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ESTIMATIONS OF SMALL TRANSFINITE DIMENSION IN SEPARABLE METRIZABLE SPACES

By

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Abstract. We improve some known inequalities describing the mutual relation between small transfinite dimension and transfinite dimension D in separable metrizable spaces. We also estimate small transfinite dimension of a product with a finite-dimensional factor, generalizing the results due to Luxemburg.

1. Introduction

All spaces considered here will be *metrizable separable*. By trind, trInd and D we denote Hurewicz's, Smirnov's and Henderson's transfinite extensions of the finite dimension dim in the class of separable metrizable spaces. The transfinite dimensions trind and trInd are the natural extension of inductive dimensions ind and Ind respectively. The transfinite dimension D is defined by a special way. Let us recall that. Let $\alpha = \lambda(\alpha) + n(\alpha)$ be the natural decomposition of the ordinal number α into the sum of the limit ordinal number $\lambda(\alpha)$ and the non-negative integer $n(\alpha)$. Define $D(\emptyset) = -1$. For a non-empty space X, the *D-dimension* $D(X)$ of X is defined to be the smallest ordinal number α such that there exists a closed cover $\{A_\beta : \beta \leq \lambda(\alpha)\}$ of X satisfying the following conditions:

(a) The union $\bigcup\{A_\beta : \delta \leq \beta \leq \lambda(\alpha)\}$ is closed for every $\delta \leq \lambda(\alpha)$.

(b) For every $x \in X$ the set $\{\beta \leq \lambda(\alpha) : x \in A_\beta\}$ has a largest element.

(c) $\dim A_\beta < \infty$ for every $\beta < \lambda(\alpha)$, and $\dim A_{\lambda(\alpha)} = n(\alpha)$.

We refer the reader to [3] for basic results on these dimensions.

It is well known ([3, Theorem 7.3.16]) that for any space X, the inequality

$$\operatorname{trind} X \leq D(X) + 1 \tag{1}$$

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holds, and ([3, Theorem 7.3.17]) if the space X has large transfinite dimension trInd, then

$$\operatorname{trind} X \leq \operatorname{trInd} X \leq D(X). \tag{2}$$

In ([5, Theorem 8.2]) Luxemburg sharpened (1) as follows: If X is an infinite-dimensional compact space, then the inequality

$$\operatorname{trind} X \leq \lambda(D(X)) + \left[\frac{n(D(X)) + 3}{2}\right] \tag{3}$$

holds. As a corollary of this result one gets the inequality

$$\operatorname{trind} X < D(X) \tag{4}$$

for any compact space X with $\lambda(D(X)) \geq \omega_0$ and $n(D(X)) \geq 4$ ([5, Corollary 8.2]). In section 2, we improve (1) for the case $n(D(X)) \geq 2$ and (3) for $n(D(X)) \geq 5$ and generalize (4) for non-compact spaces.

It is difficult to determine the behaviour of small transfinite dimension of products. In this point of view, Luxemburg proved that there exists a compact space X (it is Smirnov's compact space S^{ω_0+2}) such that $\operatorname{trind}(X \times Y) < \operatorname{trind} X + \operatorname{ind} Y$ for any finite-dimensional space Y with $\operatorname{ind} Y \geq 1$ ([5, Theorem 7.2]). (Recall that for any finite-dimensional space B we have always the equality $\operatorname{ind}(B \times I) = \operatorname{ind} B + 1$, where I is the closed interval $[0,1]$.) Since the Smirnov's compactum S^{ω_0+2} has $\operatorname{trind} S^{\omega_0+2} = \omega_0 + 2$ ([5, Theorem 7.1]), this result can be reformulated in stronger form as follows. There exists a compact space X (it is Smirnov's compact space S^{ω_0}) such that

$$\operatorname{trind}(X \times Y) < \operatorname{trind} X + \operatorname{ind} Y \tag{5}$$

for any finite-dimensional space Y with $\operatorname{ind} Y \geq 3$. (We notice that for any finite-dimensional space B, $\operatorname{ind}(B \times I^3) = \operatorname{ind} B + 3$.) In section 3, we show that (5) holds for more general spaces.

2. Mutual Relation between Small Transfinite Dimension and D-Dimension

In [2], new finite sum theorems for small transfinite dimension trind were proved. The following three sum theorems are useful in the paper.

THEOREM A ([2] Theorem 3.1). *Let X be a space represented as $X = X_1 \cup X_2$, where X_i is closed in X, and* $\operatorname{trind} X_i \leq \alpha_i$, $i = 1, 2$. *Then,*

$$\operatorname{trind} X \leq \begin{cases} \max\{\alpha_1, \alpha_2\}, & \text{if } \lambda(\alpha_1) \neq \lambda(\alpha_2), \\ \max\{\alpha_1, \alpha_2\} + 1, & \text{if } \lambda(\alpha_1) = \lambda(\alpha_2). \end{cases}$$

More generally, if $X = \bigcup_{k=1}^{n+1} X_k$, *where each* X_k *is closed in* X, *and* $\max\{\operatorname{trind} X_k : k = 1, 2, \dots n+1\} \leq \alpha$, *then* $\operatorname{trind} X \leq \alpha + m$, *where* m *is an integer such that* $0 \leq n \leq 2^m - 1$.

We need the following two notions which are natural generalizations of the free union of finite number of spaces. Recall that a decomposition $X = F \cup \bigcup_{i=1}^{\infty} E_i$ of a space X into disjoint sets is called *A-special* (*B-special*) if E_i is clopen in X (E_i is clopen in X and $\lim_{n\to\infty} \operatorname{diam}(E_i) = 0$, where $\operatorname{diam}(A)$ is the diameter of A).

THEOREM B ([2] Lemma 3.4). *Let* $X = F \cup \bigcup_{i=1}^{\infty} E_i$ *be a B-special decomposition of a space X. If* $\sup\{\operatorname{trind} F, \operatorname{trind} E_i : i \in \mathbb{N}\} \leq \alpha$, *then* $\operatorname{trind} X \leq \alpha$.

Recall from [2, Lemma 2.2] that if $X = F \cup \bigcup_{i=1}^{\infty} E_i$ is an A-special decomposition of a compact space X with $\operatorname{ind} F = n \geq 0$, then X can be represented as $X = \bigcup_{k=1}^{n+1} Z_k$, where Z_k is closed in X, and Z_k admits a B-special decomposition $Z_k = F \cup \bigcup_{j=1}^{\infty} E_j^k$ with $E_j^k \subset E_i$ for a finite number of indexes j for every i.

THEOREM C ([2] Corollary 3.11). *Let X be a compact space and* α *an ordinal number* $\geq \omega_0$, *then we have the following.*

(a) *If* $X = F \cup \bigcup_{i=1}^{\infty} E_i$ *is an A-special decomposition of X such that* $\operatorname{ind} F = n \geq 0$, $\sup_{i\to\infty} \operatorname{trind} E_i \leq \alpha$ *and* $n \leq 2^m - 1$ *for some integer m, then* $\operatorname{trind} X \leq \alpha + m$.

(b) *If F is a closed subset of the space X such that* $\operatorname{ind} F = n \geq 0$, $\sup\{\operatorname{trind}_x X : x \in X \setminus F\} \leq \alpha$ *and* $n \leq 2^m - 1$ *for some integer m, then* $\operatorname{trind} X \leq \alpha + m + 1$.

REMARK 2.1. We notice that the term 1 in the right side of the estimation from Theorem C (b) is essential. In fact, there exists a compact space Y such that $\operatorname{trind} Y = \omega_0 + 1$ and $\operatorname{trInd} Y = \omega_0 + 2$ (cf. [3, Problem 7.1.G]). Choose two disjoint closed subsets A and B of Y such that any partition L between A and B has $\operatorname{trInd} L \geq \omega_0 + 1$. Since every compact space has $\operatorname{trInd} = \omega_0$ if and only if it has $\operatorname{trind} = \omega_0$ (cf. [3, Proposition 7.1.22]), it follows that $\operatorname{trind} L \geq \omega_0 + 1$. Let X be the quotient space Y/A with π as the quotient mapping from Y to X. Then it follows that $\operatorname{trind} X = \operatorname{trInd} X = \omega_0 + 2$ and the compact space X is the one-point compactification of the space $Z = X \setminus \pi(A)$ with $\operatorname{trind} Z = \omega_0 + 1$ and $\pi(A)$ as the compactification point. Then we have $\operatorname{ind} \pi(A) = 0$, $m = 0$ and $\operatorname{trind} X = (\omega_0 + 1) + 1$.

Concerning on the space Y, it seems to be interesting to evaluate $\operatorname{trind}(Y \times I)$. We do not know it.

Now, we improve the estimation (3) for $n(D(X)) \geq 5$ as follows.

THEOREM 2.2. *Let X be a compact space with $D(X) = \alpha \geq \omega_0$. Then* $\operatorname{trind} X \leq \lambda(\alpha) + m + 1$, *where m is an integer such that $0 \leq n(\alpha) \leq 2^m - 1$.*

PROOF. Recall from the definition of D-dimension that there exists a closed cover $\{A_\beta\}_{\beta \leq \lambda(\alpha)}$ of the space X such that

(a) the union $\bigcup\{A_\beta : \delta \leq \beta \leq \lambda(\alpha)\}$ is closed for every $\delta \leq \lambda(\alpha)$;

(b) for every $x \in X$ the set $\{\beta \leq \lambda(\alpha) : x \in A_\beta\}$ has a largest element;

(c) $\dim A_\beta < \infty$ for every $\beta < \lambda(\alpha)$, and $\dim A_{\lambda(\alpha)} = n(\alpha)$.

By the properties (a) and (b), for every $x \in X \backslash A_{\lambda(\alpha)}$ there exists an open neighborhood Ox of x in X such that $D(Ox) < \lambda(\alpha)$. By the estimation (1), we have that $\operatorname{trind}_x X \leq \operatorname{trind}(Ox) < \lambda(\alpha)$ for this point x. By Theorem C (b), we have $\operatorname{trind} X \leq \lambda(\alpha) + m + 1$. $\square$

The following table helps us to understand how we improve the estimination from (3).

Table 1. Comparison of estimations (3) and Theorem 2.2

$n(D(X))$	trind X in (3)	trind X in Theorem 2.2
0	$\lambda(D(X)) + 1$	$\lambda(D(X)) + 1$
1	$\lambda(D(X)) + 2$	$\lambda(D(X)) + 2$
2	$\lambda(D(X)) + 2$	$\lambda(D(X)) + 3$
3	$\lambda(D(X)) + 3$	$\lambda(D(X)) + 3$
4	$\lambda(D(X)) + 3$	$\lambda(D(X)) + 4$
5	$\lambda(D(X)) + 4$	$\lambda(D(X)) + 4$
6	$\lambda(D(X)) + 4$	$\lambda(D(X)) + 4$
7	$\lambda(D(X)) + 5$	$\lambda(D(X)) + 4$
8	$\lambda(D(X)) + 5$	$\lambda(D(X)) + 5$
9	$\lambda(D(X)) + 6$	$\lambda(D(X)) + 5$
...	...	...
15	$\lambda(D(X)) + 9$	$\lambda(D(X)) + 5$
16	$\lambda(D(X)) + 9$	$\lambda(D(X)) + 6$
...	...	...
31	$\lambda(D(X)) + 17$	$\lambda(D(X)) + 6$
32	$\lambda(D(X)) + 17$	$\lambda(D(X)) + 7$
...	...	...
...	...	...
...	...	...

Now it is natural to repeat the following question.

QUESTION 2.3 (cf. [5] p. 345). *Do there exist compact spaces* X_α *with* $\operatorname{trind} X_\alpha = \operatorname{trInd} X_\alpha$ *for every* $\alpha < \omega_1$?

Let us improve now the estimation (1) for $n(D(X)) \geq 2$.

THEOREM 2.4. *Let* X *be a space and* $D(X) = \alpha \geq \omega_0$. *Then* $\operatorname{trind} X \leq \lambda(\alpha) + m + 1$, *where* m *is an integer such that* $0 \leq n(\alpha) + 1 \leq 2^m - 1$.

PROOF. Recall (cf. [3, p. 361]) that there exists a metrizable compactification bX of X such that $D(X) \leq D(bX) \leq D(X) + 1$. By Theorem 2.2, it follows that $\operatorname{trind} bX \leq \lambda(\alpha) + m + 1$, where m is an integer such that $0 \leq n(\alpha) + 1 \leq 2^m - 1$. Note that $\operatorname{trind} X \leq \operatorname{trind} bX$. □

We refer the reader to Table 2 for the comparisions of estimations between (1) and ours.

As a corollary to Theorem 2.4, we have an estimation similar to (4) for non-compact spaces.

COROLLARY 2.5. *For any space* X *with* $\lambda(D(X)) \geq \omega_0$ *and* $n(D(X)) \geq 5$ *we have* $\operatorname{trind} X < D(X)$.

Table 2. Comparison of estimations (1) and Theorem 2.4

$n(D(X))$	trind X in (1)	trind X in Theorem 2.4
0	$\lambda(D(X)) + 1$	$\lambda(D(X)) + 2$
1	$\lambda(D(X)) + 2$	$\lambda(D(X)) + 3$
2	$\lambda(D(X)) + 3$	$\lambda(D(X)) + 3$
3	$\lambda(D(X)) + 4$	$\lambda(D(X)) + 4$
4	$\lambda(D(X)) + 5$	$\lambda(D(X)) + 4$
5	$\lambda(D(X)) + 6$	$\lambda(D(X)) + 4$
6	$\lambda(D(X)) + 7$	$\lambda(D(X)) + 4$
7	$\lambda(D(X)) + 8$	$\lambda(D(X)) + 5$
8	$\lambda(D(X)) + 9$	$\lambda(D(X)) + 5$
...	...	...
14	$\lambda(D(X)) + 15$	$\lambda(D(X)) + 5$
15	$\lambda(D(X)) + 16$	$\lambda(D(X)) + 6$
...	...	...
31	$\lambda(D(X)) + 32$	$\lambda(D(X)) + 7$
32	$\lambda(D(X)) + 33$	$\lambda(D(X)) + 7$
...	...	...
...	...	...
...	...	...

3. Small Transfinite Dimension of Products

At first we generalize (5) for a space admitting a B-special decomposition. We notice that the Smirnov's compactum S^{ω_0} has a B-special decomposition as follows: $S^{\omega_0} = \{\text{a point}\} \cup \bigcup_{n=0}^{\infty} I^n$ such that $\operatorname{ind}\{\text{a point}\} = 0 < \omega_0$, $\operatorname{ind} I^n < \omega_0$ and $\sup_{n\to\infty} \operatorname{trind} I_n = \omega_0$.

THEOREM 3.1. *Let X be a space and λ a limit ordinal number $\geq \omega_0$. If X admits a B-special decomposition $F \cup \bigcup_{i=1}^{\infty} E_i$ such that* $\operatorname{trind} F < \lambda$, $\operatorname{trind} E_i < \lambda$ *for each i and* $\sup_{i\to\infty} \operatorname{trind} E_i = \lambda$, *then* $\operatorname{trind}(X \times Y) < \operatorname{trind} X + \operatorname{ind} Y$ *for any finite dimensional space Y with* $\operatorname{ind} Y \geq 3$.

PROOF. Let $\operatorname{ind} Y = n \geq 3$ and Z be a compactification of Y such that $\operatorname{ind} Z = n$. By the Ostrand's Theorem ([3, Theorem 3.2.4]), it follows that for every $\varepsilon = \frac{1}{k}$, $k = 1, 2, \dots$ there exist disjoint finite systems $\mathscr{B}_i^{\varepsilon}$, $i = 1, \dots, n+1$, of closed sets with $\operatorname{diam} B < \varepsilon$ for every $B \in \mathscr{B}_i^{\varepsilon}$ and every i such that $Z = \bigcup_{i=1}^{n+1} (\bigcup \mathscr{B}_i^{\varepsilon})$. With help of these systems one can observe that the product $X \times Z$ can be written as the union $\bigcup_{i=1}^{n+1} Z_i$, where every Z_i admits the B-special decomposition $(F \times Z) \cup \bigcup \{E_k \times B : B \in \mathscr{B}_i^{1/k}, k = 1, 2, \dots\}$. Note that, by [5, Proposition 6.1], the inequalities $\operatorname{trind}(F \times Z) \leq \operatorname{trind} F + \operatorname{ind} Z < \lambda$ and $\operatorname{trind}(E_k \times B) \leq \operatorname{trind} E_k + \operatorname{ind} B < \lambda$ are valid. By Theorem B, we have $\operatorname{trind} Z_i = \lambda$ for every i and $\operatorname{trind} X = \lambda$. By Theorem A, we get that $\operatorname{trind}(X \times Z) \leq \lambda + m$, where m is an integer such that $0 \leq n \leq 2^m - 1$. Since for any $n \geq 3$ we can take m such as $m < n$ if $n \geq 3$, we have $\operatorname{trind}(X \times Y) \leq \operatorname{trind}(X \times Z) < \operatorname{trind} X + \operatorname{ind} Y$. □

We need the following simple lemma to show Theorem 3.3.

LEMMA 3.2. *For every integer m there exists an integer $k(m)$ such that for every $k \geq k(m)$ the inequality $q + 1 < k$ holds for every q satisfying the inequality $2^{q-1} \leq m + k \leq 2^q - 1$.*

PROOF. For each m there is a natural number $l(m)$ such that $2^{l-1} - (l+2) \geq m$ for each $l \geq l(m)$. We put $k(m) = 2^{l(m)}$. Let $k \geq k(m)$ and q be such that $2^{q-1} \leq m + k \leq 2^q - 1$. Since $2^q > 2^q - 1 \geq m + k \geq k(m) = 2^{l(m)}$, it follows that $q \geq l(m)$. By the choice of $l(m)$, we have $k \geq 2^{q-1} - m \geq q + 2$. Hence $q + 1 < k$. □

We have another generalization of (5).

THEOREM 3.3. *Let X be an infinite-dimensional compact space with* $\operatorname{trind} X = \alpha$. *Let also the subspace*

$$F = X \backslash \{x \in X\colon \textit{there exists an open neighborhood } Ox \textit{ of } x \textit{ with } \operatorname{trind} Ox < \lambda(\alpha)\}$$

of X be finite-dimensional. Then there exists an integer $k(\operatorname{ind} F)$ such that $\operatorname{trind}(X \times Y) < \operatorname{trind} X + \operatorname{ind} Y$ *for any finite dimensional space Y with* $\operatorname{ind} Y \geq k(\operatorname{ind} F)$.

PROOF. Put $\operatorname{ind} F = m \geq 0$. Let $k(m)$ be as in Lemma 3.2, Y a space with $\operatorname{ind} Y = k \geq k(m)$ and Z a compactification of Y such that $\operatorname{ind} Z = k$. It is known that $\operatorname{ind}(F \times Z) = l \leq m + k$. Observe that $\operatorname{trind}_x(X \times Z) < \lambda(\alpha)$ for every $x \in (X \times Z) \backslash (F \times Z)$. So by Theorem C (b), we have $\operatorname{trind}(X \times Z) \leq \lambda(\alpha) + q + 1$, where q is any integer such that $0 \leq l \leq 2^q - 1$. We choose a natural number q such that $2^{q-1} \leq m + k \leq 2^q - 1$. Then it follows from Lemma 3.2 that $\operatorname{trind}(X \times Y) \leq \operatorname{trind}(X \times Z) \leq \lambda(\alpha) + q + 1 < \lambda(\alpha) + k \leq \lambda(\alpha) + n(\alpha) + k = \operatorname{trind} X + \operatorname{ind} Y$. $\square$

Recall ([5, Definition 1.3]) that an ordinal number $\alpha > \omega_0$ is called *invariant* if $\alpha = \omega_0^{\omega_0} \cdot \gamma$ for some γ. It is evident that for any two invariant numbers $\alpha, \beta > \omega_0$, $\alpha + \beta$, $\alpha(+)\beta$ are invariant too, where $+$ denotes the usual sum of ordinals and $(+)$ denotes the natural one. We refer the reader to [4] for definitions.

THEOREM 3.4. *Let α and β be invariant ordinal numbers $> \omega_0$ and i, j be two non-negative integers such that $i + j \leq 2$. Then*

$$\operatorname{trind}(S^{\alpha+i} \times S^{\beta+j}) = \operatorname{trind} S^{\alpha+i}(+)\operatorname{trind} S^{\beta+j} = (\alpha(+)\beta) + (i + j).$$

PROOF. By [1, Corollary 2], we have $\operatorname{trind}(S^{\alpha+i} \times S^{\beta+j}) = \operatorname{trind} S^{(\alpha+i)(+)(\beta+j)}$. Observe that $(\alpha + i)(+)(\beta + j) = (\alpha(+)\beta) + (i + j))$ and $\alpha(+)\beta$ is invariant. So by [5, Theorem 7.1], we have $\operatorname{trind} S^{\alpha+i} = \alpha + i$, $\operatorname{trind} S^{\beta+j} = \beta + j$ and $\operatorname{trind} S^{(\alpha(+)\beta)+(i+j))} = (\alpha(+)\beta) + (i + j))$. $\square$

Because of the last theorem, the condition of finite-dimensionality of the space Y in Theorems 3.1 and 3.3 can not be omitted. Nevertheless there exist the following generalizations of these theorems on the infinite-dimensional case.

COROLLARY 3.5. *Let Z be a space with* $\operatorname{trind} Z = \alpha$, *where α is a limit ordinal number (in particular* 0), *and let X, Y be the same spaces as either from Theorem* 3.1 (*or Theorem* 3.3). *Then* $\operatorname{trind}(X \times (Z \times Y)) < (\operatorname{trind} X(+)\operatorname{trind} Z) + \operatorname{ind} Y$.

PROOF. It follows from [6, Theorem 2.32] and Theorem 3.1 (or Theorem 3.3) that $\operatorname{trind}(X \times (Z \times Y)) = \operatorname{trind}((X \times Y) \times Z) \leq \operatorname{trind}(X \times Y)(+)\operatorname{trind} Z < (\operatorname{trind} X + \operatorname{ind} Y)(+)\operatorname{trind} Z = (\operatorname{trind} X(+)\operatorname{trind} Z) + \operatorname{ind} Y$. □

REMARK 3.6. Observe that if n, m are non-negative integers and $m(n) = \min\{m : n \leq 2^m - 1\}$ then $m(n) = [\log_2 n] + 1$. Thus Theorems A, C, 2.2 and 2.4 can be reformulated in terms of log-function.

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