ON THE CHOW-FORMS OF ELLIPTIC NORMAL CURVES OF DEGREE 4

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Abstract. In this note, we shall aim, using explicit theta constants, to describe the Chow-form of an elliptic normal curve of degree 4 embedded in P^3 by using theta functions, at any point of the upper half plane. In this case, the embedded elliptic curve is a complete intersection which is defined by two quadratic forms. In order to calculate its Chow-form, we use the elimination theory. Our main result is Th. $1'$ (§2). In our case, Chow-forms are divisors of the Grassmann variety of lines in P^3. From the theorem, we see that the Chow-forms of elliptic normal curves of degree 4 lie on a linear subspace M (cf. 2.3) of dimension 23. §2 is the main part of this note.

In §3, using the theory of elliptic modular forms of level 4, we consider the geometric meaning of Th. $1'$. Then our theorem shows that the Chow point is given by modular forms of weight 2 (of weight 1 in the traditional sense) and of level 4. The compactification of the moduli space of level 4 is isomorphic to a plane conic. Thus, we see that the Chow points of the projective elliptic normal curves of degree 4 determined by the points of the upper half plane form a rational curve of degree 4 in M which is essentially the image of the 2-uple embedding of the above plane conic (Th. 2, §3. cf. Comments after the proof of Th. 2). Furthermore, we see that the correspondence which assigns the Chow point to a point of the upper half plane can be extended to the inequivalent cusps of the principal congruence subgroup $\Gamma(4)$ of level 4 (Cor. 1, §3). Finally, we give a remark about the structure of the corresponding component of the Chow variety parametrizing 1-cycles of degree 4 in P^3 (Cor. 2, §3).

§1. Situations

This § is the part of grounds on which this note is founded.

First, we prepare some notations. Z, Q, R and C are as usual the ring of rational integers, the field of rational numbers, field of real numbers and the field of complex numbers, respectively.

H is the upper half plane of C, i.e., $H = \{\tau \in C \mid \mathrm{Im}(\tau) > 0\}$.

We put $e(t) = \exp(2\pi\sqrt{-1}t)$ in general. For $k = (k_1 \ k_2)$ with k_1 and $k_2 \in R$, we define a holomorphic function $\vartheta_k(\tau|z)$ on $H \times C$ by

$$\vartheta_k(\tau|z) = \sum_{r \in Z} e\left[\frac{1}{2}(k_1 + r)\tau(k_1 + r) + (k_1 + r)(z + k_2)\right] \quad ((\tau, z) \in H \times C).$$

For any $\tau \in H$, we denote by Λ_τ, E_τ a lattice subgroup $Z + Z\tau$ of C, a complex torus C/Λ_τ.

Let τ be an arbitrary point of H. For simplicity, for $i, j \in Z$, we put $\vartheta_{ij}(\tau|z) = \vartheta_{i/2\ j/2}(\tau|z)$. When τ is fixed, we denote by $\vartheta_{ij}(0)$ the theta constants $\vartheta_{ij}(\tau|0)$. If $(i, j) \neq (1, 1)$, for any $\tau \in H$, $\vartheta_{ij}(\tau|0) \neq 0$. Moreover, we have Jacobi's identity

$$\vartheta_{00}(0)^4 = \vartheta_{01}(0)^4 + \vartheta_{10}(0)^4.$$

We define a holomorphic embedding $\phi : E_\tau \to P^3$ by

$$\forall z \in C; \quad \phi(z) = (\vartheta_{00}(\tau|2z), \vartheta_{01}(\tau|2z), \vartheta_{10}(\tau|2z), \vartheta_{11}(\tau|2z)).$$

Since $\{\vartheta_{i/4\ 0}(\tau|z) | 0 \leq i \leq 3\}$ is linearly independent and determines the same very ample line bundle as $\{\vartheta_{ij}(\tau|2z) \mid i, j = 0, 1\}$ (e.g. using $(\theta 5)$ in [5], p. 50), the embedding ϕ is the usual embedding of the torus E_τ ([6]) followed by a projective transformation.

Let X_τ be the image curve $\phi(E_\tau) \subset P^3$, and let (x_0, x_1, x_2, x_3) be the homogeneous coordinates in P^3 corresponding to the above map.

The curve X_τ is non-singular of degree 4 and defined by

$$\begin{cases} \vartheta_{00}(0)^2 x_0^2 = \vartheta_{01}(0)^2 x_1^2 + \vartheta_{10}(0)^2 x_2^2 \\ \vartheta_{00}(0)^2 x_3^2 = \vartheta_{10}(0)^2 x_1^2 - \vartheta_{01}(0)^2 x_2^2 \end{cases}. \tag{1}$$

This is proved using the addition theorem of theta functions ([8]).

§2. Chow-forms

In this section, by using the classical elimination theory, we compute the Chow-form of the curve X_τ which is defined by (1). Since the Chow-form $P_X(U)$

of a projective variety X of dimension r and of degree m in a projective space P^n can be expressed as a homogeneous polynomial of degree m in the determinants

$$\begin{vmatrix} U_{0i_0} & \cdots & U_{0i_r} \\ & \cdots & \\ U_{ri_0} & \cdots & U_{ri_r} \end{vmatrix}$$

with sequences of indeterminates $U_\alpha = (U_{\alpha 0}, \ldots, U_{\alpha n})$ $(0 \le \alpha \le r)$ ([2] or [4] or [7]), we shall express the Chow-form of X_τ as a homogeneous polynomial in the determinants as above of the indeterminates, whose coefficients are theta constants.

2.1. In this item, forgetting (1), a little generally, we consider the Chow-form corresponding to our case.

Let a_0, a_1, a_2 be non-zero elements of $C^\times = C - \{0\}$. We consider two quadratic forms f_1, f_2:

$$f_1 := a_0^2 x_0^2 - a_1^2 x_1^2 - a_2^2 x_2^2,$$

$$f_2 := a_0^2 x_3^2 - a_2^2 x_1^2 + a_1^2 x_2^2.$$

Let $V(f_1), V(f_2)$ be the quadrics in P^3 defined by $f_1 = 0, f_2 = 0$.
As to the irreducibility of an algebraic subset $V(f_1) \cap V(f_2)$ in P^3, we have

PROPOSITION. *Let Σ be the pencil of quadrics defined by $V(f_1)$ and $V(f_2)$. If*

$$a_0^4 = a_1^4 + a_2^4, \tag{2}$$

then Σ is separable. Hence, the intersection $V(f_1) \cap V(f_2)$ is an irreducible curve of degree 4 in P^3.

The first part is proved along the analogous way as [9], pp. 73 ~ 74, and the irreducibility of the intersection $V(f_1) \cap V(f_2)$ is a direct conclusion of the separability of Σ ([9]).

Therefore, in that case, the Chow-form of the curve $V(f_1) \cap V(f_2)$ is irreducible.

Now, assuming that a_0, a_1 and a_2 satisfy (2), we compute the Chow-form of the curve $X = V(f_1) \cap V(f_2)$.

Let

$$u^{(0)} = (u_0^{(0)}, u_1^{(0)}, u_2^{(0)}, u_3^{(0)}), \quad u^{(1)} = (u_0^{(1)}, u_1^{(1)}, u_2^{(1)}, u_3^{(1)})$$

be two sequences of independent variables.

We consider the following system of equations:

$$\begin{cases} f_1 := a_0^2 x_0^2 - a_1^2 x_1^2 - a_2^2 x_2^2 = 0 \\ f_2 := a_0^2 x_3^2 - a_2^2 x_1^2 + a_1^2 x_2^2 = 0 \\ f_3 \equiv u_0^{(0)} x_0 + \cdots + u_3^{(0)} x_3 = 0 \\ f_4 \equiv u_0^{(1)} x_0 + \cdots + u_3^{(1)} x_3 = 0. \end{cases} \tag{3}$$

Then a G.C.D. of the resutant system $\mathcal{R}$ of the system of equations (3) is a power of the Chow-form P_X of X ([10]).

In our case, by A. Hurwitz ([3]), the resultant system $\mathcal{R}$ consists of one resultant which is a G.C.D. of Trägheitsformen of (3). But, as the following procedure shows, if we use the Plücker coordinates of the independent variables, then we can perform the elimination using only Sylvester's determinant without A. Hurwitz's theorem.

We put

$$p_{ij} = u_i^{(0)} u_j^{(1)} - u_j^{(0)} u_i^{(1)} \quad (0 \le i < j \le 3) \tag{4}$$

Plücker's relation is

$$p_{03} p_{12} = p_{02} p_{13} - p_{01} p_{23} \tag{a}$$

Put $s = x_2$, $t = x_3$.

From $f_3 = f_4 = 0$, we get

$$\begin{cases} p_{01} x_0 = s p_{12} + t p_{13} \\ p_{01} x_1 = -s p_{02} - t p_{03} \end{cases}$$

We eliminate x_0, x_1 from f_1, f_2.

First we have

$$p_{01}^2 f_1 = (a_0^2 p_{12}^2 - a_1^2 p_{02}^2 - a_2^2 p_{01}^2) s^2 + 2(a_0^2 p_{12} p_{13} - a_1^2 p_{02} p_{03}) st + (a_0^2 p_{13}^2 - a_1^2 p_{03}^2) t^2,$$

$$p_{01}^2 f_2 = (a_1^2 p_{01}^2 - a_2^2 p_{02}^2) s^2 - 2a_2^2 p_{02} p_{03} st + (a_0^2 p_{01}^2 - a_2^2 p_{03}^2) t^2.$$

We define

$$F_1 := (a_0^2 p_{12}^2 - a_1^2 p_{02}^2 - a_2^2 p_{01}^2) s^2 + 2(a_0^2 p_{12} p_{13} - a_1^2 p_{02} p_{03}) st + (a_0^2 p_{13}^2 - a_1^2 p_{03}^2) t^2,$$

$$\overset{\text{put}}{=} c_0 s^2 + c_1 st + c_2 t^2,$$

$$F_2 := (a_1^2 p_{01}^2 - a_2^2 p_{02}^2) s^2 - 2a_2^2 p_{02} p_{03} st + (a_0^2 p_{01}^2 - a_2^2 p_{03}^2) t^2,$$

$$\overset{\text{put}}{=} d_0 s^2 + d_1 st + d_2 t^2,$$

i.e.,

$$
\begin{cases}
c_0 = a_0^2 p_{12}^2 - a_1^2 p_{02}^2 - a_2^2 p_{01}^2 \\
c_1 = 2(a_0^2 p_{12} p_{13} - a_1^2 p_{02} p_{03}) \\
c_2 = a_0^2 p_{13}^2 - a_1^2 p_{03}^2 \\
d_0 = a_1^2 p_{01}^2 - a_2^2 p_{02}^2 \\
d_1 = -2a_2^2 p_{02} p_{03} \\
d_2 = a_0^2 p_{01}^2 - a_2^2 p_{03}^2
\end{cases}
\tag{b}
$$

Now, we consider the system of equations:

$$
\begin{cases}
F_1 = c_0 s^2 + c_1 st + c_2 t^2 = 0 \\
F_2 = d_0 s^2 + d_1 st + d_2 t^2 = 0
\end{cases}
\tag{5}
$$

Then we have the equivalence: (3) has a non-trivial solution $\Leftrightarrow$ (5) has a non-trivial solution.

Let $R(F_1, F_2)$ be the Sylvester's determinant of (5), i.e.,

$$
R(F_1, F_2) = (c_0 d_2 - c_2 d_0)^2 + (c_0 d_1 - d_0 c_1)(c_2 d_1 - d_2 c_1)
$$

$$
= c_2^2 d_0^2 - c_1 c_2 d_0 d_1 + c_0 c_2 d_1^2 + c_1^2 d_0 d_2 - 2c_0 c_2 d_0 d_2 - c_0 c_1 d_1 d_2 + c_0^2 d_2^2. \tag{c}
$$

Now, using (b), we compute each term in the right side of (c).

$$
c_2^2 d_0^2 = a_1^8 p_{01}^4 p_{03}^4 - 2a_1^6 a_2^2 p_{01}^2 p_{02}^2 p_{03}^4 + a_1^4 a_2^4 p_{02}^4 p_{03}^4 - 2a_0^2 a_1^6 p_{01}^4 p_{03}^2 p_{13}^2
$$

$$
+ 4a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2 - 2a_0^2 a_1^2 a_2^4 p_{02}^4 p_{03}^2 p_{13}^2 + a_0^4 a_1^4 p_{01}^4 p_{13}^4
$$

$$
- 2a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^2 p_{13}^4 + a_0^4 a_2^4 p_{02}^4 p_{13}^4. \tag{c1}
$$

$$
c_1 c_2 d_0 d_1 = -4a_1^6 a_2^2 p_{01}^2 p_{02}^2 p_{03}^4 + 4a_1^4 a_2^4 p_{02}^4 p_{03}^4
$$

$$
+ 4a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02} p_{03}^3 p_{12} p_{13} - 4a_0^2 a_1^2 a_2^4 p_{02}^3 p_{03}^3 p_{12} p_{13}
$$

$$
+ 4a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2 - 4a_0^2 a_1^2 a_2^4 p_{02}^4 p_{03}^2 p_{13}^2
$$

$$
- 4a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02} p_{03} p_{12} p_{13}^3 + 4a_0^4 a_2^4 p_{02}^3 p_{03} p_{12} p_{13}^3. \tag{c2}
$$

$$
c_0 c_2 d_1^2 = 4a_1^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^4 + 4a_1^4 a_2^4 p_{02}^4 p_{03}^4 - 4a_0^2 a_1^2 a_2^4 p_{02}^2 p_{03}^4 p_{12}^2
$$

$$
- 4a_0^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2 - 4a_0^2 a_1^2 a_2^4 p_{02}^4 p_{03}^2 p_{13}^2
$$

$$
+ 4a_0^4 a_2^4 p_{02}^2 p_{03}^2 p_{12}^2 p_{13}^2. \tag{c3}
$$

$$c_1^2 d_0 d_2 = 4a_0^2 a_1^6 p_{01}^4 p_{02}^2 p_{03}^2 - 4a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^4 p_{03}^2$$

$$- 4a_1^6 a_2^2 p_{01}^2 p_{02}^2 p_{03}^4 + 4a_1^4 a_2^4 p_{02}^4 p_{03}^2 - 8a_0^4 a_1^4 p_{01}^4 p_{02} p_{03} p_{12} p_{13}$$

$$+ 8a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^3 p_{03} p_{12} p_{13} + 8a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02} p_{03}^3 p_{12} p_{13}$$

$$- 8a_0^2 a_1^2 a_2^4 p_{02}^3 p_{03}^3 p_{12} p_{13} + 4a_0^6 a_1^2 p_{01}^4 p_{12}^2 p_{13}^2 - 4a_0^6 a_2^2 p_{01}^2 p_{02}^2 p_{12}^2 p_{13}^2$$

$$- 4a_0^4 a_1^2 a_2^2 p_{01}^2 p_{03}^2 p_{12}^2 p_{13}^2 + 4a_0^4 a_2^4 p_{02}^2 p_{03}^2 p_{12}^2 p_{13}^2. \tag{c4}$$

$$c_0 c_2 d_0 d_2 = a_0^2 a_1^4 a_2^2 p_{01}^6 p_{03}^2 + a_0^2 a_1^6 p_{01}^4 p_{02}^2 p_{03}^2 - a_0^2 a_1^2 a_2^4 p_{01}^4 p_{02}^2 p_{03}^2$$

$$- a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^4 p_{03}^2 - a_1^4 a_2^4 p_{01}^4 p_{03}^4 - a_1^6 a_2^2 p_{01}^2 p_{02}^2 p_{03}^4$$

$$+ a_1^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^4 + a_1^4 a_2^4 p_{02}^4 p_{03}^4 - a_0^4 a_1^4 p_{01}^4 p_{03}^2 p_{12}^2$$

$$+ a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^2 p_{03}^2 p_{12}^2 + a_0^2 a_1^4 a_2^2 p_{01}^2 p_{03}^4 p_{12}^2 - a_0^2 a_1^2 a_2^4 p_{02}^2 p_{03}^4 p_{12}^2$$

$$- a_0^4 a_1^2 a_2^2 p_{01}^6 p_{13}^2 - a_0^4 a_1^4 p_{01}^4 p_{02}^2 p_{13}^2 + a_0^4 a_2^4 p_{01}^4 p_{02}^2 p_{13}^2$$

$$+ a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^4 p_{13}^2 + a_0^2 a_1^2 a_2^4 p_{01}^4 p_{03}^2 p_{13}^2 + a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2$$

$$- a_0^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2 - a_0^2 a_1^4 a_2^2 p_{02}^4 p_{03}^2 p_{13}^2 + a_0^6 a_1^2 p_{01}^4 p_{12}^2 p_{13}^2$$

$$- a_0^6 a_2^2 p_{01}^2 p_{02}^2 p_{12}^2 p_{13}^2 - a_0^4 a_1^2 a_2^2 p_{01}^2 p_{03}^2 p_{12}^2 p_{13}^2$$

$$+ a_0^4 a_2^4 p_{02}^2 p_{03}^2 p_{12}^2 p_{13}^2. \tag{c5}$$

$$c_0 c_1 d_1 d_2 = -4a_0^2 a_1^2 a_2^4 p_{01}^4 p_{02}^2 p_{03}^2 - 4a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^4 p_{03}^2$$

$$+ 4a_1^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^4 + 4a_1^4 a_2^4 p_{02}^4 p_{03}^4 + 4a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^2 p_{03}^2 p_{12}^2$$

$$- 4a_0^2 a_1^2 a_2^4 p_{02}^2 p_{03}^4 p_{12}^2 + 4a_0^4 a_2^4 p_{01}^4 p_{02} p_{03} p_{12} p_{13}$$

$$+ 4a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^3 p_{03} p_{12} p_{13} - 4a_0^2 a_2^6 p_{01}^2 p_{02} p_{03}^3 p_{12} p_{13}$$

$$- 4a_0^2 a_1^2 a_2^4 p_{02}^3 p_{03}^3 p_{12} p_{13} - 4a_0^6 a_2^2 p_{01}^2 p_{02} p_{03} p_{12}^3 p_{13}$$

$$+ 4a_0^4 a_2^4 p_{02} p_{03}^3 p_{12}^3 p_{13}. \tag{c6}$$

$$c_0^2 d_2^2 = a_0^4 a_2^4 p_{01}^8 + 2a_0^4 a_1^2 a_2^2 p_{01}^6 p_{02}^2 + a_0^4 a_1^4 p_{01}^4 p_{02}^4 - 2a_0^2 a_2^6 p_{01}^6 p_{03}^2$$

$$- 4a_0^2 a_1^2 a_2^4 p_{01}^4 p_{02}^2 p_{03}^2 - 2a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^4 p_{03}^2 + a_2^8 p_{01}^4 p_{03}^4$$

$$+ 2a_1^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^4 + a_1^4 a_2^4 p_{02}^4 p_{03}^4 - 2a_0^6 a_2^2 p_{01}^6 p_{12}^2 - 2a_0^6 a_1^2 p_{01}^4 p_{02}^2 p_{12}^2$$

$$+ 4a_0^4 a_2^4 p_{01}^4 p_{03}^2 p_{12}^2 + 4a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^2 p_{03}^2 p_{12}^2 - 2a_0^2 a_2^6 p_{01}^2 p_{03}^4 p_{12}^2$$

$$- 2a_0^2 a_1^2 a_2^4 p_{02}^2 p_{03}^4 p_{12}^2 + a_0^8 p_{01}^4 p_{12}^4 - 2a_0^6 a_2^2 p_{01}^2 p_{03}^2 p_{12}^4$$

$$+ a_0^4 a_2^4 p_{03}^4 p_{12}^4. \tag{c7}$$

By substituting (c1),...,(c7) into the right side of (c), we have

$$R(F_1, F_2) = c_2^2 d_0^2 - c_1 c_2 d_0 d_1 + c_0 c_2 d_1^2 + c_1^2 d_0 d_2 - 2c_0 c_2 d_0 d_2 - c_0 c_1 d_1 d_2 + c_0^2 d_2^2$$

$$= \{ a_1^8 p_{01}^4 p_{03}^4 - 2a_1^6 a_2^2 p_{01}^2 p_{02}^2 p_{03}^4 + a_1^4 a_2^4 p_{02}^4 p_{03}^4 - 2a_0^2 a_1^6 p_{01}^4 p_{03}^2 p_{13}^2$$

$$+ 4a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2 - 2a_0^2 a_1^2 a_2^4 p_{02}^4 p_{03}^2 p_{13}^2 + a_0^4 a_1^4 p_{01}^4 p_{13}^4$$

$$- 2a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^2 p_{13}^4 + a_0^4 a_2^4 p_{02}^4 p_{13}^4 \}$$

$$- \{ -4a_1^6 a_2^2 p_{01}^2 p_{02}^2 p_{03}^4 + 4a_1^4 a_2^4 p_{02}^4 p_{03}^4$$

$$- 4a_0^2 a_1^4 a_2^2 p_{01}^3 p_{02} p_{03}^2 p_{13} p_{23} + 4a_0^2 a_1^2 a_2^4 p_{01} p_{02}^3 p_{03}^2 p_{13} p_{23}$$

$$+ 8a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2 - 8a_0^2 a_1^2 a_2^4 p_{02}^4 p_{03}^2 p_{13}^2$$

$$- 4a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02} (p_{02} p_{13} - p_{01} p_{23}) p_{13}^3 + 4a_0^4 a_2^4 p_{02}^3 (p_{02} p_{13} - p_{01} p_{23}) p_{13}^3 \}$$

$$+ \{ 4a_1^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^4 + 4a_1^4 a_2^4 p_{02}^4 p_{03}^4$$

$$- 4a_0^2 a_1^2 a_2^4 p_{02}^2 p_{03}^2 (-2p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2)$$

$$- 4a_0^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2 - 8a_0^2 a_1^2 a_2^4 p_{02}^4 p_{03}^2 p_{13}^2$$

$$+ 4a_0^4 a_2^4 p_{02}^2 (p_{02}^2 p_{13}^2 - 2p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2) p_{13}^2 \}$$

$$+ \{ 4a_0^2 a_1^6 p_{01}^4 p_{02}^2 p_{03}^2 - 4a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^4 p_{03}^2$$

$$- 4a_1^6 a_2^2 p_{01}^2 p_{02}^2 p_{03}^4 + 4a_1^4 a_2^4 p_{02}^4 p_{03}^4 - 8a_0^4 a_1^4 p_{01}^4 p_{02} (p_{02} p_{13} - p_{01} p_{23}) p_{13}$$

$$+ 8a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^3 (p_{02} p_{13} - p_{01} p_{23}) p_{13}$$

$$+ 8a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02} p_{03}^2 (p_{02} p_{13} - p_{01} p_{23}) p_{13}$$

$$- 8a_0^2 a_1^2 a_2^4 p_{02}^3 p_{03}^2 (p_{02} p_{13} - p_{01} p_{23}) p_{13} + 4a_0^6 a_1^2 p_{01}^4 p_{12}^2 p_{13}^2$$

$$- 4a_0^6 a_2^2 p_{01}^2 p_{02}^2 p_{12}^2 p_{13}^2 - 4a_0^4 a_1^2 a_2^2 p_{01}^2 (p_{02}^2 p_{13}^2 - 2p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2) p_{13}^2$$

$$+ 4a_0^4 a_2^4 p_{02}^2 (p_{02}^2 p_{13}^2 - 2p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2) p_{13}^2 \}$$

$$- 2 \{ a_0^2 a_1^4 a_2^2 p_{01}^6 p_{03}^2 + a_0^2 a_1^6 p_{01}^4 p_{02}^2 p_{03}^2 - a_0^2 a_1^2 a_2^4 p_{01}^4 p_{02}^2 p_{03}^2$$

$$- a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^4 p_{03}^2 - a_1^4 a_2^4 p_{01}^4 p_{03}^4 - a_1^6 a_2^2 p_{01}^4 p_{02}^2 p_{03}^4$$

$$+ a_1^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^4 + a_1^4 a_2^4 p_{02}^4 p_{03}^4$$

$$- a_0^4 a_1^4 p_{01}^4 (p_{02}^2 p_{13}^2 - 2 p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2)$$

$$+ a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^2 (p_{02}^2 p_{13}^2 - 2 p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2)$$

$$+ a_0^2 a_1^4 a_2^2 p_{01}^2 p_{03}^2 (p_{02}^2 p_{13}^2 - 2 p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2)$$

$$- a_0^2 a_1^2 a_2^4 p_{02}^2 p_{03}^2 (p_{02}^2 p_{13}^2 - 2 p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2)$$

$$- a_0^4 a_1^2 a_2^2 p_{01}^6 p_{13}^2 - a_0^4 a_1^4 p_{01}^4 p_{02}^2 p_{13}^2 + a_0^4 a_2^4 p_{01}^4 p_{02}^2 p_{13}^2$$

$$+ a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^4 p_{13}^2 + a_0^2 a_1^2 a_2^4 p_{01}^4 p_{03}^2 p_{13}^2 + a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2$$

$$- a_0^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^2 p_{13}^2 - a_0^2 a_1^2 a_2^4 p_{02}^4 p_{03}^2 p_{13}^2 + a_0^6 a_1^2 p_{01}^4 p_{12}^2 p_{13}^2$$

$$- a_0^6 a_2^2 p_{01}^2 p_{02}^2 p_{12}^2 p_{13}^2 - a_0^4 a_1^2 a_2^2 p_{01}^2 (p_{02}^2 p_{13}^2 - 2 p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2) p_{13}^2$$

$$+ a_0^4 a_2^4 p_{02}^2 (p_{02}^2 p_{13}^2 - 2 p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2) p_{13}^2 \}$$

$$- \{ -4 a_0^2 a_1^2 a_2^4 p_{01}^4 p_{02}^2 p_{03}^2 - 4 a_0^2 a_1^4 a_2^2 p_{01}^4 p_{02}^4 p_{03}^2$$

$$+ 4 a_1^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^4 + 4 a_1^4 a_2^4 p_{02}^4 p_{03}^4$$

$$+ 4 a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^2 (p_{02}^2 p_{13}^2 - 2 p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2)$$

$$- 4 a_0^2 a_1^2 a_2^4 p_{02}^2 p_{03}^2 (p_{02}^2 p_{13}^2 - 2 p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2)$$

$$+ 4 a_0^4 a_2^4 p_{01}^4 p_{02} (p_{02} p_{13} - p_{01} p_{23}) p_{13}$$

$$+ 4 a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^3 (p_{02} p_{13} - p_{01} p_{23}) p_{13}$$

$$- 4 a_0^2 a_2^6 p_{01}^2 p_{02} p_{03}^2 (p_{02} p_{13} - p_{01} p_{23}) p_{13}$$

$$- 4 a_0^2 a_1^2 a_2^4 p_{02}^3 p_{03}^2 (p_{02} p_{13} - p_{01} p_{23}) p_{13}$$

$$- 4 a_0^6 a_2^2 p_{01}^2 p_{02} (p_{02} p_{13} - p_{01} p_{23}) p_{12}^2 p_{13}$$

$$+ 4 a_0^4 a_2^4 p_{02} (p_{02}^3 p_{13}^3 - 3 p_{01} p_{02}^2 p_{13}^2 p_{23} + 3 p_{01}^2 p_{02} p_{13} p_{23}^2 - p_{01}^3 p_{23}^3) p_{13} \}$$

$$+ \{ a_0^4 a_2^4 p_{01}^8 + 2 a_0^4 a_1^2 a_2^2 p_{01}^6 p_{02}^2 + a_0^4 a_1^4 p_{01}^4 p_{02}^4 - 2 a_0^2 a_2^6 p_{01}^6 p_{03}^2$$

$$- 4 a_0^2 a_1^2 a_2^4 p_{01}^4 p_{02}^2 p_{03}^2 - 2 a_0^2 a_1^4 a_2^2 p_{01}^2 p_{02}^4 p_{03}^2 + a_2^8 p_{01}^4 p_{03}^4$$

$$+ 2 a_1^2 a_2^6 p_{01}^2 p_{02}^2 p_{03}^4 + a_1^4 a_2^4 p_{02}^4 p_{03}^4 - 2 a_0^6 a_2^2 p_{01}^6 p_{12}^2 - 2 a_0^6 a_1^2 p_{01}^4 p_{02}^2 p_{12}^2$$

$$+ 4 a_0^4 a_2^4 p_{01}^4 (p_{02}^2 p_{13}^2 - 2 p_{01} p_{02} p_{13} p_{23} + p_{01}^2 p_{23}^2)$$

$$+ 4a_0^4 a_1^2 a_2^2 p_{01}^2 p_{02}^2 (p_{02}^2 p_{13}^2 - 2p_{01}p_{02}p_{13}p_{23} + p_{01}^2 p_{23}^2)$$

$$- 2a_0^2 a_2^6 p_{01}^2 p_{03}^2 (p_{02}^2 p_{13}^2 - 2p_{01}p_{02}p_{13}p_{23} + p_{01}^2 p_{23}^2)$$

$$- 2a_0^2 a_1^2 a_2^4 p_{02}^2 p_{03}^2 (p_{02}^2 p_{13}^2 - 2p_{01}p_{02}p_{13}p_{23} + p_{01}^2 p_{23}^2) + a_0^8 p_{01}^4 p_{12}^4$$

$$- 2a_0^6 a_2^2 p_{01}^2 (p_{02}^2 p_{13}^2 - 2p_{01}p_{02}p_{13}p_{23} + p_{01}^2 p_{23}^2) p_{12}^2$$

$$+ a_0^4 a_2^4 (p_{02}^4 p_{13}^4 - 4p_{01}p_{02}^3 p_{13}^3 p_{23} + 6p_{01}^2 p_{02}^2 p_{13}^2 p_{23}^2$$

$$- 4p_{01}^3 p_{02}p_{13}p_{23}^3 + p_{01}^4 p_{23}^4) \}. \tag{d}$$

If we use (2) and (a), the right side of (d) is simplified, and we get

$$R(F_1, F_2) = a_0^2 p_{01}^4 [a_0^4 p_{03}^4 - 2a_0^2 a_1^2 p_{03}^2 p_{13}^2 + a_1^4 p_{13}^4 + 2a_0^2 a_1^2 p_{02}^2 p_{03}^2$$

$$- 4a_1^4 p_{02}^2 p_{13}^2 + 4a_1^4 p_{01}p_{02}p_{13}p_{23} + 2a_0^2 a_1^2 p_{12}^2 p_{13}^2 - 2a_1^2 a_2^2 p_{23}^2 p_{13}^2$$

$$- 2a_0^2 a_2^2 p_{01}^2 p_{03}^2 + 2a_1^4 p_{01}^2 p_{23}^2 - 2a_0^2 a_2^2 p_{03}^2 p_{23}^2 + 2a_1^2 a_2^2 p_{01}^2 p_{13}^2$$

$$- 2a_2^4 p_{02}^2 p_{13}^2 - 2a_1^2 a_2^2 p_{02}^2 p_{23}^2 + a_2^4 p_{01}^4 + 2a_1^2 a_2^2 p_{01}^2 p_{02}^2 + a_1^4 p_{02}^4$$

$$- 2a_0^2 a_2^2 p_{01}^2 p_{12}^2 - 2a_0^2 a_1^2 p_{02}^2 p_{12}^2 - 4a_2^4 p_{01}p_{02}p_{13}p_{23} + 4a_2^4 p_{01}^2 p_{23}^2$$

$$+ a_0^4 p_{12}^4 - 2a_0^2 a_2^2 p_{12}^2 p_{23}^2 + a_2^4 p_{23}^4].$$

This is a resultant of (5). The Chow-form P_X appears in $R(F_1, F_2)$ as a divisor and it is an irreducible homogeneous polynomial of degree 4 in the Plücker coordinates. Now, we take a set of indeterminates $U_j^{(i)}$ and denote by capitals P_{ij} the expressions which result if in the Plücker coordinates p_{ij} the $u_j^{(i)}$ are replaced by $U_j^{(i)}$. Thus we get

THEOREM 1. *Let $P_X(U)$ be the Chow-form of the curve X. Then $P_X(U)$ is given by*

$$P_X(U) = a_0^4 P_{03}^4 - 2a_0^2 a_1^2 P_{03}^2 P_{13}^2 + a_1^4 P_{13}^4 + 2a_0^2 a_1^2 P_{02}^2 P_{03}^2 - 4a_1^4 P_{02}^2 P_{13}^2$$

$$+ 4a_1^4 P_{01}P_{02}P_{13}P_{23} + 2a_0^2 a_1^2 P_{12}^2 P_{13}^2 - 2a_1^2 a_2^2 P_{23}^2 P_{13}^2 - 2a_0^2 a_2^2 P_{01}^2 P_{03}^2$$

$$+ 2a_1^4 P_{01}^2 P_{23}^2 - 2a_0^2 a_2^2 P_{03}^2 P_{23}^2 + 2a_1^2 a_2^2 P_{01}^2 P_{13}^2 - 2a_2^4 P_{02}^2 P_{13}^2$$

$$- 2a_1^2 a_2^2 P_{02}^2 P_{23}^2 + a_2^4 P_{01}^4 + 2a_1^2 a_2^2 P_{01}^2 P_{02}^2 + a_1^4 P_{02}^4 - 2a_0^2 a_2^2 P_{01}^2 P_{12}^2$$

$$- 2a_0^2 a_1^2 P_{02}^2 P_{12}^2 - 4a_2^4 P_{01}P_{02}P_{13}P_{23} + 4a_2^4 P_{01}^2 P_{23}^2 + a_0^4 P_{12}^4$$

$$- 2a_0^2 a_2^2 P_{12}^2 P_{23}^2 + a_2^4 P_{23}^4.$$

2.2. We go back to the situations as in §1. We fix an arbitrary point τ of H, and so we write simply X instead of X_τ.

In the equations (1) in §1, we put

$$a_0 = \vartheta_{00}(0), \quad a_1 = \vartheta_{01}(0), \quad a_2 = \vartheta_{10}(0).$$

Then

$$X = V(f_1) \cap V(f_2)$$

under the notations in 2.1. Moreover, Jacobi's identity is

$$a_0^4 = a_1^4 + a_2^4.$$

Therefore, from Th. 1, we get

THEOREM 1'. *Let $P_X(U)$ be the Chow-form of the projective elliptic normal curve $X = \phi(E_\tau)$. Then $P_X(U)$ is given by*

$$P_X(U) = \vartheta_{00}(0)^4 P_{03}^4 - 2\vartheta_{00}(0)^2\vartheta_{01}(0)^2 P_{03}^2 P_{13}^2 + \vartheta_{01}(0)^4 P_{13}^4$$

$$+ 2\vartheta_{00}(0)^2\vartheta_{01}(0)^2 P_{02}^2 P_{03}^2 - 4\vartheta_{01}(0)^4 P_{02}^2 P_{13}^2 + 4\vartheta_{01}(0)^4 P_{01}P_{02}P_{13}P_{23}$$

$$+ 2\vartheta_{00}(0)^2\vartheta_{01}(0)^2 P_{12}^2 P_{13}^2 - 2\vartheta_{01}(0)^2\vartheta_{10}(0)^2 P_{23}^2 P_{13}^2$$

$$- 2\vartheta_{00}(0)^2\vartheta_{10}(0)^2 P_{01}^2 P_{03}^2 + 2\vartheta_{01}(0)^4 P_{01}^2 P_{23}^2$$

$$- 2\vartheta_{00}(0)^2\vartheta_{10}(0)^2 P_{03}^2 P_{23}^2 + 2\vartheta_{01}(0)^2\vartheta_{10}(0)^2 P_{01}^2 P_{13}^2 - 2\vartheta_{10}(0)^4 P_{02}^2 P_{13}^2$$

$$- 2\vartheta_{01}(0)^2\vartheta_{10}(0)^2 P_{02}^2 P_{23}^2 + \vartheta_{10}(0)^4 P_{01}^4 + 2\vartheta_{01}(0)^2\vartheta_{10}(0)^2 P_{01}^2 P_{02}^2$$

$$+ \vartheta_{01}(0)^4 P_{02}^4 - 2\vartheta_{00}(0)^2\vartheta_{10}(0)^2 P_{01}^2 P_{12}^2 - 2\vartheta_{00}(0)^2\vartheta_{01}(0)^2 P_{02}^2 P_{12}^2$$

$$- 4\vartheta_{10}(0)^4 P_{01}P_{02}P_{13}P_{23} + 4\vartheta_{10}(0)^4 P_{01}^2 P_{23}^2 + \vartheta_{00}(0)^4 P_{12}^4$$

$$- 2\vartheta_{00}(0)^2\vartheta_{10}(0)^2 P_{12}^2 P_{23}^2 + \vartheta_{10}(0)^4 P_{23}^4.$$

Let G be the Grassmann variety of lines in $\mathbf{P}^3$. By Γ_4, we mean the homogeneous part of degree 4 of the homogeneous coordinate ring Γ of G. The vector space Γ_4 has a base consisting of the standard monomials of degree 4, and it is of dimension 105 ([2], p. 387).

The above expression of $P_X(U)$ is unique, in the sense that the monomials appearing in the expression are standard.

2.3. We denote by $c(X)$ the Chow point of $X(= \phi(E_\tau))$. By the beginning of this §, $c(X)$ is a point of the projective space $\mathbf{P}(\Gamma_4)$. Let M be the linear subspace

of $P(\Gamma_4)$ spanned by the monomials appearing in the expression of $P_X(U)$, i.e.,

$$M := \{\{P_{03}^4, P_{03}^2 P_{13}^2, P_{13}^4, P_{02}^2 P_{03}^2, P_{02}^2 P_{13}^2, P_{01} P_{02} P_{13} P_{23}, P_{12}^2 P_{13}^2, P_{23}^2 P_{13}^2, P_{01}^2 P_{03}^2,$$

$$P_{01}^2 P_{23}^2, P_{03}^2 P_{23}^2, P_{01}^2 P_{13}^2, P_{02}^2 P_{13}^2, P_{02}^2 P_{23}^2, P_{01}^4, P_{01}^2 P_{02}^2, P_{02}^4, P_{01}^2 P_{12}^2, P_{02}^2 P_{12}^2,$$

$$P_{01} P_{02} P_{13} P_{23}, P_{01}^2 P_{23}^2, P_{12}^4, P_{12}^2 P_{23}^2, P_{23}^4\}\}.$$

Then the theorem shows that:

(1) M contains not only $c(X)$, but also $c(\phi(E_\tau))$ for $\forall \tau \in H$,

(2) identifying M with $\boldsymbol{P}^{23}$, a system of homogeneous coordinates of $c(X)$ is given by

$$c(X) = (\vartheta_{00}(0)^4, -2\vartheta_{00}(0)^2 \vartheta_{01}(0)^2, \vartheta_{01}(0)^4, 2\vartheta_{00}(0)^2 \vartheta_{01}(0)^2, -4\vartheta_{01}(0)^4, 4\vartheta_{01}(0)^4,$$

$$2\vartheta_{00}(0)^2 \vartheta_{01}(0)^2, -2\vartheta_{01}(0)^2 \vartheta_{10}(0)^2, -2\vartheta_{00}(0)^2 \vartheta_{10}(0)^2, 2\vartheta_{01}(0)^4,$$

$$-2\vartheta_{00}(0)^2 \vartheta_{10}(0)^2, 2\vartheta_{01}(0)^2 \vartheta_{10}(0)^2, -2\vartheta_{10}(0)^4, -2\vartheta_{01}(0)^2 \vartheta_{10}(0)^2,$$

$$\vartheta_{10}(0)^4, 2\vartheta_{01}(0)^2 \vartheta_{10}(0)^2, \vartheta_{01}(0)^4, -2\vartheta_{00}(0)^2 \vartheta_{10}(0)^2, -2\vartheta_{00}(0)^2 \vartheta_{01}(0)^2,$$

$$-4\vartheta_{10}(0)^4, 4\vartheta_{10}(0)^4, \vartheta_{00}(0)^4, -2\vartheta_{00}(0)^2 \vartheta_{10}(0)^2, \vartheta_{10}(0)^4). \tag{6}$$

In the following, we fix the identification $M = \boldsymbol{P}^{23}$.

§3. Application to the Moduli Theory

In this section where the author owes to [8] to a great extent, we consider the totality of the Chow point $c(\phi(E_\tau))$ $(\tau \in H)$.

3.1. We start with recalling some facts from the theory of elliptic modular functions.

The homogeneous modular group $SL(2, \boldsymbol{Z})$ acts on H by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

For any positive integer N, the principal congruence subgroup $\Gamma(N)$ of level N of $SL(2, \boldsymbol{Z})$ is, by definition

$$\Gamma(N) = \mathrm{Ker}(SL(2, \boldsymbol{Z}) \xrightarrow{\mathrm{mod}\, N} SL(2, \boldsymbol{Z}/N\boldsymbol{Z})).$$

By a modular form of weight k (k a positive integer) and level N, we mean a holomorphic function $f(\tau)$ on H such that

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N); \quad f(\gamma \tau) = (c\tau + d)^k f(\tau)$$

and that $f(\tau)$ is bounded at $Q \cup \{\infty\}$ (for further details, [8]).

We are concerned with $\Gamma(4)$.

The inequivalent cusps of $\Gamma(4)$ are $\infty, 0, 1/2, 1, 2$ and 3.

We denote the extended upper half plane by $\tilde{H}$, i.e.,

$$\tilde{H} := H \cup Q \cup \{\infty\}.$$

The compactification $\widetilde{H/\Gamma(4)}$ of $H/\Gamma(4)$ is set-theoretically

$$\widetilde{H/\Gamma(4)} = \tilde{H}/\Gamma(4) = H/\Gamma(4) \cup \left\{ \infty, 0, \frac{1}{2}, 1, 2, 3 \right\}.$$

The followings are known ([8]):

(i). $\vartheta_{ij}(\tau|0)^2$ are modular forms on the upper half plane H of weight 1 of level 4,

(ii). The map $\theta' : H \to P^2$ defined by

$$\tau \mapsto \theta'(\tau) = (\vartheta_{00}(\tau|0)^2, \vartheta_{01}(\tau|0)^2, \vartheta_{10}(\tau|0)^2)$$

induces a holomorphic embedding $\theta : H/\Gamma(4) \to P^2$.

Let (y_0, y_1, y_2) be the homogeneous coordinates in P^2 corresponding to the above map, and let C_0 be a plane conic defined by

$$y_0^2 - y_1^2 - y_2^2 = 0.$$

We define an open set C_0' of C_0 by

$$C_0' := C_0 - \{(1, 0, \pm 1), (1, \pm 1, 0), (0, 1, \pm i)\}.$$

Then, $\text{Im}(\theta) = C_0'$.

(iii). The holomorphic map θ is extended to the compactification $\widetilde{H/\Gamma(4)}$ so that the following diagram is commutative:

$$
\begin{array}{ccc}
\widetilde{H/\Gamma(4)} & \xrightarrow{\sim} & C_0 \\
\cup & & \cup \\
H/\Gamma(4) & \xrightarrow{\sim} & C_0'
\end{array}.
$$

We also denote by θ the extended holomorphic map.

Let p be any one of the inequivalent cusps $\infty, 0, 1/2, 1, 2$ and 3. Then $\theta(p)$ is given by the following table:

p	∞	0	1/2	1	2	3
$\theta(p)$	$(1,1,0)$	$(1,0,1)$	$(1,-1,0)$	$(0,1,i)$	$(1,0,-1)$	$(0,1,-i)$

3.1.1. For every $\tau \in H$, we denote by $c(\tau)$ the Chow-point $c(\phi(E_\tau))$ of the projective elliptic normal curve $\phi(E_\tau)$. By 2.3, $c(\tau) \in M = \boldsymbol{P}^{23}$ and $c(\tau)$ is obtained from replacing $\vartheta_{ij}(0)$ by $\vartheta_{ij}(\tau|0)$ in (6). Therefore, a system of homogeneous coordinates of $c(\tau)$ is given by modular forms of weight 2 and of level 4.

We are interested in the subset $\{c(\tau)|\tau \in H\}$ of M. Let C be the closure of the set $\{c(\tau)|\tau \in H\}$ in the Zariski topology.

We define a holomorphic map $c : H \to C$ by $\tau \mapsto c(\tau)$.

THEOREM 2. *C is a non-singular rational curve of degree 4 in M having the following properties*:
   (1) $\{c(\tau) \,|\, \tau \in H\} \subset C$ *is a Zariski open subset.*
   (2) $\exists$ *an isomorphism*: $C_0 \xrightarrow{\sim} C$ *such that the diagram*:

$$
\begin{array}{ccc}
H & \xrightarrow{\;c\;} & C \\
\downarrow & & \uparrow{\scriptstyle\wr} \\
H/\Gamma(4) & \xrightarrow{\;\theta\;} & C_0
\end{array}
$$

is commutative.

PROOF. Let C_0 be the plane conic defined in (ii) of 3.1. By C', we mean the image of the 2-uple embedding, which is denoted by ρ, of C_0. C' is a non-singular rational curve of degree 4 in $\boldsymbol{P}^5$ (which lies on a hyperplane of $\boldsymbol{P}^5$).
   Then, $\forall \tau \in H$;

$$\rho(\theta'(\tau)) = (\vartheta_{00}(\tau|0)^4, \vartheta_{00}(\tau|0)^2\vartheta_{01}(\tau|0)^2, \vartheta_{00}(\tau|0)^2\vartheta_{10}(\tau|0)^2, \vartheta_{01}(\tau|0)^4,$$

$$\vartheta_{01}(\tau|0)^2\vartheta_{10}(\tau|0)^2, \vartheta_{10}(\tau|0)^4).$$

While, by (6),
$\exists \pi' : M \to \boldsymbol{P}^4$ a composition of a linear projection and a (trivial) projective transformation such that $\forall \tau \in H$;

$$\pi'(c(\tau)) = (\vartheta_{00}(\tau|0)^4, \vartheta_{00}(\tau|0)^2\vartheta_{01}(\tau|0)^2, \vartheta_{00}(\tau|0)^2\vartheta_{10}(\tau|0)^2, \vartheta_{01}(\tau|0)^4,$$

$$\vartheta_{01}(\tau|0)^2\vartheta_{10}(\tau|0)^2, \vartheta_{10}(\tau|0)^4)$$

$$= \rho(\theta'(\tau)).$$

π' is not unique, and so we fix such an π'.

Conversely, $\rho(\theta'(\tau))$ determines $c(\tau)$. Hence, π' induces a bijection

$$\{c(\tau) \mid \tau \in H\} \xrightarrow{\sim} \{\rho(\theta'(\tau)) \mid \tau \in H\} = \rho(C_0').$$

Since $\rho(C_0')$ is a Zariski open set of C', C becomes an algebraic curve which is isomorphic to C_0, using the bijection.   Q.E.D.

This theorem shows that for any $\tau \in H$, the Chow point $c(\tau)$ of the projective elliptic normal curve $\phi(E_\tau)$ is essentially given by the image of the 2-uple embedding of the point $(\vartheta_{00}(\tau|0)^2, \vartheta_{01}(\tau|0)^2, \vartheta_{10}(\tau|0)^2)$.

In our case, the appearance of the 2-uple embedding is primarily based on the fact that elliptic normal curves of degree 4 in P^3 are complete intersections by quadratic forms.

It seems to the author that: when we consider elliptic normal curves of degree $r > 4$ embedded in P^{r-1} by using theta functions, the level of the corresponding principal congruence subgroup will be biggeer than 4. Therefore if r is sufficiently large, the genus of the corresponding curve C is bigger than 0. Moreover the genus of C will be bigger as long as r is larger.

3.1.2. Next we consider the compactification of $H/\Gamma(4)$.

By the abuse of notations, we denote by ρ the isomorphism $C_0 \xrightarrow{\sim} C$ in the Th. 2.

When p is any one of the inequivalent cusps, let X_p be the specialization of the projective curve X_τ ($\tau \in H$) over the specialization p of τ. The equations for X_p are given by the following table ([8], p. 57):

p	∞	0	$1/2$	1	2	3
equations	$x_0^2 = x_1^2$ $x_2^2 = -x_3^2$	$x_0^2 = x_2^2$ $x_1^2 = x_3^2$	$x_0^2 = -x_1^2$ $x_2^2 = x_3^2$	$x_0^2 = -ix_3^2$ $x_1^2 = -ix_2^2$	$x_0^2 = -x_2^2$ $x_1^2 = -x_3^2$	$x_0^2 = ix_3^2$ $x_1^2 = ix_2^2$

Thus, for any p, X_p is a 1-cycle of degree 4, consisting of 4 lines.

Since Chow-forms are compatible with specializations, the six Chow-forms P_{X_∞}, P_{X_0}, $P_{X_{1/2}}$, P_{X_1}, P_{X_2} and P_{X_3} are obtained by specializations.

Let $c(p)$ be the Chow-point $c(X_p)$ of the cycle X_p. Then, $(c(p), \theta(p))$ is a specialization of $(c(\tau), \theta(\tau))$. Therefore $c(p)$ lies on the curve C.

Let $\Gamma(4)(p)$ be the orbit of p under $\Gamma(4)$. Then

$$Q \cup \{\infty\} = \Gamma(4)(\infty) \cup \Gamma(4)(0) \cup \Gamma(4)\left(\frac{1}{2}\right) \cup \Gamma(4)(1) \cup \Gamma(4)(2) \cup \Gamma(4)(3)$$

Now, we define a map

$$c : \tilde{H} \to C$$

by

$$c(\tau) := \begin{cases} c(\tau), & \text{if } \tau \in H \\ c(p), & \text{if } \tau \in \Gamma(4)(p) \end{cases}.$$

We note that for any $p' \in \Gamma(4)(p)$, the limiting cycle $X_{p'}$ of X_τ, as $\tau \to p'$, is uniquely determined and then $X_{p'} = X_p$.

Summarizing we have

COROLLARY 1. *The diagram*:

$$
\begin{array}{ccc}
\tilde{H} & \xrightarrow{\;c\;} & C \\
\big\downarrow & & \big\uparrow{\scriptstyle p} \\
\widetilde{H/\Gamma(4)} & \xrightarrow{\;\sim\;} & C_0
\end{array}
$$

is commutative.

Now, we recall the topology on $\tilde{H}$. A fundamental system of neighbourhoods $\mathfrak{B}(x)$ of x $(x \in \tilde{H})$ is defined by

$$\mathfrak{B}(x) := \begin{cases} \text{the usual one,} & \text{if } x \in H \\ \text{the set of all } \{p, \text{ the interior of a circle in} \\ H \text{ tangent to the real axis at } p\}, & \text{if } \infty \neq x = p \notin H \\ \text{the set of all } \{\infty, \tau \mid \tau \in H, \mathrm{Im}(\tau) > c\} \ (c > 0), & \text{if } x = \infty \end{cases}.$$

The topological space $\widetilde{H/\Gamma(4)} = \tilde{H}/\Gamma(4)$ is defined to be the quotient space of $\tilde{H}$. Hence, c becomes a continuous map.

3.2. Finally, we consider the Chow variety $\mathscr{C}$ of the 1-cycles of degree 4 in $\mathbf{P}^3$. $\mathscr{C}$ is an algebraic set in the 104-dimensional projective space $P(\Gamma_4)$. Let Σ be the component of $\mathscr{C}$ whose general point corresponds to an irreducible curve which is a complete intersections of two quadrics. Σ is of dimension 16 ([1]).

Σ is stable under the canonical action of $PGL(4)$.

PROOF. Let $c \in \Sigma$ be any point and let X be the 1-cycle of degree 4 in $\mathbf{P}^3$ corresponding to c.

If X is an irreducible curve, then $\forall \gamma \in PGL(4)$, X^γ is also irreducible.

Hence, $c^\gamma = c(X^\gamma) \in \Sigma$.

Next, we assume that X is not irreducible. Then there is an irreducible curve X_0 s.t. $c_0 = c(X_0) \in \Sigma$ and that c is a specialization of c_0.

Then c^γ is also a specialization of c_0^γ.

Since $c_0^\gamma \in \Sigma$, we have $c^\gamma \in \Sigma$.   Q.E.D.

As to the structure of Σ, we have the following:

COROLLARY 2. *Let C be the rational curve as in Th. 2. Then there is a natural dominating morphism:*

$$SL(4, C) \times C \to \Sigma$$

induced by the canonical action.

PROOF.   By Th. 2, $C(\subset M)$ lies on Σ. Let $\sigma : SL(4, C) \times \Sigma \to \Sigma$ be the canonical action.

Let $f : SL(4, C) \times C \to \Sigma$ be the morphism induced by σ, $(\gamma, c) \mapsto c^\gamma$.

Let $\rho : C_0 \xrightarrow{\sim} C$ be the isomorphism.

$C_0' = C_0 - \{6 \text{ cusps}\}$ is the open set of C_0. $\rho(C_0')$ is an open set of C.

We denote by f' the restriction of f to $SL(4, C) \times \rho(C_0')$.

Now, for any $\tau \in H$, the projective curve $X_\tau = \phi(E_\tau)$ is a non-singular curve embedded by a complete linear system of degree 4. Hence X_τ is Chow-stable (Th. 4.15 in [7]). In other words, the orbit of the Chow point $c(X_\tau)$ by $SL(4, C)$ is closed and the stabilizer is finite.

$$\therefore \dim \operatorname{Im} f' = \dim SL(4, C) + \dim \rho(C_0') = 16 = \dim \Sigma.$$

Hence, f dominates Σ.   Q.E.D.

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