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ATOMIC MAPPINGS CAN SPOIL LIGHTNESS OF OPEN MAPPINGS

By

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Abstract. We study continua on which each nonconstant open mapping is light. W. Makuchowski asked if this property is preserved under atomic mappings. It is known that this is true under an additional assumption of arcwise connectedness of the domain continuum. We show that in general this is not true.

1. Introduction

It is known that each open mapping defined on an arc or on a simple closed curve is light, see [9, Theorems 1.2 and 1.3, p. 184]. These results have been extended in several ways. For example in [3, Theorem 5, p. 214] it is shown that if a domain space is locally dendritic and the range one has no isolated points, then each nonconstant open mapping is light. In particular, this is the case when the domain space is a continuum being a local dendrite. Obviously each local dendrite, as a locally connected continuum, is arcwise connected. In a further study of the subject, W. Makuchowski asked in [8, Question 2.6, p. 782] whether there is a non-arcwise connected continuum X such that each nonconstant open mapping defined on X is light. Answering this question in the affirmative, we exhibited in [4, Theorems 16 and 41] two uncountable families of such continua, each having some additional properties.

In [8, Question 2.10, p. 783] W. Makuchowski asked whether the considered property of continua (that each nonconstant open mapping defined on any of them is light) is preserved under atomic mappings. Answering this question in the negative, we present an uncountable family of continua having the property, and

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atomic mappings defined on the continua so that the range spaces do not have the property.

All spaces considered in the paper are assumed to be metric and all mappings are continuous. Given a space X and its subset S, we denote by $\operatorname{cl} S$ the closure of S, by $\operatorname{bd} S$ its boundary in X, and by $\operatorname{int} S$ its interior in X. The symbols $\boldsymbol{Z}$ and $\boldsymbol{N}$ denote the sets of all integers and of all positive integers, respectively, and $\boldsymbol{R}$ stands for the set of all real numbers. We will also use notation $(-\infty, \infty)$ instead of $\boldsymbol{R}$.

A *continuum* means a compact connected space. A continuum homeomorphic to the unit circle is called a *simple closed curve*. A subset S of a space X is said to be *arcwise connected* provided that for every two points p and q of S there exists in X an arc A with end points p and q such that $A \subset S$. An arc $A \subset X$ with end points p and q is said to be a *free arc* in X provided that $A \setminus \{p, q\}$ is an open subset of X. We will use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [6, §51, I, p. 254]. In particular, a point p of a subset S of a space X is called an *end point* of S provided that it is of order one in S, i.e., for each $\varepsilon > 0$ there is a neighborhood U of p such that $\operatorname{diam} U < \varepsilon$ and card $(S \cap \operatorname{bd} U) = 1$. A *ray* means a one-to-one image of the closed half-line $[0, +\infty)$, and the image of 0 is called the *end point of the ray*.

For an arbitrary class $\mathfrak{M}$ of mappings between continua, a mapping $f : X \to Y$ is said to be *hereditarily* $\mathfrak{M}$ provided that for each subcontinuum $S \subset X$ the partial mapping $f|S : S \to f(S) \subset Y$ is in $\mathfrak{M}$.

A mapping $f : X \to Y$ is said to be:

- *interior at a point* $p \in X$ provided that for each open neighborhood U of p in X the point $f(p)$ is an interior point of the image $f(U)$ in Y;
- *atomic* provided that for each subcontinuum $K \subset X$ with nondegenerate image $f(K)$ the equality $f^{-1}(f(K)) = K$ holds;
- *monotone* provided that for each subcontinuum Q of Y the inverse image $f^{-1}(Q)$ is connected;
- *open* provided that for each open subset U of X its image $f(U)$ is an open subset of Y;
- *simple* provided that card $f^{-1}(y) \leq 2$ for each point $y \in Y$;
- *light* provided that for each point $y \in Y$ each component of the inverse image $f^{-1}(y)$ is a singleton (equivalently, if $f^{-1}(f(x))$ is totally disconnected for each $x \in X$; note that if the inverse images of points are compact, this condition is equivalent to the property that they are zero-dimensional).

Obviously a mapping is open if and only if it is interior at each point of its domain. It is known that each atomic mapping of a continuum is hereditarily monotone, [7, (4.14), p. 17]. The reader is referred to [7, Chapters 3 and 4, especially Table II, p. 28] to see various interrelations between these classes of mappings. Simple mappings were defined in [2, p. 84]. It is evident that each simple mapping is light.

Let L denote the class of all continua X such that each nonconstant open mapping defined on X is light, and let $\mathfrak{M}$ be an arbitrary class of mappings between continua that contains the class of homeomorphisms. Consider classes $\mathfrak{M}$ for which the implication holds:

$$\text{if } f \in \mathfrak{M} \quad \text{and} \quad X \in \mathsf{L}, \quad \text{then } f(X) \in \mathsf{L}. \tag{1}$$

Recall the following assertions (see [8, Propositions 2.7–2.9, p. 783]).

(1.1) *The implication* (1) *holds if* $\mathfrak{M}$ *is the class of open mappings.*

(1.2) *The implication* (1) *holds if* $\mathfrak{M}$ *is the class of atomic mappings and the continuum* X *is arcwise connected.*

(1.3) *The implication* (1) *does not hold if* $\mathfrak{M}$ *is the class of hereditarily monotone mappings.*

W. Makuchowski in [8, Question 2.10, p. 783] asked whether the implication (1) is true if the continuum X is not arcwise connected and $\mathfrak{M}$ is the class of atomic mappings. Answering this question in the negative, we construct in the present paper an uncountable family of non-arcwise connected continua for which (1) does not hold.

Let us recall the following two (still open) problems (see [3, Problem 3, p. 214 and Problem 11, p. 217]).

Problem 1.4. What topological spaces X and Y have the property that each open mapping from X onto Y is light?

Problem 1.5. Characterize all continua X being in the class L.

Note that the family of continua in L that is defined in the present paper gives a partial answer to Problems 1.4 and 1.5.

2. Crooked Spirals

If C is a dense subspace of a compact space Z, then Z is called a *compactification* of C, and $Z \setminus C$ is called the *remainder* of C in Z (see e.g. [1, p. 34]).

It is known that if C is a locally compact, noncompact, separable metric space, then each continuum is a remainder of C in some compactification of C, [1, Theorem, p. 35]. Taking as C a one-to-one image of the real half-line $[0, \infty)$ we conclude the following statement, which will play the key role in our considerations.

STATEMENT 2.1. *Each nondegenerate continuum B is a remainder of a ray C in some compactification of C, and then $Y = B \cup C$ is a continuum having C as an arc-component with $B = \mathrm{cl}\, C \setminus C$.*

Let B^- and B^+ are two disjoint continua. A set C is called a *spiral from B^- to B^+* if

(S1) there is a one-to-one surjective mapping $g : (-\infty, \infty) \to C$ of the open interval $(-\infty, \infty)$ onto C;

(S2) $B^+ \cup g([0, \infty))$ is a compactification of $g([0, \infty))$ having B^+ as the remainder of $g([0, \infty))$ in $B^+ \cup g([0, \infty))$;

(S3) $B^- \cup g((-\infty, 0])$ is a compactification of $g((-\infty, 0])$ having B^- as the remainder of $g((-\infty, 0])$ in $B^- \cup g((-\infty, 0])$;

(S4) $(B^- \cup B^+) \cap C = \emptyset$.

If conditions (S1)–(S4) are satisfied, then we say that the above mentioned mapping $g : (-\infty, \infty) \to C$ *describes the spiral C from B^- to B^+*, and the union

$$X = B^- \cup C \cup B^+$$

is named a *spiral-arc from B^- to B^+*. Obviously each arc is a spiral-arc (with its ends as degenerate continua B^- and B^+); and the $\sin(1/x)$-curve $\mathrm{cl}\{(x, \sin(1/x)) : x \in (0, 1]\}$ is a spiral-arc from a singleton $\{(1, \sin 1)\}$ to the closed segment $\{(0, y) : y \in [-1, 1]\}$.

A spiral C from a continuum B^- to a nondegenerate continuum B^+ is said to be *crooked* (*from B^- to B^+*) if for each point $a \in B^+$ there exists an arbitrarily small neighborhood U of a such that for each $b \in B^+ \cap U$ there exists in $U \cap C$ a sequence of arcs J_n with end points α_n, β_n satisfying

(CS1) $\lim \alpha_n = \lim \beta_n \in \{a, b\}$;

(CS2) $\{a, b\} \subset \liminf J_n$.

A simple example of a crooked spiral is the following.

EXAMPLE 2.2. Let $B^- = \{(1, 0)\}$ and $B^+ = \{(0, y) \in \mathbf{R}^2 : y \in [0, 1]\}$. Take a sequence of all rationals $\{r_n\}$ in $(0, 1)$, and for each $n \in \mathbf{N}$ let L_n be the union of

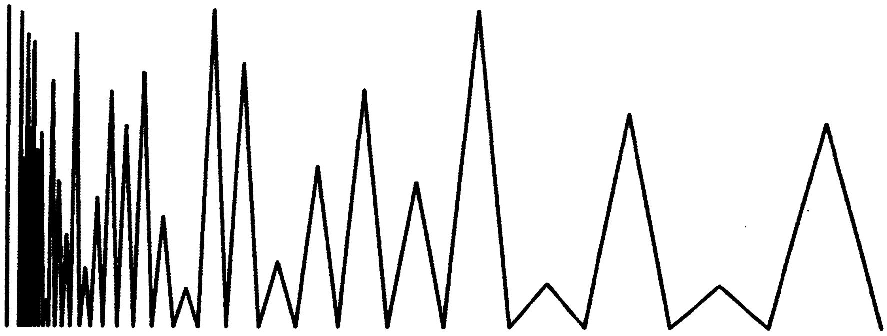

Figure 1 (Example 2.2)

two segments, the first one joining points $(1/n, 0)$ with $((1/n + 1/(n+1))/2, r_n)$, and the second one joining $(1/(n+1), 0)$ with $((1/n + 1/(n+1))/2, r_n)$. Put $C = \bigcup\{L_n : n \in N\}$. Then C is a crooked spiral from B^- to B^+. The continuum $X = B^- \cup C \cup B^+$ is pictured in Fig. 1. $\square$

According to the above definition of a crooked spiral, a spiral C which is crooked from B^- to B^+ need not be crooked from B^+ to B^- even if both B^- and B^+ are nondegenerate.

Note the following simple observation.

(2.3) Let X be a spiral-arc from B^- to B^+. Then each nondegenerate subcontinuum of X with the empty interior is contained in $B^- \cup B^+$.

Observe that if the nondegenerate continua B^- and B^+ in a spiral-arc X are arcwise connected, then X has exactly three arc-components, namely B^-, B^+ and C. Since an arbitrary continuum can be taken as B^- and B^+ in the definition of a spiral-arc according to Statement 2.1, and since the family of all (nonhomeomorphic) subcontinua of the Hilbert cube is uncountable, the family of all spiral-arcs is uncountable. But even if the continua B^- and B^+ are fixed, we have uncountably many nonhomeomorphic 'crooked' compactifications of the real half-line with B^+ as the remainder. Summarizing, we conclude that

(2.4) *The family of all (crooked) spiral-arcs is uncountable.*

Let H denote the class of all continua X such that each nonconstant open mapping defined on X is a homeomorphism. Thus $\mathsf{H} \subset \mathsf{L}$, and an arc is in $\mathsf{L} \setminus \mathsf{H}$.

Remarks 2.5. (a) In a conversation with the second named author, W. Makuchowski asked if there exists an arclike continuum X which is in H. Recall that a continuum X is said to be *arclike* provided that it is the inverse limit of arcs with surjective bonding mappings, or—equivalently—provided that for each $\varepsilon > 0$ there is a surjective ε-mapping f of X onto an arc Y (i.e., such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in Y$). Obviously any such a continuum cannot be arcwise connected (and, consequently, cannot be locally connected, because each either arcwise or locally connected arclike continuum is an arc, which is not in H). The question is answered in the affirmative: an uncountable family of such continua has been constructed in [4, (32) and Theorem 41].

(b) Note that if the continua B^- and B^+ in the definition of the spiral-arc X are arclike, then X is arclike (and vice versa).

The main property of the family of all spiral-arcs is formulated in the following theorem.

THEOREM 2.6. *Let $X = B^- \cup C \cup B^+$ be a spiral-arc with a crooked spiral C from a continuum B^- to a nondegenerate continuum B^+, and let $f : X \to Y$ be an open nonconstant surjective mapping.*

(2.7) *If B^- and B^+ are homeomorphic, then f is either a homeomorphism or a '2-folding' mapping, i.e., X is the one point union of two copies Y^- and Y^+ of Y with the common point in C and both $f|Y^-$ and $f|Y^+$ are homeomorphisms onto Y.*

(2.8) *If B^- and B^+ are not homeomorphic, then f is a homeomorphism, and thus $X \in \mathsf{H}$.*

(2.9) *If C is not a crooked spiral from B^+ to B^-, then f is a homeomorphism, and thus $X \in \mathsf{H}$.*

In any case the mapping f is simple, and therefore $X \in \mathsf{L}$.

A proof of Theorem 2.6 will be given in Section 3. Now we present some consequences of the theorem.

EXAMPLE 2.10. *There exists an arclike spiral-arc $X \in \mathsf{H} \subset \mathsf{L}$ and an atomic mapping $f : X \to f(X)$ such that $f(X) \notin \mathsf{L}$.*

PROOF. Let $\{r_n\}_{-\infty}^{+\infty}$ be a double-sequence containing all rationals in $[0,1]$ satisfying $r_n = 1$ for $n \leq 0$. Let $\{x_n\}_{-\infty}^{+\infty}$ be a monotone double-sequence in $(-1,1)$ satisfying

$$\lim_{n\to-\infty} x_n = -1, \quad \lim_{n\to+\infty} x_n = 1, \quad x_0 = 0.$$

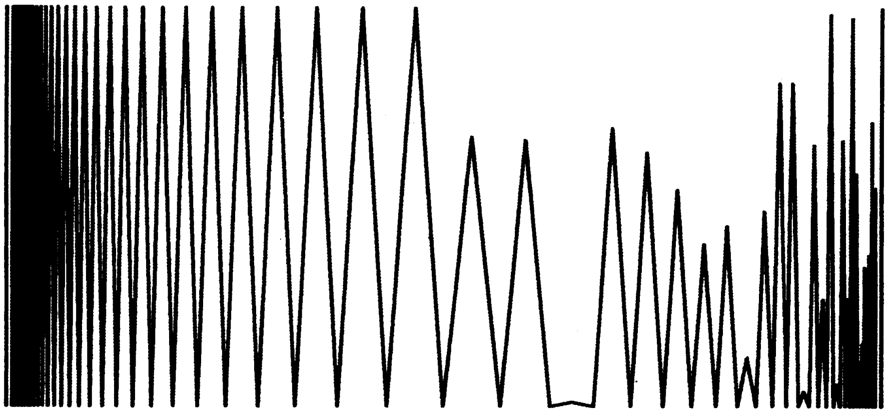

Figure 2 (Example 2.10)

Define a piecewise linear mapping $g : (-1, 1) \to [0, 1]$ as a one which is linear on each segment $[x_n, (x_n + x_{n+1})/2]$ and $[(x_n + x_{n+1})/2, x_{n+1}]$, with $g(x_n) = 0$, $g((x_n + x_{n+1})/2) = r_n$ for each $n \in \boldsymbol{Z}$. Put

$$B^- = \{(-1, y) \in \boldsymbol{R}^2 : y \in [0, 1]\}, \quad B^+ = \{(1, y) \in \boldsymbol{R}^2 : y \in [0, 1]\},$$

$$C = \{(x, g(x)) \in \boldsymbol{R}^2 : x \in (-1, 1)\}.$$

Set $X = B^- \cup C \cup B^+$. The continuum X is pictured in Fig. 2. Evidently, X is arclike.

Observe that the spiral C is crooked from B^- to B^+ while it is not crooked from B^+ to B^-. Then $X \in \mathsf{H} \subset \mathsf{L}$ due to Theorem 2.6. Define a mapping $f : X \to f(X) = Y$ by the conditions

(i) $f((x, y)) = (x, y)$ for $x \in [-1, 0)$;

(ii) $f((x, y)) = (x, 0)$ for $x \in [0, 1]$.

Clearly, f is atomic. Observe that $f(X)$ is homeomorphic to the $\sin(1/x)$-curve, which is not in L (see [4, Remark 18]). The proof is then complete. □

Remarks 2.11. (a) Example 2.10 shows that all three above mentioned questions of Makuchowski, viz. Questions 2.6 and 2.10 of [4, p. 782 and 783] and the one formulated in Remark 2.5, can be answered by just one example, namely the continuum X of Example 2.10.

(b) The same example shows that the assumption of arcwise connectedness of the continuum X in (1.2) is essential.

As it was observed in Remark 2.5 arclike continua in H are (obviously) not arcwise connected. So, one can ask if there are arclike members of H which are 'relatively close' to arcwise connected continua in the following sense. A continuum X is said to have the *arc approximation property* provided that for each subcontinuum K of X and for each point $p \in K$ there is a sequence of arcwise connected continua K_n of X such that $p \in K_n$ for each $n \in \mathbf{N}$ and $K = \operatorname{Lim} K_n$ (see [5, Section 3, p. 113]). More precisely, the following question is interesting.

QUESTION 2.12. Does there exist in the class H an arclike continuum having the arc approximation property?

Note that no spiral-arc $X = B^- \cup C \cup B^+$ has the arc approximation property because if a continuum has the property, then each arc component of the continuum is dense (see [5, Proposition 3.10, p. 116]), while arc components of the spiral-arc X contained in the union $B^- \cup B^+$ are not dense in X.

PROBLEM 2.13. Characterize arclike continua being in the class H.

3. Proofs

We start with recalling a well known result, see [9, Chapter 8, (7.31), p. 147].

STATEMENT 3.1. *The order of a point is never increased under an open mapping.*

The next two propositions concern mappings of continua that contain a spiral-arc in a special way.

PROPOSITION 3.2. *Let a continuum X contain a spiral C from a continuum to a nondegenerate subcontinuum B in such a way that $B \cup C$ is an open subset of X. If a nonconstant mapping $f : X \to f(X) = Y$ is open, then $f(B)$ is not a singleton.*

PROOF. Assume on the contrary that $f(B)$ is a singleton. We claim that

(3.3) there exists a spiral $C' \subset C$ from a singleton p' to B such that f is injective on C'.

To see this, let $g : \mathbf{R} \to C$ be a homeomorphism describing the spiral C. Suppose on the contrary that for each $n \in \mathbf{N}$ the restriction $f|g([n, \infty))$ is not injective. This means that for each $n \in \mathbf{N}$ there are $s_n,\ t_n \in \mathbf{R}$ with $n \le s_n < t_n$ such that $f(g(s_n)) = f(g(t_n))$. Then $f(g([s_n, \infty))) = f(g((s_n, \infty)))$. Since $B \cup C$ is open in X by assumption, it follows that $B \cup g((s_n, \infty)) \subset B \cup C$ is open as well, and therefore $f(B \cup g([s_n, \infty))) = f(B \cup g((s_n, \infty)))$ is both closed and open subset of Y, so it equals Y. Since the sequence of continua $g([n,\infty)) \cup B$ approaches B, we infer from continuity of f that $f(g([n,\infty))) \cup f(B)$ is a null-sequence of continua. On the other hand, each term of this sequence equals Y, a contradiction, because f is nonconstant. Therefore (3.3) holds.

We may now assume that f is injective on C by (3.3). Put $f(B) = \{r\}$. Since each point of C is of order 2 in X, it follows from Statement 3.1 that $f(B \cup C)$ contains a free arc J having r as one of its end points. Since B is nondegenerate, we can choose two distinct points b_1 and b_2 of B. Let U_1 and U_2 be disjoint open neighborhoods of b_1 and b_2, respectively, in X, contained in $B \cup C$. Then their images $f(U_1)$ and $f(U_2)$ are open subsets of Y each of which contains r. Further, since $C \cap U_1$ and $C \cap U_2$ are nonempty open subsets of X, their images $f(C \cap U_1)$ and $f(C \cap U_2)$ are open in J. Then since $r \in f(\operatorname{cl} C) \subset \operatorname{cl} f(C)$, the sets $\{r\} \cup f(C \cap U_1)$ and $\{r\} \cup f(C \cap U_2)$ are open neighborhoods of r in J. Let $s \in f(C \cap U_1) \cap f(C \cap U_2)$. Thus there are points $x_1 \in C \cap U_1$ and $x_2 \in C \cap U_2$ such that $f(x_1) = f(x_2) = s$. Since U_1 and U_2 are disjoint, $x_1 \neq x_2$, contrary to the injectivity of f on C. This contradiction finishes the proof. □

PROPOSITION 3.4. *Let a continuum X contain a crooked spiral C from a continuum to a nondegenerate subcontinuum B in such a way that $B \cup C$ is an open subset of X. If a mapping $f : X \to f(X) = Y$ is such that $f(X)$ is either an arc or a simple closed curve, then f is not open.*

PROOF. Suppose on the contrary that f is open. Composing f with an open projection from a circle onto a unit segment we may assume that $f(X) = Y = [0,1]$. By Proposition 3.2 we see that $f(B)$ is not a singleton. Take $a \in B$ such that $0 \neq f(a) \neq 1$. Then arbitrarily close to a there are points in B which are not mapped to $f(a)$. Then there is an open set W containing a such that $f(W) \subset (0,1)$.

By the crookedness of C there is a neighborhood $U \subset W$ of a such that for any fixed point $b \in U \cap B$ with $f(b) \neq f(a)$ there exists in $U \cap C$ a sequence of arcs J_n with end points α_n, β_n satisfying conditions (CS1)–(CS2).

Let $\varepsilon = |f(b) - f(a)|/3$. By the continuity of f there are in U open neighborhoods V_a and V_b of a and of b respectively, such that $f(V_a) \subset \{y \in (0,1) : |y - f(a)| < \varepsilon\}$ and $f(V_b) \subset \{y \in (0,1) : |y - f(b)| < \varepsilon\}$.

In the sequence $\{J_n\}$ of arcs in C one can find an arc $J \subset C \cap U$ such that J meets both V_a and V_b and its end points α and β are contained in one of the mentioned sets; we may assume that they are contained in V_a. Note that $f(J)$ is an arc in $Y = [0,1]$ not containing the end points of Y. Moreover, $f(J)$ meets both $f(V_a)$ and $f(V_b)$. Hence, at least one of the end points of $f(J)$, denote it by y, is different from both $f(\alpha)$ and $f(\beta)$. But then f is not interior at that point $x \in J \setminus \{\alpha, \beta\}$ which is mapped onto y. We have obtained a contradiction with the openness of f. □

REMARK 3.5. The assumption that $B \cup C$ is an open subset of X made in Propositions 3.2 and 3.4 implies that the spiral C is open in X. Thus it is natural to ask if the conclusions of these propositions hold if the assumption of openness of $B \cup C$ is replaced by a weaker one, namely by openness of C.

PROOF 3.6. (Proof of Theorem 2.6.) Let $X = B^- \cup C \cup B^+$ be a spiral-arc with a crooked spiral C from a continuum B^- to a nondegenerate continuum B^+, and let $f : X \to Y$ be an open nonconstant surjective mapping.

First observe that, since any point of C is of order 2 in X, the following assertion holds by Statement 3.1.

(3.7) The image $f(C)$ is an arcwise connected open subset of Y that is composed of points of orders at most 2 in Y.

Therefore $f(C)$ is homeomorphic to one of the following four sets:

(a) the open interval $(0, 1)$;
(b) the half-open interval $[0, 1)$;
(c) the closed interval $[0, 1]$;
(d) the circle.

Second, note that since C is an open subset of X, its image $f(C)$ is open in Y. Thus, if either (c) or (d) holds, $f(C)$ is both open and closed subset of Y, so it equals Y, whence f cannot be open by Proposition 3.4. Therefore cases (c) and (d) cannot hold, and thus we have the following assertion

(3.8) Only cases (a) and (b) are possible.

The next assertion is obvious.

(3.9) If $f(C)$ is homeomorphic to the open interval $(0, 1)$ (i.e., if case (a) holds), then f is injective on C.

In the next assertion we study case (b).

(3.10) If $f(C)$ is homeomorphic to the half-open interval $[0, 1)$ (i.e., if case (b) holds), then there exists $c \in \mathbf{R}$ such that f is injective on both $g((-\infty, c])$ and $g([c, \infty))$, and $f(C) = f(g((-\infty, c])) = f(g([c, \infty)))$.

To see this, let $h : f(C) \to [0,1)$ be the above mentioned homeomorphism. Choose $c \in \boldsymbol{R}$ such that $h(f(g(c))) = 0$. Suppose on the contrary that $f|g([c,\infty))$ is not injective. This means that there are $s,t \in \boldsymbol{R}$ with $c \le s < t$ such that $f(g(s)) = f(g(t)) = y$ for some $y \in Y$. Then $g([s,t])$ is an arc in C. Let A be a component of $g([s,t]) \backslash f^{-1}(y)$. Thus $\operatorname{cl} A$ is an arc in C. If the end points of $\operatorname{cl} A$ are a and b, then $f(a) = f(b) = y$. Therefore $f(\operatorname{cl} A)$ is an arc in $f(C)$, and $f(A)$ is its dense and connected subset which does not contain the point y. It follows that y is an end point of the arc $f(\operatorname{cl} A)$. Let y' be the other end point of this arc. Thus $y' \in f(A)$, and if $x' \in A \cap f^{-1}(y')$, then f is not interior at x', a contradiction with openness of f. We have shown that f is injective on $g([c,\infty))$. Similarly we can show that f is injective on $g((-\infty,c])$. Note further that the point c is uniquely determined.

To show the equality suppose on the contrary that $f(g([c,\infty)))$ is a proper subset of $f(g((-\infty,c]))$ (the other possibility can be considered in the same way). Since B^+ is the remainder of $g([c,\infty))$ in $B^+ \cup g([c,\infty))$ according to (S2), the set $f(B^+)$ must be a singleton. Thereby, according to Proposition 3.2, the mapping f cannot be open, a contradiction. Therefore (3.10) is proved.

(3.11) $f(C) \cap (f(B^-) \cup f(B^+)) = \varnothing$.

Suppose on the contrary that this intersection is not empty. By symmetry we can assume $f(C) \cap (f(B^+) \neq \varnothing$. Then there are points $x_1 \in C$ and $x_2 \in B^+$ such that $f(x_1) = f(x_2)$. Let $t = g^{-1}(x_1) \in \boldsymbol{R}$. If case (b) holds, we can take that $t \in [c,\infty)$ by (3.10). If $t = c$, then f is injective on $g[c,\infty))$ and $f(C) = f(g([c,\infty)))$ according to (3.10). Otherwise there is $\varepsilon > 0$ such that f is injective on $g([t-\varepsilon,\infty))$. Consider the arc $A = g([t-\varepsilon, t+\varepsilon])$ (if case (a) holds, $\varepsilon > 0$ is arbitrary). So A is a free arc in $C \subset X$, whence by openness of f its image $f(A)$ is a free arc in Y. Since $x_2 \in B^+$, then, according to (S2), there is a sequence of points $c_m \in C$ tending to x_2. Then the numbers $t_m = g^{-1}(c_m) \in \boldsymbol{R}$ tend to infinity, whence $t + \varepsilon < t_m$ for almost all $m \in \boldsymbol{N}$. Since $f(x_1) = f(x_2) \in f(g((t-\varepsilon, t+\varepsilon)))$, we have $f(c_m) \in f(g((t-\varepsilon, t+\varepsilon))) \subset f(A)$, whence it follows that f is not injective on $g([c,\infty))$ (or on C, in case (a)), contrary to either (3.9) or (3.10). Thus (3.11) is shown.

(3.12) The partial mappings $f|B^-$ and $f|B^+$ are injective.

By symmetry it is enough to show that $f|B^+$ is injective. Striving for a contradiction, suppose that there are two distinct points $x_1, x_2 \in B^+$ such that $f(x_1) = f(x_2)$. Choose disjoint open neighborhoods U_1 of x_1 and U_2 of x_2. By (S2) there is a sequence of points $c_m \in U_1 \cap C$ (or even $c_m \in U_1 \cap g([c,\infty))$, if (b) holds) converging to x_1. Then the open set $f(U_1) \cap f(U_2)$ contains almost all points $f(c_m)$. Fix one of them, c_{m_0}. Thus there is a point $d \in U_2$ with $f(c_{m_0}) = f(d)$. If $d \in C$, then f is not injective on C (in case (b) we may assume that $d \in g([c, \infty))$ by (3.10), and then f is not injective on $g([c, \infty)))$. In both cases we get a contradiction, either with (3.9) or with (3.10). Thus $d \in U_2 \cap B^+$, whence we see that $f(C) \cap f(B^+) \neq \emptyset$, a contradiction again, now with (3.11). Therefore (3.12) is established.

The sequential assertion concerns case (a).

(3.13) If $f(C)$ is homeomorphic to the open interval $(0, 1)$ (i.e., if case (a) holds), then $f(B^-) \cap f(B^+) = \emptyset$.

Indeed, if not, there are points $x^- \in B^-$ and $x^+ \in B^+$ such that $f(x^-) = f(x^+) = y$. Let U^- and U^+ be disjoint open neighborhoods of the points x^- and x^+, respectively. By the openness of f the set $f(U^-) \cap f(U^+)$ is an open neighborhood of y. Take a point $c' \in U^+ \cap C$ such that $f(c') \in f(U^-)$. Thus there is a point $d \in U^-$ with $f(c') = f(d)$. Since f is injective on C, it follows that $d \in B^-$. Therefore $f(C) \cap f(B^-) \neq \emptyset$, contrary to (3.11). So (3.13) is proved.

Now we can summarize the considered cases. If (a) holds, then by (3.9), (3.11), (3.12) and (3.13) we infer that f is injective on the whole X, i.e., f is a homeomorphism. In case (b) the mapping f is '2-folding' on C according to (3.10), and it is a homeomorphism on B^- and on B^+ by (3.12), which are glue together under f by its continuity, again by (3.10). Observe that in this case B^- and B^+ are homeomorphic. Notice also that in this case $Y^+ = g([c, \infty)) \cup B^+$ is homeomorphic to $Y^- = g((-\infty, c]) \cup B^-$. But then C is a crooked spiral from B^+ to B^- due to our assumption in Theorem 2.6. Since the discussed cases cover all possibilities, Theorem 2.6 is shown. □

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