology. Then, it is easily checked that X is a discretely star-Lindelöf T_1-space. To show that $X \times (\omega + 1)$ is not discretely star-Lindelöf, we write $Y = \bigcup_{n \in \omega} Y_n$, where $Y_m \cap Y_n = \varnothing$ if $m \neq n$ and $|Y_n| = \omega_1$ for each $n \in \omega$. Put $G_y = \{y\} \cup \omega_2$ for each $y \in Y$ and define

$$\mathscr{U} = \bigcup_{n \in \omega} \bigcup_{y \in Y_n} (\{G_y \times \{i\} : i \leq n\} \cup \{G_y \times \{j : n < j \leq \omega\}\}).$$

Then, $\mathscr{U}$ is an open cover of $X \times (\omega + 1)$. Let B be a countable discrete closed set in $X \times (\omega + 1)$. Then, $B \cap (\omega_2 \times (\omega + 1))$ is finite since $\omega_2 \times (\omega + 1)$ is countably compact. Hence, there exists $n \in \omega$ such that $B \cap (\omega_2 \times \{n\}) = \varnothing$. Since $|Y_n| = \omega_1$, we can find $y \in Y_n$ such that $\langle y, n \rangle \notin B$. Then, $B \cap (G_y \times \{n\}) = \varnothing$, which implies that $\langle y, n \rangle \notin St(B, \mathscr{U})$, because $G_y \times \{n\}$ is the only element of $\mathscr{U}$ containing the point $\langle y, n \rangle$. Hence, $X \times (\omega + 1)$ is not discretely star-Lindelöf. □

Let N be the discrete space of non-negative integers. The space N^{ω_1} is star-Lindelöf since it is separable, but is not ω_1-compact by Mycielski [11]. The following problem, however, still remains open.

Problem 4.5. Is the product N^{ω_1} discretely star-Lindelöf?

There is another star-covering property, called the property (a), which is closely related to discretely star-Lindelöf spaces. Matveev [8] defined a space X to have the *property* (a) if for every open cover $\mathscr{U}$ of X and for every dense subset D of X, there exists a closed (in X) discrete subset F of D such that $St(F, \mathscr{U}) = X$. It is obvious that every T_1-space X satisfies the following condition: For every open cover $\mathscr{U}$ of X, there exists a closed discrete set F in X such that $St(F, \mathscr{U}) = X$. Both discretely star-Lindelöf spaces and spaces with the property (a) were defined by strengthening this condition. Every uncountable discrete space has the property (a), but is not discretely star-Lindelöf. On the other hand, the space $X = \omega_1 \times (\omega_1 + 1)$ is discretely star-Lindelöf since it is countably compact, while it is known ([7, Example 1.5]) that X does not have the property (a). We conclude this paper with the following result due to Ohta under his permission.

Example 4.6 (Ohta). The product N^{ω_1} does not have the property (a).

Proof. We consider the open cover $\mathscr{U} = \{U(\alpha, \beta, k) : \{\alpha, \beta\} \subseteq \omega_1, \alpha \neq \beta, k \in N\}$ of N^{ω_1}, where $U(\alpha, \beta, k) = \{x \in N^{\omega_1} : x(\alpha) = x(\beta) = k\}$. Also, we define the set $D = \{x \in N^{\omega_1} : |\mathrm{supp}(x)| < \omega\}$, where $\mathrm{supp}(x) = \{\alpha < \omega_1 : x(\alpha) \neq 0\}$. Then, D is a dense σ-compact subset of N^{ω_1}. We show that $St(F, \mathscr{U}) \neq N^{\omega_1}$ for every closed (in N^{ω_1}) discrete subset F of D. For this end, let F be such a subset of D. Then, F is at most countable since D is σ-compact, which implies that the set $S = \bigcup\{\mathrm{supp}(x) : x \in F\}$ is also at most countable. Hence, we can find a point $y \in N^{\omega_1}$ such that $y|_S : S \to N$ is one-to-one and $y(\alpha) = 1$ for each $\alpha \in \omega_1 \setminus S$. Now, let $U(\alpha,\beta,k) \in \mathscr{U}$ such that $U(\alpha,\beta,k) \cap F \neq \emptyset$. It remains to show that $y \notin U(\alpha,\beta,k)$. We distinguish two cases: If $\{\alpha,\beta\} \subseteq S$, then $y \notin U(\alpha,\beta,k)$ since $y(\alpha) \neq y(\beta)$ by the definition of y. If $\{\alpha,\beta\} \nsubseteq S$, then we may assume that $\alpha \notin S$. Since there is a point $x \in U(\alpha,\beta,k) \cap F$, we have $k = x(\alpha) = 0$ because $\alpha \notin \mathrm{supp}(x)$. Hence, if $y \in U(\alpha,\beta,k)$, then $y(\alpha) = 0$, which contradicts the definition of y. Consequently, $y \notin U(\alpha,\beta,k)$. □

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