TSUKUBA J. MATH.
Vol. 24 No. 1 (2000), 35–58

BOUNDARY VALUE PROBLEMS RELATED TO DIFFERENTIAL OPERATORS WITH COEFFICIENTS OF GENERALIZED HERMITE OPERATORS

By

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1. Introduction

Let us define a generalized Hermite operator $L = (L_1, \dots, L_n)$ by

$$L_j = D_{x_j}^2 + V_j(x_j) \quad (1 \le j \le n),$$

where $V_j(s)$ is a $C^\infty(R)$-function satisfying the following conditions: there exist $\delta_j > 0$, $c_0 > 0$ and $C_k > 0$ such that

$$\begin{cases} V_j(s) \ge c_0(1+|s|)^{2\delta_j} & (s \in R), \\ |D_s^k V_j(s)| \le C_k(1+|s|)^{2\delta_j} & (s \in R) \ (k \in I_+ = \{0, 1, 2, \dots\}). \end{cases}$$

Then there exists a complete orthonomal system $\{\phi_{j,k}(s)\}_{k \in I_+}$ in $L^2(R)$, such that

$$L_j \phi_{j,k}(s) = \lambda_{j,k}\phi_{j,k}(s), \quad \phi_{j,k} \in S(R),$$

$$0 < \lambda_{j,0} \le \lambda_{j,1} \le \cdots \le \lambda_{j,k} \le \dots, \quad \sum_{k=0}^{\infty} \lambda_{j,k}^{-p_0} < +\infty,$$

where S(R) is the L. Schwartz space of rapidly decreasing functions in R ([2]). Let us define differential operators with coefficients of generalized Hermite operators by

$$P(D_t, L) = P_m(L)D_t^m + P_{m-1}(L)D_t^{m-1} + \cdots + P_0(L),$$

$$P_j(L) = \sum_{|\beta| \le M} a_{j,\beta}L^\beta = \sum_{|\beta| \le M} a_{j,\beta}L_1^{\beta_1} \cdots L_n^{\beta_n},$$

$$Q_k(D_t, L) = Q_{k,M}(L)D_t^M + Q_{k,M-1}(L)D_t^{M-1} + \cdots + Q_{k,0}(L),$$

$$Q_{k,j}(L) = \sum_{|\beta| \le M} b_{k,j,\beta}L^\beta = \sum_{|\beta| \le M} b_{k,j,\beta}L_1^{\beta_1} \cdots L_n^{\beta_n},$$

Received October 19, 1998
Revised May 6, 1999

where $a_{j,\beta}, b_{k,j,\beta}$ are constants and m, M are non-negative integers $(M \geq m)$. In the previous paper [1], we have already considered the Cauchy problem

$$(A)\begin{cases} P(D_t, L)u(t,x) = f(t,x) & (0 < t < T, x \in R^n), \\ D_t^k u(t,x)|_{t=0} = g_k(x) & (x \in R^n, 0 \leq k \leq m-1), \end{cases}$$

if P is an evolution differential operator with coefficients of generarized Hermite operators.

In this paper, we will consider the boundary value problem:

$$(B)\begin{cases} P(D_t, L)u(t,x) = f(t,x) & (t > 0, x \in R^n), \\ Q_k(D_t, L)u(t,x)|_{t=0} = g_k(x) & (x \in R^n, 0 \leq k \leq r-1), \\ u(t,x) \in S([0,\infty), S'(R^n)), \end{cases}$$

for given data $f(t,x) \in S([0,\infty), S'(R^n))$ and $g_k(x) \in S'(R^n)$ $(0 \leq k \leq r-1)$, where r is an integer $(0 \leq r \leq m)$, which will be explained later. $S(R^n)$ is the L. Schwartz space of rapidly decreasing functions in R^n ([2]). $S'(R^n)$ is the conjugate space of $S(R^n)$. $S([0,\infty), S'(R^n))$ is a set of mappings such that

$$u : [0,\infty) \ni t \to u(t,x) \in S'(R^n),$$

satisfying

$$u_\phi(t) = \langle u(t,x), \phi(x) \rangle \in S([0,\infty)),$$

for any $\phi(x) \in S(R^n)$.

Now denote

$$\begin{aligned} \Lambda &= \{\lambda_\alpha | \alpha \in I_+^n\} = \{(\lambda_{1,\alpha_1}, \dots, \lambda_{n,\alpha_n}) | \alpha \in I_+^n\} \\ &= \{(\lambda_{10}, \dots, \lambda_{n0}), (\lambda_{11}, \lambda_{20}, \dots, \lambda_{n0}), (\lambda_{10}, \lambda_{21}, \dots, \lambda_{n0}), \dots\}, \\ P(\tau, \lambda) &= P_m(\lambda)\tau^m + P_{m-1}(\lambda)\tau^{m-1} + \cdots + P_0(\lambda) \\ &= P_m(\lambda)(\tau - \tau_1(\lambda)) \cdots (\tau - \tau_m(\lambda)), \\ Q_k(\tau, \lambda) &= Q_{k,M}(\lambda)\tau^M + Q_{k,M-1}(\lambda)\tau^{M-1} + \cdots + Q_{k,0}(\lambda). \end{aligned}$$

$P(D_t, L)$ is called separative, iff

(I) there exist $C_1 > 0$ and $p_1 > 0$ such that

$$|P_m(\lambda)| \geq C_1 |\lambda|^{-p_1} \quad (\lambda \in \Lambda),$$

(II) there exist $C_2 > 0$, $p_2 > 0$ and r $(0 \leq r \leq m)$ such that

$$\operatorname{Im} \tau_j(\lambda) \geq C_2 |\lambda|^{-p_2} \quad (1 \leq j \leq r, \quad \lambda \in \Lambda),$$

$$\operatorname{Im} \tau_j(\lambda) \leq 0 \quad (r+1 \leq j \leq m, \quad \lambda \in \Lambda).$$

Especially, $P(D_t, L)$ is called uniformly separative, iff

(I) there exist $C_1 > 0$ and $p_1 > 0$ such that

$$|P_m(\lambda)| \geq C_1|\lambda|^{-p_1} \quad (\lambda \in \Lambda),$$

(II$'$) there exists r $(1 \leq r \leq m)$ and $\mu > 0$ such that

$$\operatorname{Im} \tau_j(\lambda) \geq \mu \quad (1 \leq j \leq r, \quad \lambda \in \Lambda),$$

$$\operatorname{Im} \tau_j(\lambda) \leq 0 \quad (r+1 \leq j \leq m, \quad \lambda \in \Lambda).$$

In case when $P(D_t, L)$ is separative, we define

$$P_+(\tau, \lambda) = \begin{cases} (\tau - \tau_1(\lambda)) \cdots (\tau - \tau_r(\lambda)) & (r \neq 0), \\ 1 & (r = 0), \end{cases}$$

$$P_-(\tau, \lambda) = \begin{cases} (\tau - \tau_{r+1}(\lambda)) \cdots (\tau - \tau_m(\lambda)) & (r \neq m), \\ 1 & (r = m). \end{cases}$$

and

$$R(\lambda) = det\left(\frac{1}{2\pi i}\oint_\gamma \frac{Q_k(\tau, \lambda)\tau^l}{P_+(\tau, \lambda)} d\tau\right)_{k, l = 0, \ldots, r-1},$$

where γ is a closed curve on τ-plane, enclosing all zeros of $P_+(\tau, \lambda)$. We say that $\{P(D_t, L), Q_k(D_t, L) \ (k = 0, 1, \ldots, r-1)\}$ satisfies the Lopatinski condition, iff

(III) there exists $C_3 > 0$ and $p_3 > 0$ such that

$$|R(\lambda)| \geq C_3|\lambda|^{-p_3} \quad (\lambda \in \Lambda).$$

The following Theorem 1 and Theorem 2 will be obtained on the base of lemmas established in [1].

THEOREM 1. *Assume that* $P(D_t, L)$ *is uniformly separative and that* $\{P(D_t, L), Q_k(D_t, L) \ (k = 0, \ldots, r-1)\}$ *satisfies the Lopatinski condition. Let* $0 < \eta < \min(\mu, 1)$. *Suppose that*

$$e^{\eta t} f(t, x) \in S([0, \infty), S'(R^n)), \quad g_k(x) \in S'(R^n) \ (0 \leq k \leq r-1).$$

Then there exists a unique solution $u(t, x)$ *of the problem* (B), *and* $e^{\eta t} u(t, x)$ *belongs to* $S([0, \infty), S'(R^n))$.

THEOREM 2. *Assume that* $P(D_t, L)$ *is separative and that* $\{P(D_t, L), Q_k(D_t, L)$ $(k = 0, \ldots, r-1)\}$ *satisfies the Lopatinski condition. Let* $0 < \eta < 1$. *Suppose that*

$$e^{\eta t} f(t, x) \in S([0, \infty), S'(R^n)), \quad g_k(x) \in S'(R^n) \ (0 \leq k \leq r-1).$$

Then there exists a unique solution $u(t,x)$ *of the problem* (B), *where* $u(t,x)$ *belongs to* $S([0,\infty), S'(R^n))$.

2. Preparations

The following Lemma 1 $\sim$ Lemma 3 have been established in [1].

LEMMA 1. *Set* $\phi_\alpha(x) = \phi_{1,\alpha_1}(x_1)\cdots\phi_{n,\alpha_n}(x_n)$, *then* $\phi_\alpha(x)$ *is an eigenfunction of* $L^\beta = L_1^{\beta_1}\cdots L_n^{\beta_n}$, *corresponding to an eigenvalue* $\lambda_\alpha^\beta = \lambda_{1,\alpha_1}^{\beta_1}\cdots\lambda_{n,\alpha_n}^{\beta_n}$, *and moreover*

1) *there exists* $p_0 > 0$ *such that*

$$\sum_{\alpha\in I_+^n} |\lambda_\alpha|^{-p_0} < +\infty,$$

2) $\{\phi_\alpha(x)\}_{\alpha\in I_+^n}$ *is a complete orthonomal system of* $L^2(R^n)$,

3) $\phi_\alpha(x)\in S(R^n)$ *and there exist* $C(l)>0$ *and* $p(l)>0$ *such that*

$$\|\phi_\alpha\|_l \le C(l)|\lambda_\alpha|^{p(l)} \quad (\alpha\in I_+^n)$$

for any $l\in I_+$, *where*

$$\|\phi\|_l := \sum_{|\beta|+|\gamma|\le l} \sup_x |x^\beta D_x^\gamma \phi|.$$

Here we call $\{\phi_\alpha(x)\}_{\alpha\in I_+^n}$ *a family of generalized Hermite functions.*

Let $f\in S'(R^n)$. Set

$$a(f) = \{a_\alpha(f), \alpha\in I_+^n\}, \quad a_\alpha(f) = \langle f, \phi_\alpha\rangle,$$

where $a(f)$ is called a generalized Hermite coefficient of f. Let s be a set of complex multi-sequences $a = \{a_\alpha, \alpha\in I_+^n\}$ such that

$$|a|_h := \sup_{\alpha\in I_+^n} |a_\alpha|\,|\lambda_\alpha|^{h/2} < +\infty$$

for any $h\in I_+$.

LEMMA 2. *The mapping* $H: S(R^n)\ni f\to a(f)\in s$ *is linear and continuous. More precisely, there exists* $C_h>0$ *such that*

$$|a(f)|_{2h} \le C_h\|f\|_{2n+2(\delta+1)h} \quad \left(\delta = \max_j \delta_j\right)$$

for any $h(\in I_+)$. *Conversely,*

$$s \ni a \to f(x) := \sum_{\alpha} a_\alpha \phi_\alpha(x) \in S(R^n)$$

is linear and continuous, where $a(f) = a$. *More precisely, there exist* $C_l > 0$ *and* $p(l) > 0$ *such that*

$$\|f\|_l \leq C_l |a|_{2p(l)+2p_0}$$

for any $l \in I_+$.

Let s' be the conjugate space of s, namely, let s' be a set of all linear continuous functionals $b : s \ni a \to \langle b, a\rangle \in C$. More precisely, there exists $h > 0$ and $C > 0$ such that

$$|\langle b, a\rangle| \leq C|a|_h \quad (a \in s).$$

LEMMA 3. *Let* $T \in S'(R^n)$, *and put*

$$b = \{b_\alpha | \alpha \in I_+^n\}, \quad b_\alpha = \langle T, \phi_\alpha\rangle.$$

Then

1) *there exists* $h > 0$ *such that*

$$|b|_{-h} := \sup_{\alpha} |b_\alpha|\, |\lambda_\alpha|^{-h/2} < +\infty,$$

2) *the mapping* $s \ni a \to \sum_\alpha a_\alpha b_\alpha \in C$ *belongs to* s',

3) *it holds*

$$\langle T, f\rangle = \sum_{\alpha} b_\alpha a_\alpha(f),$$

where $a_\alpha(f) = \langle f, \phi_\alpha\rangle$ *for any* $f \in S(R^n)$. *Conversely, let* $b \in s'$. *Then* $T : S(R^n) \ni f \to \langle b, a(f)\rangle$ *belongs to* $S'(R^n)$.

LEMMA 4. 1) *Suppose* $u(t,x)$ *belongs to* $S([0,\infty), S'(R^n))$. *Set* $u_\alpha(t) = \langle u(t,x), \phi_\alpha(x)\rangle$. *Then there exist* $C(j,k) > 0$ *and* $p(j,k) > 0$ *such that*

$$|t^j D_t^k u_\alpha(t)| \leq C(j,k)|\lambda_\alpha|^{p(j,k)} \quad (t \in [0,\infty), \quad \alpha \in I_+^n).$$

2) *Conversely, suppose* $u_\alpha(t)$ *belongs to* $S([0,\infty))(\alpha \in I_+)$, *where*

$$|t^j D_t^k u_\alpha(t)| \leq C(j,k)|\lambda_\alpha|^{p(j,k)} \quad (t \in [0,\infty), \quad \alpha \in I_+^n).$$

Then $u(t,x) = \sum_{\alpha \in I_+^n} u_\alpha(t)\phi_\alpha(x)$ *belongs to* $S([0,\infty), S'(R^n))$.

PROOF. 1) Since $u(t,x)$ belongs to $S([0,\infty), S'(R^n))$,

$$u_\phi(t) = \langle u(t,x), \phi(x)\rangle \in S([0,\infty))$$

for any $\phi \in S(R^n)$. Namely,

$$\|u_\phi(t)\|_l = \sum_{j+k\le l} \sup_{t\in[0,\infty)} |t^j D_t^k u_\phi(t)|$$

$$= \sum_{j+k\le l} \sup_{t\in[0,\infty)} |\langle t^j D_t^k u(t,x), \phi(x)\rangle| < +\infty$$

for any $\phi \in S(R^n)$. Therefore, $\{t^j D_t^k u(t,x) | t \in [0,\infty)\}$ is a bounded set in $S'(R^n)$ in the sense of simple topology for any j, k. By using the fundamental Lemma of Fréchet space ([3]), there exist $C(j,k) > 0$ and $l(j,k) > 0$ such that

$$|\langle t^j D_t^k u(t,x), \phi\rangle| \le C(j,k)\|\phi\|_{l(j,k)} \quad (t \in [0,\infty), \quad \phi \in S(R^n)).$$

Besides, since

$$\|\phi_\alpha\|_l \le C(l)|\lambda_\alpha|^{p(l)}$$

from 3) of Lemma 1, we have

$$|t^j D_t^k u_\alpha(t)| \le C(j,k)|\lambda_\alpha|^{p(j,k)} \quad (t \in [0,\infty), \quad \alpha \in I_+^n).$$

2) Conversely, suppose $u_\alpha(t) \in S([0,\infty))$ satisfy

$$|t^j D_t^k u_\alpha(t)| \le C(j,k)|\lambda_\alpha|^{p(j,k)} \quad (t \in [0,\infty), \quad \alpha \in I_+^n)$$

for any $j, k \in I_+$. Let $f \in S(R^n)$ and set $a_\alpha(f) = \langle f, \phi_\alpha\rangle$. By using Lemma 2, we have

$$\sum_{\alpha\in I_+^n} |a_\alpha(f)|\,|t^j D_t^k u_\alpha(t)| \le C(j,k)\sum_\alpha |a_\alpha(f)|\,|\lambda_\alpha|^{p(j,k)}$$

$$\le C'(j,k)\sup_\alpha |a_\alpha(f)|\,|\lambda_\alpha|^{p(j,k)+p_0}$$

$$\le C''(j,k)\|f\|_{2n+2(\delta+1)(p(j,k)+p_0)}.$$

Hence $\sum_\alpha a_\alpha(f)u_\alpha(t)$ converges in $S([0,\infty))$. Therefore

$$u(t,x) = \sum_\alpha u_\alpha(t)\phi_\alpha(x) \in S([0,\infty), S'(R^n)),$$

namely,

$$\langle u(t,x), f(x)\rangle = \sum_\alpha u_\alpha(t)\langle\phi_\alpha(x), f(x)\rangle = \sum_\alpha a_\alpha(f)u_\alpha(t) \in S([0,\infty))$$

for $f \in S(R^n)$. □

3. Ordinary Differential Operators Depending on Parameter λ

Let us consider polynomials with respect to τ depending on the parameter $\lambda(\in \Lambda)$:

$$\begin{aligned} P(\tau,\lambda) &= P_m(\lambda)\tau^m + P_{m-1}(\lambda)\tau^{m-1} + \cdots + P_0(\lambda) \\ &= P_m(\lambda)(\tau-\tau_1(\lambda))\cdots(\tau-\tau_m(\lambda)), \\ Q_k(\tau,\lambda) &= Q_{k,m}(\lambda)\tau^M + Q_{k,M-1}(\lambda)\tau^{M-1} + \cdots + Q_{k,0}(\lambda) \quad (k=0,\dots,r-1), \end{aligned}$$

where

$$\operatorname{Im}\tau_k(\lambda) > 0 \ (1 \le k \le r), \quad \operatorname{Im}\tau_k(\lambda) \le 0 \ (r+1 \le k \le m), \quad |R(\lambda)| \ne 0.$$

We define

$$\mu(\lambda) = \min_{1\le j\le r} \operatorname{Im}\tau_j(\lambda), \quad \rho(\lambda) = \max_{1\le j\le m} |\tau_j(\lambda)|.$$

LEMMA 5. *Let $r \ne m$, $0 < \eta < 1$ and $\lambda \in \Lambda$. Suppose $e^{\eta t}f(t) \in S([0,\infty))$. Then there exists a unique solution $h(t)$ of the problem:*

$$(b_-)\begin{cases} P_-(D_t,\lambda)h(t) = f(t) \quad (t>0), \\ e^{\eta t}h(t) \in S([0,\infty)), \end{cases}$$

and there exist $C_l > 0$ and $N_l > 0$, independent of η and λ, such that

$$\|e^{\eta t}h(t)\|_l \le C_l\eta^{-(m-r)(l+3)}(1+\rho(\lambda))^{(l+2)(m-r-1)}\|e^{\eta t}f(t)\|_{N_l}$$

for any l.

PROOF. 1) Let $f_1(t)$ be an extension of $f(t)$ in $C^\infty(R)$ such that

$$\|e^{\eta t}f_1\|_l \le C_l\|e^{\eta t}f\|_{k[l]}$$

for any l, where constant C_l is independent of η. Then it holds

$$\begin{aligned} \hat{f}_1(\xi+i\eta) &:= \int_{-\infty}^{+\infty} e^{-i(\xi+i\eta)t}f_1(t)\,dt \\ &= F\{e^{\eta t}f_1(t)\}(\xi) \in S_\xi \quad (\xi \in R), \end{aligned}$$

where F is the Fourier transform. By the Fourier inversion formula, we have

$$f_1(t) = e^{-\eta t}F^{-1}\{\hat{f}_1(\xi+i\eta)\}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(\xi+i\eta)t}\hat{f}_1(\xi+i\eta)\,d\xi.$$

Let $e^{\eta t}h(t) \in S(R)$ satisfy

$$P_-(D_t, \lambda)h(t) = f_1(t) \quad (t \in R),$$

then

$$P_-(\xi + i\eta, \lambda)\hat{h}(\xi + i\eta) = \hat{f}_1(\xi + i\eta) \quad (\xi \in R),$$

that is,

$$\hat{h}(\xi + i\eta) = \frac{\hat{f}_1(\xi + i\eta)}{P_-(\xi + i\eta, \lambda)} \quad (\xi \in R),$$

because $P_-(\xi + i\eta, \lambda)$ is non-zero.

2) Let us make sure that

$$\frac{\hat{f}_1(\xi + i\eta)}{P_-(\xi + i\eta, \lambda)} \in S_\xi.$$

First we remark

$$\left(\frac{d}{d\xi}\right)^j \frac{1}{P_-(\xi + i\eta, \lambda)} = \frac{\Psi_j(\xi + i\eta, \lambda)}{P_-(\xi + i\eta, \lambda)^{j+1}} \quad (j = 0, 1, \ldots),$$

where $\Psi_j(z, \lambda)$ is a polynomial of degree $j(m - r - 1)$ with respect to z and

$$|\Psi_j(z, \lambda)| \leq C_j(1 + \rho(\lambda))^{j(m-r-1)}(1 + |z|)^{j(m-r-1)}.$$

Moreover since

$$|P_-(\xi + i\eta, \lambda)| = |(\xi + i\eta - \tau_{r+1}(\lambda)) \cdots (\xi + i\eta - \tau_m(\lambda))| \geq \eta^{m-r},$$

we have

$$\left|\left(\frac{d}{d\xi}\right)^j \frac{1}{P - (\xi + i\eta, \lambda)}\right| \leq \frac{|\Psi_j(\xi + i\eta, \lambda)|}{|P_-(\xi + i\eta, \lambda)^{j+1}|}$$

$$\leq \frac{C_j(1 + \rho(\lambda))^{j(m-r-1)}(1 + |\xi + i\eta|)^{j(m-r-1)}}{\eta^{(m-r)(j+1)}}.$$

Therefore, we have

$$\left\|\frac{\hat{f}_1(\xi + i\eta)}{P_-(\xi + i\eta, \lambda)}\right\|_l = \sum_{j+k \leq l} \sup_\xi \left|\xi^j \left(\frac{d}{d\xi}\right)^k \frac{\hat{f}_1(\xi + i\eta)}{P_-(\xi + i\eta, \lambda)}\right|$$

$$\leq C_l \eta^{-(m-r)(l+1)}(1 + \rho(\lambda))^{l(m-r-1)}\|\hat{f}_1(\xi + i\eta)\|_{l(m-r)}$$

$$\leq C_l' \eta^{-(m-r)(l+1)}(1 + \rho(\lambda))^{l(m-r-1)}\|e^{\eta t} f_1\|_{l(m-r)+2} < +\infty,$$

which means

$$\frac{\hat{f}_1(\xi+i\eta)}{P_-(\xi+i\eta,\lambda)} \in S_\xi.$$

3) Set

$$h(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(\xi+i\eta)t}\frac{\hat{f}_1(\xi+i\eta)}{P_-(\xi+i\eta,\lambda)}d\xi,$$

then we have

$$\begin{aligned}\|e^{\eta t}h(t)\|_l &\le C_l\left\|\frac{\hat{f}_1(\xi+i\eta)}{P_-(\xi+i\eta,\lambda)}\right\|_{l+2}\\ &\le C_l'\eta^{-(m-r)(l+3)}(1+\rho(\lambda))^{(l+2)(m-r-1)}\|e^{\eta t}f_1\|_{(m-r)(l+2)+2}\\ &\le C_l''\eta^{-(m-r)(l+3)}(1+\rho(\lambda))^{(l+2)(m-r-1)}\|e^{\eta t}f\|_{K[(m-r)(l+2)+2]},\end{aligned}$$

and

$$P_-(D_t,\lambda)h(t)=f_1(t) \quad (t\in R),$$

which means

$$P_-(D_t,\lambda)h(t)=f(t) \quad (t>0).$$

4) Let $h(t)\in S([0,\infty))$ be a solution of $P_-(D_t,\lambda)h(t)=0$ $(t>0)$. Set

$$h_1(t)=(D_t-\tau_{r+2}(\lambda))\cdots(D_t-\tau_m(\lambda))h(t)\in S([0,\infty)),$$

then

$$P_-(D_t,\lambda)h(t)=(D_t-\tau_{r+1}(\lambda))h_1(t)=0. \quad (t>0)$$

Multiplying both sides by $e^{-i\tau_{r+1}(\lambda)t}$, we have

$$D_t(e^{-i\tau_{r+1}(\lambda)t}h_1(t))=0 \quad (t>0),$$

namely,

$$e^{-i\tau_{r+1}(\lambda)t}h_1(t)=C \quad (t>0).$$

Since $|e^{-i\tau_{r+1}(\lambda)t}|=e^{\mathrm{Im}\,\tau_{r+1}(\lambda)t}\le 1$ $(t>0)$ and $h_1(t)\in S([0,\infty))$, we have $C=0$, namely,

$$h_1(t)=(D_t-\tau_{r+2}(\lambda))\cdots(D_t-\tau_m(\lambda))h(t)=0.$$

In the same way, we have $h(t)=0$. □

Next we consider

$$(b)_+\begin{cases} P_+(D_t,\lambda)u(t)=h(t) & (t>0),\\ Q_k(D_t,\lambda)u(t)|_{t=0}=g_k & (0\le k\le r-1),\end{cases}$$

where $e^{\eta t}h(t)\in S([0,\infty))$ and $r\ge 1$. Let

$$r\neq 0,\quad 0<\eta<\min(\mu(\lambda),1),\quad d_\eta(\lambda)=\min\left(\frac{\mu(\lambda)-\eta}{2},1\right)\quad (\lambda\in\Lambda),$$

and set

$$W(t,\lambda)=\frac{1}{2\pi i}\oint_\gamma\frac{e^{it\tau}}{P_+(\tau,\lambda)}d\tau,$$

where γ is a closed curve of the boundary of the domain $\{|\tau|<\rho(\lambda)+d_\eta(\lambda)\}\cap\{\operatorname{Im}\tau>\mu(\lambda)-d_\eta(\lambda)\}$. Then the solution of $(b)_+$ can be represented as

$$u(t)=\sum_{j=0}^{r-1}b_j(\lambda)D_t^jW(t,\lambda)+i\int_o^t h(s)W(t-s,\lambda)\,ds=U(t,\lambda)+V(t,\lambda),$$

where

$$\begin{pmatrix} b_0(\lambda)\\ \vdots\\ b_{r-1}(\lambda)\end{pmatrix}=\begin{pmatrix}\frac{1}{2\pi i}\oint\frac{Q_0(\tau,\lambda)}{P_+(\tau,\lambda)}d\tau & \cdots & \frac{1}{2\pi i}\oint\frac{Q_r(\tau,\lambda)\tau^{r-1}}{P_+(\tau,\lambda)}d\tau\\ \cdots\cdots\cdots & & \\ \frac{1}{2\pi i}\oint\frac{Q_{r-1}(\tau,\lambda)}{P_+(\tau,\lambda)}d\tau & \cdots & \frac{1}{2\pi i}\oint\frac{Q_{r-1}(\tau,\lambda)\tau^{r-1}}{P_+(\tau,\lambda)}d\tau\end{pmatrix}^{-1}\begin{pmatrix}\tilde g_0(\lambda)\\ \vdots\\ \tilde g_{r-1}(\lambda)\end{pmatrix}$$

$$=R(\lambda)^{-1}\begin{pmatrix}\Delta_{11}(\lambda) & \cdots & \Delta_{r1}(\lambda)\\ \cdots\cdots\cdots & & \\ \Delta_{1r}(\lambda) & \cdots & \Delta_{rr}(\lambda)\end{pmatrix}\begin{pmatrix}\tilde g_0(\lambda)\\ \vdots\\ \tilde g_{r-1}(\lambda)\end{pmatrix},$$

where

$$\tilde g_j(\lambda)=g_j-Q_j(D_t,\lambda)V(0,\lambda),$$

$$R(\lambda)=det\left(\frac{1}{2\pi i}\oint_\gamma\frac{Q_k(\tau,\lambda)\tau^l}{P_+(\tau,\lambda)}d\tau\right)_{k,l=0,\dots,r-1}.$$

Lemma 6. *Let* $0<\eta<\min(\mu(\lambda),1)$ *and* $\lambda\in\Lambda$. *Then it hold*

i) $|D_t^kW(t,\lambda)|\le d_\eta(\lambda)^{-r}(1+\rho(\lambda))^{k+1}e^{-\mu_1(\lambda)t}$ $(\mu_1(\lambda)=\mu(\lambda)-d_\eta(\lambda))$,

ii) $|D_t^kV(t,\lambda)|\le d_\eta(\lambda)^{-r}(1+\rho(\lambda))^{k+1}\left(\sum_{j=0}^{k-r}|D_t^jh(t)|+\int_0^t|h(s)|e^{-\mu_1(\lambda)(t-s)}\,ds\right)$,

iii) $|b_k(\lambda)| \leq C|R(\lambda)|^{-1}(d_\eta(\lambda)^{-r}|\lambda|^M(1+\rho(\lambda))^{M+(1/2)r+1/2})^r$

$$\times\left(\sum_{j=0}^{r-1}|g_j| + \sum_{j=0}^{M-r}|D_t^j h(0)|\right),$$

where C is a positive constant independent of λ and η.

PROOF. i) Since

$$D_t^k W(t,\lambda) = \frac{1}{2\pi i}\oint_\gamma \frac{\tau^k e^{it\tau}}{P_+(\tau,\lambda)}\,d\tau,$$

we have

$$|D_t^k W(t,\lambda)| \leq \frac{1}{2\pi}\oint_\gamma \frac{|\tau|^k |e^{it\tau}|}{|P_+(\tau,\lambda)|}|d\tau|.$$

Since

$$|e^{it\tau}| = e^{-t\,\mathrm{Im}\,\tau} \leq e^{-\mu_1(\lambda)t},$$

$$|\tau| \leq \rho(\lambda) + d_\eta(\lambda) \leq 1 + \rho(\lambda),$$

and

$$\frac{1}{|P_+(\tau,\lambda)|} = \frac{1}{|\tau-\tau_1(\lambda)|\,|\tau-\tau_2(\lambda)|\cdots|\tau-\tau_r(\lambda)|} \leq d_\eta(\lambda)^{-r}$$

on γ, we have

$$|D_t^k W(t,\lambda)| \leq d_\eta(\lambda)^{-r}(1+\rho(\lambda))^{k+1}e^{-\mu_1(\lambda)t}.$$

ii) Since

$$D_t^j V(t,\lambda) = i\int_0^t h(s)D_t^j W(t-s,\lambda)\,ds \quad (j = 0,1,\ldots,r-1),$$

$$D_t^k V(t,\lambda) = \sum_{j=0}^{k-r} D_t^j h(t) D_t^{k-j-1} W(0,\lambda) + i\int_0^t h(s)D_t^k W(t-s,\lambda)\,ds \quad (k \geq r),$$

we have

$$|D_t^k V(t,\lambda)| \leq \sum_{j=0}^{k-r}|D_t^j h(t)|\,|D_t^{k-j-1}W(0,\lambda)| + \int_0^t |h(s)|\,|D_t^k W(t-s,\lambda)|\,ds$$

$$\leq d_\eta(\lambda)^{-r}(1+\rho(\lambda))^{k+1}\left(\sum_{j=0}^{k-r}|D_t^j h(t)| + \int_0^t |h(s)|e^{-\mu_1(\lambda)(t-s)}\,ds\right).$$

iii) Since

$$|D_t^k V(0,\lambda)| \le d_\eta(\lambda)^{-r}(1+\rho(\lambda))^{k+1}\sum_{j=0}^{k-r}|D_t^j h(0)|$$

from ii), we have

$$\begin{aligned}|Q_k(D_t,\lambda)V(0,\lambda)| &= |Q_{kM}(\lambda)D_t^M V(0,\lambda)+\cdots+Q_{k0}(\lambda)V(0,\lambda)| \\ &\le C|\lambda|^M d_\eta(\lambda)^{-r}(1+\rho(\lambda))^{M+1}\sum_{j=0}^{M-r}|D_t^j h(0)|,\end{aligned}$$

therefore

$$\begin{aligned}\sum_{j=0}^{r-1}|\tilde{g}_j| &= \sum_{j=0}^{r-1}|g_j - Q_j(D_t,\lambda)V(0,\lambda)| \\ &\le C|\lambda|^M d_\eta(\lambda)^{-r}(1+\rho(\lambda))^{M+1}\left(\sum_{j=0}^{r-1}|g_j|+\sum_{i=0}^{M-r}|D_t^i h(0)|\right).\end{aligned}$$

Since

$$\left|\frac{1}{2\pi i}\oint_\gamma \frac{Q_k(\tau,\lambda)\tau^{j-1}}{P_+(\tau,\lambda)}d\tau\right| \le C d_\eta(\lambda)^{-r}|\lambda|^M(1+\rho(\lambda))^{M+j},$$

we have

$$|\Delta_{kj}| \le C\left(d_\eta(\lambda)^{-r}|\lambda|^M(1+\rho(\lambda))^{M+((1/2)r+1)}\right)^{r-1}.$$

Therefore we have

$$\begin{aligned}|b_k(\lambda)| &\le |R(\lambda)|^{-1}\sum_{j=1}^{r}|\Delta_{jk}(\lambda)|\,|\tilde{g}_{j-1}| \\ &\le C|R(\lambda)|^{-1}\left(d_\eta(\lambda)^{-r}|\lambda|^M(1+\rho(\lambda))^{M+((1/2)r+1)}\right)^{r-1}\sum_{j=0}^{r-1}|\tilde{g}_j| \\ &\le C|R(\lambda)|^{-1}\left(d_\eta(\lambda)^{-r}|\lambda|^M(1+\rho(\lambda))^{M+(1/2)r+1/2}\right)^{r} \\ &\quad\times\left(\sum_{g=0}^{r-1}|g_j|+\sum_{j=0}^{M-r}|D_t^j h(0)|\right).\end{aligned}$$

□

LEMMA 7. *Let* $0<\eta<\min(\mu(\lambda),1)$ *and* $\lambda\in\Lambda$. *Suppose* $e^{\eta t}h(t)\in S([0,\infty))$. *Then there exists a unique solution* $u(t)$ *of* $(b)_+$, *where* $e^{\eta t}u(t)$ *belongs to* $S([0,\infty))$.

Moreover it holds

$$\|e^{\eta t}u(t)\|_l \le C_l \max(1, |R(\lambda)|^{-1})|\lambda|^{Mr} d_\eta(\lambda)^{-(r^2+r+l)}(1+\rho(\lambda))^{Mr+(1/2)r^2+(3/2)r+l}$$

$$\times\left(\sum_{j=0}^{r-1}|g_j| + \|e^{\eta t}h(t)\|_l\right) \quad (l \ge M-r+1),$$

where C_l *is a positive constant, independent of* λ *and* η.

PROOF. 1) Owing to i) and iii) of Lemma 6, we have

$$|D_t^k U(t,\lambda)| = \left|\sum_{j=0}^{r-1} b_j(\lambda) D_t^{k+j} W(t,\lambda)\right|$$

$$\le C|R(\lambda)|^{-1}\left(d_\eta(\lambda)^{-r}|\lambda|^M(1+\rho(\lambda))^{M+(1/2)r+1/2}\right)^r$$

$$\times\left(\sum_{j=0}^{r-1}|g_j| + \sum_{j=0}^{M-r} D_t^j|h(0)|\right)\sum_{j=0}^{r-1} d_\eta(\lambda)^{-r}(1+\rho(\lambda))^{k+j+1}e^{-\mu_1(\lambda)t}$$

$$\le C|R(\lambda)|^{-1}d_\eta(\lambda)^{-r(r+1)}|\lambda|^{Mr}(1+\rho(\lambda))^{Mr+(1/2)r^2+(3/2)r+k}e^{-\mu_1(\lambda)t}$$

$$\times\left(\sum_{j=0}^{r-1}|g_j| + \sum_{j=0}^{M-r}|D_t^j h(0)|\right).$$

Therefore we have

$$|t^\beta D_t^k[e^{\eta t}U(t,\lambda)]| \le \sum_{i=0}^{k}\binom{k}{i}\eta^{k-i}e^{\eta t}t^\beta|D_t^i U(t,\lambda)|$$

$$\le C|R(\lambda)|^{-1}d_\eta(\lambda)^{-r(r+1)}|\lambda|^{Mr}(1+\rho(\lambda))^{Mr+(1/2)r^2+(3/2)r+k}$$

$$\times(t^\beta e^{-d_\eta(\lambda)t})\left(\sum_{j=0}^{r-1}|g_j| + \sum_{j=0}^{M-r}|D_t^j h(0)|\right)$$

$$\le C'|R(\lambda)|^{-1}d_\eta(\lambda)^{-r(r+1)-\beta}|\lambda|^{Mr}(1+\rho(\lambda))^{Mr+(1/2)r^2+(3/2)r+k}$$

$$\times\left(\sum_{j=0}^{r-1}|g_j| + \sum_{j=0}^{M-r}|D_t^j h(0)|\right).$$

On the other hand, since

$$\begin{aligned}\|e^{\eta t}h(t)\|_l &= \sum_{\beta+k\le l}\sup_{0<t<+\infty}|t^\beta D_t^k[e^{\eta t}h(t)]| \\ &= \sum_{\beta+k\le l}\sup_{0<t<+\infty}|t^\beta e^{\eta t}(D_t-i\eta)^k h(t)| \\ &\ge \sum_{k=0}^{l}|(D_t-i\eta)^k h(0)|,\end{aligned}$$

we have

$$\sum_{k=0}^{l}|D_t^k h(0)| \le C_l\|e^{\eta t}h(t)\|_l.$$

Therefore we have

$$\begin{aligned}\|e^{\eta t}U(t,\lambda)\|_l \le{}& C_l|R(\lambda)|^{-1}d_\eta(\lambda)^{-(r^2+r+l)}|\lambda|^{Mr} \\ &\times(1+\rho(\lambda))^{Mr+(1/2)r^2+(3/2)r+l}\left(\sum_{j=0}^{r-1}|g_j|+\|e^{\eta t}h(t)\|_l\right).\end{aligned}$$

2) Owing to ii) of Lemma 6, we have

$$\begin{aligned}|t^\beta D_t^k[e^{\eta t}V(t,\lambda)]| &= |t^\beta e^{\eta t}(D_t-i\eta)^k V(t,\lambda)| \\ &\le Cd_\eta(\lambda)^{-r}(1+\rho(\lambda))^{k+1} \\ &\quad\times\left(\sum_{j=0}^{k-r}|t^\beta e^{\eta t}D_t^j h(t)|+\int_0^t t^\beta e^{\eta t}e^{-\mu_1(\lambda)(t-s)}|h(s)|\,ds\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sup_{0\le t<+\infty}\sum_{j=0}^{k-r}|t^\beta e^{\eta t}D_t^j h(t)| &= \sum_{j=0}^{k-r}\sup_{0\le t<+\infty}|t^\beta(D_t+i\eta)^j e^{\eta t}h(t)| \\ &= \sum_{j=0}^{k-r}\sup_{0\le t<+\infty}\left|t^\beta\sum_{j'=0}^{j}\binom{j}{j'}(i\eta)^{j-j'}D_t^{j'}(e^{\eta t}h(t))\right| \\ &\le C_k\|e^{\eta t}h(t)\|_{\beta+k-r}.\end{aligned}$$

and

$$\sup_{0\le t<+\infty}\int_0^t t^\beta e^{\eta t}e^{-\mu_1(\lambda)(t-s)}|h(s)|\,ds$$

$$= \sup_{0\le t<+\infty}\int_0^t \{(t-s)+s\}^\beta e^{(\eta-\mu_1(\lambda))(t-s)}e^{\eta s}|h(s)|\,ds$$

$$= \sup_{0\le t<+\infty}\int_0^t \sum_{\beta'=0}^{\beta}\binom{\beta}{\beta'}(t-s)^{\beta-\beta'}e^{-d_\eta(\lambda)(t-s)}s^{\beta'}e^{\eta s}|h(s)|\,ds$$

$$\le C_\beta\sum_{\beta'=0}^{\beta}\sup_{0\le s<+\infty}|s^{\beta'}e^{\eta s}h(s)|\sum_{\beta''=0}^{\beta}\sup_{0\le t<+\infty}\int_0^t(t-s)^{\beta''}e^{-d_\eta(\lambda)(t-s)}\,ds$$

$$\le C'_\beta d_\eta(\lambda)^{-(\beta+1)}\|e^{\eta t}h(t)\|_\beta.$$

Hence we have

$$\|e^{\eta t}V(t,\lambda)\|_l = \sum_{\beta+k\le l}\sup_{0\le t<+\infty}|t^\beta D_t^k[e^{\eta t}V(t,\lambda)]|$$

$$\le C_l d_\eta(\lambda)^{-(r+l+1)}(1+\rho(\lambda))^{l+1}\|e^{\eta t}h(t)\|_l.$$

Therefore we have

$$\|e^{\eta t}u(t)\|_l \le \|e^{\eta t}U(t,\lambda)\|_l + \|e^{\eta t}V(t,\lambda)\|_l$$

$$\le C_l\max(1,|R(\lambda)|^{-1})d_\eta(\lambda)^{-(r^2+r+l)}|\lambda|^{Mr}(1+\rho(\lambda))^{Mr+(1/2)r^2+(3/2)r+l}$$

$$\times\left(\sum_{j=0}^{r-1}|g_j| + \|e^{\eta t}h(t)\|_l\right). \qquad \square$$

Lemma 8. *Let* $0<\eta<\min(\mu(\lambda),1)$ *and* $\lambda\in\Lambda$. *Suppose* $e^{\eta t}f(t)\in S([0,\infty))$. *Then there exists a unique solution* $u(t)$ *of*

$$(b)\begin{cases} P(D_t,\lambda)u(t)=f(t) & (t>0),\\ Q_k(D_t,\lambda)u(0)=g_k & (0\le k\le r-1),\\ e^{\eta t}u(t)\in S([0,\infty)). \end{cases}$$

Moreover, it holds

$$\|e^{\eta t}u(t)\|_l \le C_l\eta^{-(m-r)(l-3)}\max(1,|R(\lambda)|^{-1})\max(1,|P_m(\lambda)|^{-1})|\lambda|^{Mr}d_\eta(\lambda)^{-(r^2+r+l)}$$

$$\times(1+\rho(\lambda))^{Mr+(1/2)r^2+(1/2)r+2m+l(m-r)}\left(\sum_{j=0}^{r-1}|g_j| + \|e^{\eta t}f(t)\|_{N_l}\right),$$

where C_l *is a positive constant, independent of* λ *and* η.

PROOF. Let $u(t)$ be a solution of (b) and set

$$h(t) = P_+(D_t, \lambda)u(t).$$

Then $h(t)$ satisfies

$$(b)_- \begin{cases} P_-(D_t, \lambda)h(t) = \dfrac{f(t)}{P_m(\lambda)} \quad (t > 0), \\ e^{\eta t}h(t) \in S([0, \infty)) \end{cases}$$

and $u(t)$ satisfies

$$(b)_+ \begin{cases} P_+(D_t)u(t) = h(t) \quad (t > 0), \\ D_t^k u(0) = g_k \quad (0 \le k \le r-1). \end{cases}$$

Conversely, let $h(t)$ be a solution of $(b)_-$ and let $u(t)$ be a solution of $(b)_+$, then $u(t)$ is a solution of (b). Therefore there exists a unique solution of (b), owing to Lemma 5 and Lemma 7. Now, let $u(t)$ be a solution of (b), then $h(t) = P_+(D_t, \lambda)u(t)$ satisfies

$$\|e^{\eta t}h(t)\|_l \le C_l \eta^{-(m-r)(l+3)}(1 + \rho(\lambda))^{(l+2)(m-r-1)} \frac{\|e^{\eta t}f(t)\|_{N_l}}{|P_m(\lambda)|}$$

from Lemma 5. Therefore we have

$$\|e^{\eta t}u(t)\|_l \le C_l \max(1, |R(\lambda)|^{-1})|\lambda|^{Mr} d_\eta(\lambda)^{-(r^2+r+l)}$$
$$\times (1 + \rho(\lambda))^{Mr+(1/2)r^2+(3/2)r+l} \left(\sum_{j=0}^{r-1} |g_j| + \|e^{\eta t}h(t)\|_l \right)$$

from Lemma 7. Here we have

$$\|e^{\eta t}u(t)\|_l \le C_l \eta^{-(m-r)(l+3)} \max(1, |R(\lambda)|^{-1}) \max(1, |P_m(\lambda)|^{-1})|\lambda|^{Mr} d_\eta(\lambda)^{-(r^2+r+l)}$$
$$\times (1 + \rho(\lambda))^{Mr+(1/2)r^2-(1/2)r+2m+l(m-r)} \left(\sum_{j=0}^{r-1} |g_j| + \|e^{\eta t}f(t)\|_{N_l} \right). \qquad \square$$

4. Proofs of Theorems

Suppose $e^{\eta t}f(t, x) \in S([0, \infty), S'(R^n))$ and $g_k(x) \in S'(R^n)$ $(0 \le k \le r-1)$. And consider the boundary value problem:

$$(B) \begin{cases} P(D_t, L)u(t, x) = f(t, x) \quad (x \in R^n, t > 0), \\ B_k(D_t, L)u(t, x)|_{t=0} = g_k(x) \quad (x \in R^n, 0 \le k \le r-1), \\ u(t, x) \in S([0, \infty), S'(R^n)). \end{cases}$$

Set

$$u_\alpha(t) = \langle u(t,x), \phi_\alpha(x)\rangle, \quad f_\alpha(t) = \langle f(t,x), \phi_\alpha(x)\rangle, \quad g_{k,\alpha} = \langle g_k(x), \phi_\alpha(x)\rangle,$$

then the problem (B) can be formally reduced to the boundary value problems of ordinary differential operators:

$$(b_\alpha)\begin{cases} P(D_t, \lambda_\alpha)u_\alpha(t) = f_\alpha(t) & (t > 0), \\ B_k(D_t, \lambda_\alpha)u_\alpha(0) = g_{k,\alpha} & (0 \le k \le r-1), \\ u_\alpha(t) \in S([0,\infty)), \end{cases}$$

where $e^{\eta t} f_\alpha(t) \in S([0,\infty))$ and $g_{k,\alpha} \in C$ $(0 \le k \le r-1)$ for any $\alpha \in I_+^n$.

PROOF OF THEOREM 1. In condition (II′), we may assume μ is so small that $0 < \mu < 1$. Let $0 < \eta < \mu$. Then we have

$$\|e^{\eta t}u_\alpha(t)\|_l \le C_l \eta^{-(m-r)(l+3)} \max(1, |R(\lambda_\alpha)|^{-1}) \max(1, |P_m(\lambda)|^{-1}) |\lambda_\alpha|^{Mr} d_\eta(\lambda_\alpha)^{-(r^2+r+l)}$$

$$\times (1 + \rho(\lambda_\alpha))^{Mr + (1/2)r^2 - (1/2)r + 2m + l(m-r)} \left(\sum_{j=0}^{r-1} |g_{j,\alpha}| + \|e^{\eta t} f_\alpha(t)\|_{N_l} \right)$$

from Lemma 8. Since $\mu(\lambda_\alpha) \ge \mu$, we have

$$d_\eta(\lambda_\alpha) = \min\left(\frac{\mu(\lambda_\alpha) - \eta}{2}, 1\right) \ge \frac{\mu - \eta}{2}.$$

Since $|P_j(\lambda)| \le C|\lambda|^M$ and $|P_m(\lambda)| \ge C_1|\lambda|^{-p_1}$ from condition (I), we have

$$\rho(\lambda) \le \sum_{j=0}^{m} \frac{|P_j(\lambda)|}{|P_m(\lambda)|} \le \sum_{j=0}^{m} \frac{C|\lambda|^M}{C_1|\lambda|^{-p_1}} \le C'|\lambda|^{p_4} \quad (p_4 = p_1 + M).$$

Moreover since

$$|R(\lambda)| \ge C_3|\lambda|^{-p_3}$$

from condition (III), we have

$$\|e^{\eta t}u_\alpha(t)\|_l \le C_l' \eta^{-(m-r)(l+3)} (\mu - \eta)^{-(r^2+r+l)} |\lambda_\alpha|^{p_l'} \left(\sum_{k=0}^{r-1} |g_{k,\alpha}| + \|e^{\eta t} f_\alpha(t)\|_{N_l} \right)$$

$$\left(p_l' = p_4\left\{ Mr + \frac{1}{2}r^2 - \frac{1}{2}r + 2m + l(m-r) \right\} + Mr + p_1 + p_3 \right).$$

On the other hand, since

$$e^{\eta t} f(t,x) \in S([0,\infty), S'(R^n)),$$

there exists $q(l) > 0$ for any l such that

$$K_l := \sup_{\alpha \in I_+^n} \|e^{\eta t} f_\alpha(t)\|_l |\lambda_\alpha|^{-q(N_l)} < +\infty$$

from 1) of Lemma 4. Moreover since $g_k(x) \in S'(R^n)$, there exists $q > 0$ such that

$$K := \sup_{\alpha \in I_+^n} \sum_{k=0}^{r-1} |g_{k,\alpha}| \, |\lambda_\alpha|^{-q} < +\infty$$

from 1) of Lemma 3. Therefore we have

$$\begin{aligned} \|e^{\eta t} u_\alpha(t)\|_l &\le C_l' \eta^{-(m-r)(l+3)} (\mu - \eta)^{-(r^2+r+l)} |\lambda_\alpha|^{p_l'} (K|\lambda_\alpha|^q + K_l |\lambda_\alpha|^{q(N_l)}) \\ &\le C_l'' \eta^{-(m-r)(l+3)} (\mu - \eta)^{-(r^2+r+l)} (K + K_l) |\lambda_\alpha|^{p_l} \end{aligned}$$

where

$$p_l = p_4 \left(Mr + \frac{1}{2} r^2 - \frac{1}{2} r + 2m + l(m - r) \right) + Mr + p_1 + p_3 + \max(q, q(N_l)).$$

Hence we have $e^{\eta t} u(t, x) \in S([0, \infty), S'(R^n))$ from Lemma 4. □

PROOF OF THEOREM 2. Let $0 < \eta < 1$. In condition (II), we may assume C_2 is so small that

$$C_2 |\lambda_0|^{-p_2} \le \eta,$$

where $\lambda_0 = (\lambda_{10}, \dots, \lambda_{n0})$. Then

$$\min(\mu(\lambda_\alpha), \eta) \ge \min(C_2 |\lambda_\alpha|^{-p_2}, \eta) = C_2 |\lambda_\alpha|^{-p_2}.$$

Owing to the result of Lemma 8, we have

$$\begin{aligned} \|e^{\xi t} u_\alpha(t)\|_l &\le C_l \xi^{-(m-r)(l+3)} \max(1, |R(\lambda_\alpha)|^{-1}) \max(1, P_m(\lambda)^{-1}) |\lambda_\alpha|^{Mr} d_\xi(\lambda_\alpha)^{-(r^2+r+l)} \\ &\quad \times (1 + \rho(\lambda_\alpha))^{Mr + (1/2)r^2 - (1/2)r + 2m + l(m-r)} \left(\sum_{k=0}^{r-1} |g_{k,\alpha}| + \|e^{\xi t} f_\alpha(t)\|_{N_l} \right) \\ &\quad (0 < \xi < \min(\mu(\lambda_\alpha), \eta), \quad \alpha \in I_+^n), \end{aligned}$$

where

$$d_\xi(\lambda_\alpha) = \min\left(\frac{\mu(\lambda_\alpha) - \xi}{2}, 1 \right).$$

Now let us specify ξ as

$$\xi = \xi_\alpha = \frac{1}{2}\min(\mu(\lambda_\alpha), \eta).$$

Then we have

$$\xi_\alpha \geq \left(\frac{1}{2}C_2\right)|\lambda_\alpha|^{-p_2},$$

$$d_{\xi_\alpha}(\lambda_\alpha) = \min\left(\frac{\mu(\lambda_\alpha) - \xi_\alpha}{2}, 1\right) \geq \frac{1}{4}C_2|\lambda_\alpha|^{-p_2}.$$

In consideration of

$$|P_m(\lambda_\alpha)|^{-1} \leq C_1^{-1}|\lambda_\alpha|^{p_1}, \quad |R(\lambda_\alpha)|^{-1} \leq C_3^{-1}|\lambda_\alpha|^{p_3},$$
$$\rho(\lambda_\alpha) \leq C_4|\lambda_\alpha|^{p_4},$$

we have

$$\|e^{\xi_\alpha t}u_\alpha(t)\|_l \leq C_l'|\lambda_\alpha|^{p_5}\left(\sum_{k=0}^{r-1}|g_{k,\alpha}| + \|e^{\xi_\alpha t}f_\alpha(t)\|_{N_l}\right),$$

where

$$p_5 = p_2(m-r)(l+3) + p_2(r^2+r+l)$$
$$+ p_4\left\{Mr + \frac{1}{2}r^2 - \frac{1}{2}r + 2m + l(m-r)\right\} + Mr + p_1 + p_3.$$

Since

$$\|e^{\eta_1 t}u(t)\|_l \leq C_l\|e^{\eta_2 t}u(t)\|_l$$

for any η_1 and η_2 $(0 \leq \eta_1 < \eta_2 \leq 1)$, we have

$$\|u_\alpha(t)\|_l \leq C_l\|e^{\xi_\alpha t}u_\alpha(t)\|_l, \quad \|e^{\xi_\alpha t}f_\alpha(t)\|_{N_l} \leq C_l\|e^{\eta t}f_\alpha(t)\|N_l.$$

Therefore we have

$$\|u_\alpha(t)\|_l \leq C_l'|\lambda_\alpha|^{p_5}\left(\sum_{k=0}^{r-1}|g_{k,\alpha}| + \|e^{\eta t}f_\alpha(t)\|_{N_l}\right).$$

In the same way as in the proof of Theorem 1, we have

$$\|u_\alpha(t)\|_l \leq C_l''|\lambda_\alpha|^{q_l}(K + K_l),$$

where

$$K_l := \sup_{\alpha \in I_+^n} \|e^{\eta t} f_\alpha(t)\|_l |\lambda_\alpha|^{-q(N_l)} < +\infty, \quad K := \sum_{k=0}^{r-1} \sup_{\alpha \in I_+^n} |g_{k,\alpha}| \, |\lambda_\alpha|^{-q} < +\infty,$$

$$q_l = p_5 + \max(q, q(N_l)).$$

Hence we have $u(t,x) \in S([0,\infty), S'(R^n))$, from Lemma 4. □

5. Examples

For the special case of

$$P(D_t, L) = D_t^2 - \sum_{|\beta| \le N} a_\beta L^\beta,$$

we can describe some criteria for $P(D_t, L)$ to be separative or uniformly separative, by checking the positions of zeros of

$$P(\tau, \lambda) = \tau^2 - \sum_{|\beta| \le N} a_\beta \lambda^\beta$$

with respect to τ. Set

$$P(\tau, \lambda) = \tau^2 - (X(\lambda) + iY(\lambda)) = (\tau - \tau_+(\lambda))(\tau - \tau_-(\lambda)),$$

where

$$X(\lambda) = \sum_{|\beta| \le N} \mathrm{Re}(a_\beta \lambda^\beta), \quad Y(\lambda) = \sum_{|\beta| \le N} \mathrm{Im}(a_\beta \lambda^\beta),$$

$$\tau_\pm(\lambda) = (X(\lambda) + iY(\lambda))^{1/2}.$$

Let us remark the facts:

i) in case of $X > 0$ and $|Y/X| < 1$, we have

$$C_1 \frac{|Y|}{|X|^{1/2}} \le |\mathrm{Im}(X + iY)^{1/2}| \le C_2 \frac{|Y|}{|X|^{1/2}},$$

ii) in case of $X < 0$ and $|Y/X| < 1$, we have

$$C_1 |X|^{1/2} \le |\mathrm{Im}(X + iY)^{1/2}| \le C_2 |X|^{1/2},$$

iii) in case of $|Y/X| \ge 1$, we have

$$C_1 |Y|^{1/2} \le |\mathrm{Im}(X + iY)^{1/2}| \le C_2 |Y|^{1/2}.$$

These facts will prove the following Proposition 1 and Proposition 2.

PROPOSITION 1. *$P(D_t, L)$ is uniformly separative, iff there exists $k > 0$ such that*

$$(X(\lambda), Y(\lambda)) \notin \Omega_k \quad (\lambda \in \Lambda),$$

where

$$\Omega_k = \{(X, Y) | Y^2 \leq kX\} \cup \{(X, Y) | X^2 + Y^2 \leq 2k^2\}.$$

PROOF. 1) Suppose $P(D_t, L)$ is uniformly separative. By the condition (II′), we have that there exists $q > 0$ such that

$$\operatorname{Im} \tau_+(\lambda) \geq q \quad (\lambda \in \Lambda).$$

We remark that

$$\Omega_k = \{(X, Y) | X > |Y|, \quad Y^2 \leq kX\} \cup \{(X, Y) | X \leq |Y|, \quad X^2 + Y^2 \leq 2k^2\}.$$

i) If $X(\lambda) > 0$, $|Y(\lambda)/X(\lambda)| < 1$ and $\lambda \in \Lambda$, we have

$$C_2 \frac{|Y(\lambda)|}{|X(\lambda)|^{1/2}} \geq q,$$

that is,

$$|Y(\lambda)|^2 \geq \left(\frac{q}{C_2}\right)^2 |X(\lambda)|.$$

ii) If $X(\lambda) < 0$, $|Y(\lambda)/X(\lambda)| < 1$ and $\lambda \in \Lambda$, we have

$$C_2 |X(\lambda)|^{1/2} \geq q,$$

that is,

$$X(\lambda)^2 + Y(\lambda)^2 \geq X(\lambda)^2 \geq \left(\frac{q}{C_2}\right)^4.$$

iii) If $|Y(\lambda)/X(\lambda)| \geq 1$ and $\lambda \in \Lambda$, we have

$$C_2 |Y(\lambda)|^{1/2} \geq q,$$

that is,

$$X(\lambda)^2 + Y(\lambda)^2 \geq Y(\lambda)^2 \geq \left(\frac{q}{C_2}\right)^4.$$

Then we have

$$(X(\lambda), Y(\lambda)) \notin \Omega_k \quad (\lambda \in \Lambda), \quad k = \frac{q^2}{\sqrt{2} C_2^2}.$$

2) Suppose $(X(\lambda), Y(\lambda)) \notin \Omega_k$ $(\lambda \in \Lambda)$. Namely,

$$Y(\lambda)^2 \geq kX(\lambda), \quad X(\lambda)^2 + Y(\lambda)^2 \geq 2k^2 \quad (\lambda \in \Lambda).$$

i) If $X(\lambda) > 0$, $|Y(\lambda)/X(\lambda)| < 1$ and $\lambda \in \Lambda$, we have

$$|\mathrm{Im}(X(\lambda) + iY(\lambda))^{1/2}| \geq C_1 \frac{|Y(\lambda)|}{|X(\lambda)|^{1/2}} \geq C_1 k^{1/2}.$$

ii) If $X(\lambda) < 0$, $|Y(\lambda)/X(\lambda)| < 1$ and $\lambda \in \Lambda$, we have

$$\begin{aligned} |\mathrm{Im}(X(\lambda) + iY(\lambda))^{1/2}| &\geq C_1 |X(\lambda)|^{1/2} \\ &\geq C_1 \left\{ \frac{1}{2}(X(\lambda)^2 + Y(\lambda)^2) \right\}^{1/4} \\ &\geq C_1 k^{1/2}. \end{aligned}$$

iii) If $|Y(\lambda)/X(\lambda)| \geq 1$, and $\lambda \in \Lambda$, we have

$$\begin{aligned} |\mathrm{Im}(X(\lambda) + iY(\lambda))^{1/2}| &\geq C_1 |Y(\lambda)|^{1/2} \\ &\geq C_1 \left\{ \frac{1}{2}(X(\lambda)^2 + Y(\lambda)^2) \right\}^{1/4} \\ &\geq C_1 k^{1/2}. \end{aligned}$$

□

PROPOSITION 2. *$P(D_t, L)$ is separative, if there exist $C > 0$ and $p > 0$ such that*

$$|Y(\lambda)| \geq C|\lambda|^{-p} \quad (\lambda \in \Lambda).$$

PROOF. First, we will pay attention that there exist $C' > 0$ such that

$$|X(\lambda)| \leq C'|\lambda|^N \quad (\lambda \in \Lambda).$$

Suppose $|Y(\lambda)| \geq C|\lambda|^{-p}$ $(\lambda \in \Lambda)$. Then,

i) if $X(\lambda) > 0$, $|Y(\lambda)/X(\lambda)| < 1$ and $\lambda \in \Lambda$, we have

$$\begin{aligned} |\mathrm{Im}(X(\lambda) + iY(\lambda))^{1/2}| &\geq C_1 \frac{|Y(\lambda)|}{|X(\lambda)|^{1/2}} \\ &\geq C_1 \frac{C|\lambda|^{-p}}{(C')^{1/2}|\lambda|^{(1/2)N}} \\ &= \frac{C_1 C}{(C')^{1/2}} |\lambda|^{-(p+(1/2)N)}, \end{aligned}$$

ii) if $X(\lambda) < 0$, $|Y(\lambda)/X(\lambda)| < 1$ and $\lambda \in \Lambda$, we have

$$|\mathrm{Im}(X(\lambda) + iY(\lambda))^{1/2}| \geq C_1|X(\lambda)|^{1/2} \geq C_1|Y(\lambda)|^{1/2} \geq C_1C^{1/2}|\lambda|^{-(1/2)p},$$

iii) if $|Y(\lambda)/X(\lambda)| \geq 1$ and $\lambda \in \Lambda$, we have

$$|\mathrm{Im}(X(\lambda) + iY(\lambda))^{1/2}| \geq C_1|Y(\lambda)|^{1/2} \geq C_1C^{1/2}|\lambda|^{-(1/2)p}. \qquad \square$$

Example 1. Let

$$P(D_t, L) = D_t^2 - \{L_1^2 + L_2^2 - L_3^2 + i(L_1 + \cdots + L_n)\},$$

then P is unifomly separative. In fact, since

$$|X| = |\lambda_1^2 + \lambda_2^2 - \lambda_3^2| \leq |\lambda|^2,$$
$$Y = \lambda_1 + \cdots + \lambda_n \geq |\lambda| \geq |\lambda_0|,$$

we have

$$(X, Y) = (\lambda_1^2 + \lambda_2^2 - \lambda_3^2, \lambda_1 + \cdots + \lambda_n) \notin \Omega_{\min(1, \frac{1}{\sqrt{2}}|\lambda_0|)}.$$

Example 2. Let

$$P(D_t, L) = D_t^2 - \{L_1^3 + L_2^3 - L_3^3 + i(L_1^2 + \cdots + L_n^2)\},$$

then P is uniformly separative. In fact, since

$$|X| = |\lambda_1^3 + \lambda_2^3 - \lambda_3^3| \leq |\lambda|^3,$$
$$Y = \lambda_1^2 + \cdots + \lambda_n^2 = |\lambda|^2 \geq |\lambda_0|^{1/2}|\lambda|^{3/2} \geq |\lambda_0|^2,$$

we have

$$(X, Y) = (\lambda_1^2 + \lambda_2^2 - \lambda_3^2, \quad \lambda_1^2 + \cdots + \lambda_n^2) \notin \Omega_{\min(|\lambda_0|, \frac{1}{\sqrt{2}}|\lambda_0|^2)}.$$

Example 3. Let

$$P(D_t, L) = D_t^2 - \{L_1^2 + L_2^2 - L_3^2 + i(L_1 - L_2)\},$$

then P is separative, if

$$L_1 = D_{x_1}^2 + x_1^2 + 1, \quad L_2 = D_{x_2}^2 + x_2^2 + 2.$$

In fact, since

$$\Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3, \quad \Lambda_1 \in \{2, 4, \ldots\}, \quad \Lambda_2 \in \{3, 5, \ldots\},$$

it holds

$$|Y| = |\lambda_1 - \lambda_2| \geq 1 \quad (\lambda \in \Lambda).$$

EXAMPLE 4. Let

$$P(D_t, L) = D_t^2 - \{L_1^2 + L_2^2 - L_3^2 + i(L_1 - L_2)\},$$

then P is separative if

$$\Lambda = \Lambda_1 \times \Lambda_2 \times \Lambda_3,$$

$$\Lambda_1 = \{\lambda_{1,k} | k = 0, 1, \dots\} = \{1, 3, 5, \dots\}, \quad \Lambda_2 = \{\lambda_{2,k} | k = 0, 1, \dots\},$$

and

$$\frac{1}{k+2} \leq |\lambda_{1,k} - \lambda_{2,k}| < 1 \quad (k = 0, 1, \dots).$$

In fact, since

$$|\lambda_{1,j} - \lambda_{2,k}| \geq |\lambda_{1,j} - \lambda_{1,k}| - |\lambda_{1,k} - \lambda_{2,k}| \geq 2 - 1 = 1 \quad (j \neq k),$$

$$|\lambda_{1,k} - \lambda_{2,k}| \geq \frac{1}{k+2} \geq \frac{1}{2(\lambda_{1,k}^2 + \lambda_{2,k}^2)^{1/2}},$$

we have

$$|\lambda_1 - \lambda_2| \geq \frac{1}{2|\lambda|} \quad (\lambda \in \Lambda). \qquad \square$$

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