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REAL HYPERSURFACES IN A COMPLEX HYPERBOLIC SPACE WITH THREE CONSTANT PRINCIPAL CURVATURES

By

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1. Introduction

Let $H_n(\boldsymbol{C})$ be a complex hyperbolic space of complex dimension n (≥ 2) with the metric of constant holomorphic sectional curvature -4 and M be a real hypersurface in $H_n(\boldsymbol{C})$ with the induced metric. We denote by $\tilde{J}$ the natural complex structure of $H_n(\boldsymbol{C})$.

S. Montiel [4] gave the following classification theorem.

THEOREM. *If M is a connected real hypersurface of $H_n(\boldsymbol{C})$ $(n \geq 3)$ with two distinct constant principal curvatures, then M is holomorphic congruent to an open part of one of the following real hypersurfaces of $H_n(\boldsymbol{C})$: a geodesic hypersphere in $H_n(\boldsymbol{C})$; a tube around $H_{n-1}(\boldsymbol{C})$ in $H_n(\boldsymbol{C})$; a tube of radius $ln(2+\sqrt{3})$ around $H_n(\boldsymbol{R})$ in $H_n(\boldsymbol{C})$; a horosphere in $H_n(\boldsymbol{C})$.*

Moreover, J. Berndt [1] classified all real hypersurfaces with constant principal curvatures in $H_n(\boldsymbol{C})$ under the assumption:

(C) The structure vector field is principal.

In this paper we prove that Berndt's theorem holds without the condition (C) for the case where the number of constant principal curvature is three and $n \geq 3$. More precisely,

MAIN THEOREM. *Let M is a connected real hypersurface in $H_n(\boldsymbol{C})$ $(n \geq 3)$ with three distinct constant principal curvatures. Then M is holomorphic congruent to an open part of one of the following hypersurfaces:*

(a) *a tube of radius $r \in \boldsymbol{R}^+$ around $H_k(\boldsymbol{C})$ for a $k \in \{1, \dots, n-2\}$,*

(b) *a tube of radius $r \in \boldsymbol{R}^+ \backslash \{ln(2+\sqrt{3})\}$ around $H_n(\boldsymbol{R})$.*

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2. Preliminaries

Let $n \geq 3$ and $H_n(\boldsymbol{C})$ be a complex hyperbolic space with the metric of constant holomorphic sectional curvature $4c$ $(c < 0)$ and M be a real hypersurface in $H_n(\boldsymbol{C})$ with the induced metric. Choose a local field $\{e_1, \dots, e_{2n}\}$ of orthonormal frame in such a way that, restricted to M, the vectors $e_1, \dots, e_{2n-1}$ are tangent to M. Hereafter let the indices i, j, k, l run through from 1 to $2n-1$ unless otherwise stated. We denote by θ_i, θ_{ij} and Θ_{ij} the canonical 1-forms, the connection forms and curvature form of M respectively. Then they satisfy

$$(2.1) \qquad d\theta_i = -\sum_j \theta_{ij} \wedge \theta_j, \quad \theta_{ij} + \theta_{ji} = 0,$$

$$(2.2) \qquad d\theta_{ij} = -\sum_k \theta_{ik} \wedge \theta_{kj} + \Theta_{ij}.$$

Let $\tilde{J}$ be the natural complex structure of $H_n(\boldsymbol{C})$ and (J_{ij}, f_k) be the almost contact structure of M, i.e., $\tilde{J}(e_i) = \sum_j J_{ji} e_j + f_i e_{2n}$. Then (J_{ij}, f_k) satisfies

$$(2.3) \qquad \sum_k J_{ik} J_{kj} = f_i f_j - \delta_{ij}, \quad \sum_j f_j J_{ji} = 0,$$

$$\sum_i f_i^2 = 1, \quad J_{ij} + J_{ji} = 0.$$

The vector field $\sum_i f_i e_i$ is called the structure vector field of M.

Let ϕ_i be 1-forms of M such that $\sum_i \phi_i \theta_i$ is the second fundamental form of M for e_{2n}. Then the parallelism of $\tilde{J}$ implies

$$(2.4) \qquad dJ_{ij} = \sum_k (J_{ik}\theta_{kj} - J_{jk}\theta_{ki}) - f_i \phi_j + f_j \phi_i,$$

$$(2.5) \qquad df_i = \sum_k (f_k \theta_{ki} - J_{ki} \phi_k).$$

The equation of Gauss is given by

$$(2.6) \qquad \Theta_{ij} = \phi_i \wedge \phi_j + c\theta_i \wedge \theta_j + c\sum_{k,l} (J_{ik}J_{jl} + J_{ij}J_{kl})\theta_k \wedge \theta_l.$$

The equation of Codazzi is given by

$$(2.7) \qquad d\phi_i = -\sum_j \phi_j \wedge \theta_{ji} + c\sum_{j,k} (f_j J_{ik} + f_i J_{jk})\theta_j \wedge \theta_k.$$

3. Formulas

In this section we assume that all principal curvatures $x_1, \dots, x_{2n-1}$ (not necessarily distinct) of M for e_{2n} are constant. We may set $\phi_i = x_i\theta_i$. Then by (2.1) and (2.7) we can write the connection forms θ_{ij} in the form

$$(3.1) \qquad (x_i - x_j)\theta_{ij} = c\sum_k (A_{ijk} + f_iJ_{jk} + f_jJ_{ik})\theta_k$$

where $A_{ijk} = A_{jik} = A_{ikj}$ (cf. [5]). In particular, we have

$$(3.2) \qquad A_{ijk} = -f_iJ_{jk} - f_jJ_{ik} \quad \text{if } x_i = x_j,$$

$$(3.3) \qquad f_iJ_{jk} = 0 \quad \text{if } x_i = x_j = x_k.$$

We quote an important formula,

$$(3.4) \qquad 2c^2 \sum_k^{x_k \neq x_i} \frac{(A_{ijk} + f_kJ_{ij} + f_iJ_{kj})^2}{x_k - x_i}$$
$$- 2c^2 \sum_k^{x_k \neq x_j} \frac{(A_{ijk} + f_kJ_{ji} + f_jJ_{ki})^2}{x_k - x_j}$$
$$- 6c(x_i - x_j)J_{ij}^2 + 3c(x_if_j^2 - x_jf_i^2) - (x_i - x_j)(c + x_ix_j) = 0$$

(cf. [5]).

For an index i, we denote by $[i]$ the set of indices j with $x_i = x_j$. Then it is obvious that the vector $F_i = \sum_{j \in [i]} f_je_j$ is independent of the choice of orthonormal frame $\{e_j \| j \in [i]\}$ for the eigenspace belonging to x_i. Therefore for any index i we can indicate a special index i' so that the vector F_i linearly depends on $e_{i'}$. In other words, we can choose an orthonormal frame for the eigenspace belonging to x_i so that $f_j = 0$ for $j \in [i]\backslash\{i'\}$. In the same way, for J_{jk} (j is any index and fixed and k is the index that $x_k \neq x_{i'}$), we can indicate a special index k' and choose an orthonormal frame for the eigenspace belonging $x_{k'}$ so that $J_{jl} = 0$ for $l \in [k]\backslash\{k'\}$.

Hereafter we assume that dim $M \geq 5$ and that M has three distinct constant principal curvatures x, y, and z. Let $m(x), m(y)$ and $m(z)$ be the multiplicities of x, y and z respectively. We shall make use of the following convention on the range of indices:

$$1 \leq a, b, c \leq m(x), \quad m(x) + 1 \leq r, s, t \leq m(x) + m(y)$$
$$m(x) + m(y) + 1 \leq u, v, w \leq 2n - 1.$$

Now, we quote a Lemma.

LEMMA 3.1 ([5]). *If* $f_a f_r f_u \neq 0$ *then*

$$f_a \sum_r f_r J_{rb} - f_b \sum_r f_r J_{ra} = 0, \quad f_a \sum_u f_u J_{ub} - f_b \sum_u f_u J_{ua} = 0,$$

(3.5)
$$f_r \sum_a f_a J_{as} - f_s \sum_a f_a J_{ar} = 0, \quad f_r \sum_u f_u J_{us} - f_s \sum_u f_u J_{ur} = 0,$$

$$f_u \sum_a f_a J_{av} - f_v \sum_a f_a J_{au} = 0, \quad f_u \sum_r f_r J_{rv} - f_v \sum_r f_r J_{ru} = 0.$$

4. Proof of Main Theorem

It is sufficient to prove that two of f_a, f_r and f_u are 0. For this, first, suppose that $f_a f_r f_u \neq 0$. Then, from (3.3), $J_{ab} = J_{rs} = J_{uv} = 0$. We quote equations which are obtained by taking the exterior derivative of $J_{ab} = 0$, $J_{rs} = 0$ and $J_{uv} = 0$.

(4.1)
$$2c(y-z)\sum_u (f_a J_{bu} - f_b J_{au}) J_{uc} - (z-x)(x^2 - yx + 2c)(f_a \delta_{bc} - f_b \delta_{ac}) = 0,$$

(4.2)
$$2c(y-z)\sum_r (f_a J_{br} - f_b J_{ar}) J_{rc} - (x-y)(x^2 - zx + 2c)(f_a \delta_{bc} - f_b \delta_{ac}) = 0,$$

(4.3)
$$2c(z-x)\sum_a (f_r J_{sa} - f_s J_{ra}) J_{at} - (x-y)(y^2 - zy + 2c)(f_r \delta_{st} - f_s \delta_{rt}) = 0,$$

(4.4)
$$2c(z-x)\sum_u (f_r J_{su} - f_s J_{ru}) J_{ut} - (y-z)(y^2 - xy + 2c)(f_r \delta_{st} - f_s \delta_{rt}) = 0,$$

(4.5)
$$2c(x-y)\sum_r (f_u J_{vr} - f_v J_{ur}) J_{rw} - (y-z)(z^2 - xz + 2c)(f_u \delta_{vw} - f_v \delta_{uw}) = 0,$$

(4.6)
$$2c(x-y)\sum_a (f_u J_{va} - f_v J_{ua}) J_{aw} - (z-x)(z^2 - yz + 2c)(f_u \delta_{vw} - f_v \delta_{uw}) = 0,$$

$$2c(y-z)\left(\sum_r f_r^2 - \sum_{a,u} J_{au}^2\right) - (z-x)(x^2 - yx + 2c)(m(x)-1) = 0, \tag{4.7}$$

$$2c(y-z)\left(\sum_u f_u^2 - \sum_{a,r} J_{ar}^2\right) - (x-y)(x^2 - zx + 2c)(m(x)-1) = 0, \tag{4.8}$$

$$2c(z-x)\left(\sum_u f_u^2 - \sum_{a,r} J_{ar}^2\right) - (x-y)(y^2 - zy + 2c)(m(y)-1) = 0, \tag{4.9}$$

$$2c(z-x)\left(\sum_a f_a^2 - \sum_{r,u} J_{ru}^2\right) - (y-z)(y^2 - xy + 2c)(m(y)-1) = 0, \tag{4.10}$$

$$2c(x-y)\left(\sum_a f_a^2 - \sum_{r,u} J_{ru}^2\right) - (y-z)(z^2 - xz + 2c)(m(z)-1) = 0, \tag{4.11}$$

$$2c(x-y)\left(\sum_r f_r^2 - \sum_{a,u} J_{au}^2\right) - (z-x)(z^2 - yz + 2c)(m(z)-1) = 0, \tag{4.12}$$

(cf. [5]).

LEMMA 4.1. *If* $f_a f_r f_u \neq 0$ *then* $m(x)$, $m(y)$, $m(z) \geq 2$.

PROOF. At first, we assume $m(x) = 1$. Then from (4.7) and (4.8) we have

$$\sum_r f_r^2 = \sum_u J_{au}^2, \quad \sum_u f_u^2 = \sum_r J_{ar}^2, \tag{4.13}$$

which imply

$$\begin{aligned} m(y) &= \sum_r J_{ar}^2 + \sum_{r,u} J_{ru}^2 + \sum_r f_r^2 \\ &= \sum_u f_u^2 + \sum_{r,u} J_{ru}^2 + \sum_u J_{au}^2 \\ &= m(z). \end{aligned}$$

Hence $m(y) = m(z) \geq 2$ since $\dim M \geq 5$. Then from (4.9) and (4.12) we have $y^2 - zy + 2c = 0$ and $z^2 - yz + 2c = 0$, and so

$$z = -y, \quad y^2 = -c. \tag{4.14}$$

On the other hand, from (4.3), we have $(f_r J_{sa} - f_s J_{ra})J_{at} = 0$. Multiply the equation by J_{tu} and sum over t. Then, since $\sum_t J_{at}J_{tu} = f_a f_u \neq 0$, we have

$$f_r J_{sa} - f_s J_{ra} = 0. \tag{4.15}$$

Similarly from (4.6), we have

$$f_u J_{va} - f_v J_{ua} = 0. \tag{4.16}$$

Here we indicate a special index r' (resp. u') and choose an orthonormal frame $\{e_r\}$ (resp. $\{e_u\}$) for the eigenspace belonging y (resp. z) so that $f_s = 0$ if $s \neq r'$ (resp. $f_v = 0$ if $v \neq u'$). Then $f_{r'} f_{u'} \neq 0$. Put $r = r'$, $s' \neq r'$ in (4.15) and $u = u'$, $v \neq u'$ in (4.16) to get

$$J_{sa} = J_{va} = 0 \quad s \neq r', v \neq u'. \tag{4.17}$$

From (2.3) and (4.17) we have

$$0 = f_a J_{sa} = -f_{u'} J_{su'},$$

hence

$$J_{su'} = 0 \quad s \neq r'. \tag{4.18}$$

Similarly, we have

$$J_{r'v} = 0 \quad v \neq u'. \tag{4.19}$$

From (2.3) and (4.17), we have

$$\sum_u J_{su} J_{ua} = f_s f_a = 0, \quad \sum_u f_u J_{su} = 0 \tag{4.20}$$

$$\sum_u J_{su} J_{ur} = f_s f_r - \delta_{sr} = -\delta_{sr}$$

if $s \neq r'$. We shall take the exterior derivative of $J_{sa} = 0$ $(s \neq r')$. From (4.17), (4.18), (4.19) and (4.20), we have

$$-J_{ar'}\theta_{r's} + \sum_u J_{su}\theta_{ua} - J_{au'}\theta_{u's} + y f_a \theta_s = 0 \tag{4.21}$$

We shall take the exterior derivative of $f_s = 0$ $(s \neq r')$. From (4.17), (4.18), (4.19) and (4.20), we have

$$f_a \theta_{as} + f_{r'}\theta_{r's} + f_{u'}\theta_{u's} - \sum_u J_{us}\phi_u = 0 \tag{4.22}$$

Canceling $\theta_{r's}$ from (4.21) and (4.22), we get

$$J_{ar'}\left\{\frac{c}{x-y}f_a\sum_u(A_{asu}+f_aJ_{su})\theta_u\right.$$

$$+\frac{c}{z-y}f_{u'}\left(A_{asu'}\theta_a-2\sum_u f_{u'}J_{us}\theta_u\right)-z\sum_u J_{us}\theta_u\bigg\}$$

$$+f_{r'}\left\{\frac{c}{z-x}\sum_r\left(\sum_u J_{su}A_{aru}-f_a\delta_{sr}\right)\theta_r\right.$$

$$\left.-\frac{c}{z-y}\left(J_{au'}A_{asu'}\theta_a-2\sum_v f_{u'}J_{au'}J_{vs}\theta_v\right)+y\sum_r f_a\delta_{sr}\theta_r\right\}=0.$$

Taking account of the coefficients of θ_a, we have

$$(f_{u'}J_{ar'}-f_{r'}J_{au'})A_{asu'}=0.$$

Here we assert $f_{u'}J_{ar'}-f_{r'}J_{au'}\neq 0$. If not so, multiplying $(f_{u'}J_{ar'}-f_{r'}J_{au'})=0$ by $f_{u'}$, we have

$$0=f_{u'}^2J_{ar'}-f_{r'}f_{u'}J_{au'}=(f_{u'}^2+f_{r'}^2)J_{ar'}.$$

Hence

$$J_{ar'}=0.$$

But (2.3) implies that

$$0=J_{u'a}J_{ar'}=\sum_k J_{u'k}J_{kr'}=f_{u'}f_{r'},$$

which contradicts $f_{r'}f_{u'}\neq 0$. Hence $A_{asu'}=0$. Putting $i=s$ $(\neq r')$ and $j=u'$ in (3.4), we get

$$\frac{2c^2}{z-y}\sum_u f_{u'}^2J_{us}^2+3cyf_{u'}^2-(y-z)(c+yz)=0$$

by (3.2) and (4.18). From this equation, (4.14) and $\sum_u J_{us}^2=1$, we have

$$8c^2(f_{u'}^2-1)=0.$$

Hence $f_{u'}^2=1$, which contradicts (2.3) and $f_{r'}\neq 0$.

We can prove similarly in the case where $m(y)=1$ or $m(z)=1$. Q.E.D.

Now multiply (4.1) (resp. (4.2)) by J_{cr} (resp. J_{cu}) and sum over c. Then by Lemma 3.1 we have

$$(x^2 - yx + 2c)(f_a J_{br} - f_b J_{ar}) = 0. \tag{4.23}$$

Similarly from (4.3), we have

$$(z^2 - xz + 2c)(f_u J_{va} - f_v J_{ua}) = 0. \tag{4.24}$$

Since $x^2 - yx + 2c \neq 0$ or $x^2 - zx + 2c \neq 0$, we may assume $x^2 - yx + 2c \neq 0$. Then (4.2) and (4.23) imply $x^2 - zx + 2c = 0$. Hence $z^2 - xz + 2c \neq 0$. In fact, if $z^2 - xz + 2c = 0$, then $x = -z$ and it follows from (4.10) and (4.11) that $y^2 - xy + 2c = 0$. Hence $y^2 + zy + 2c = 0$. From (4.8), (4.9) and $x^2 - xz + 2c = 0$ we have $y^2 - zy + 2c = 0$. Then we have $yz = 0$, which contradicts $c \neq 0$. Hence (4.6) and (4.24) imply $z^2 - yz + 2c = 0$. Then (4.7) and (4.12) imply $x^2 - yx + 2c = 0$. From $x^2 - xz + 2c = 0$ we have $x = 0$, which contradicts $c \neq 0$. We can prove similarly if $x^2 - zx + 2c \neq 0$.

Owing to the above result, we may set $f_a = 0$.

Next, we prove $f_r f_u = 0$. For this, we suppose that $f_r f_u \neq 0$.

We need to consider three cases.

Case 1: $m(y), m(z) \geq 2$. Then $J_{rs} = J_{uv} = 0$. Here we indicate a special index r' (resp. u') and choose an orthonormal frame $\{e_r\}$ (resp. $\{e_u\}$) so that $f_s = 0$ if $s \neq r'$ (resp. $f_v = 0$ if $v \neq u'$). Then, from (2.3), we have

$$0 = \sum_i f_i J_{ir} = f_{u'} J_{u'r}.$$

Hence

$$J_{ru'} = 0. \tag{4.25}$$

Similarly

$$J_{r'u} = 0. \tag{4.26}$$

If $m(x) \geq 2$, then we choose an orthonormal frame $\{e_a\}$ so that $J_{1r'} \neq 0$, $J_{ar'} = 0$ if $a \neq 1$. Then, from (2.3), we have

$$0 = \sum_i J_{ai} J_{ir'} = J_{a1} J_{1r'}.$$

Hence $J_{1a} = 0$ for any a. Similarly we get

(4.27) $$J_{1s} = J_{1v} = 0 \quad \text{if } s \neq r', v \neq u',$$

(4.28) $$J_{au'} = 0 \quad \text{if } a \neq 1,$$

from (2.3). Taking the exterior derivative of $J_{r'u} = 0$ and $f_s = 0$ $(s \neq r')$, we have

$$J_{r'1}\theta_{1u} - \sum_a J_{ua}\theta_{ar'} - \sum_s J_{us}\theta_{sr'} - f_{r'}\phi_u + f_u\phi_{r'} = 0,$$

$$f_{r'}\theta_{r's} + f_{u'}\theta_{u's} - \sum_b J_{bs}\phi_b - \sum_v J_{vs}\phi_v = 0.$$

Canceling $\theta_{r's}$ from these equations, we get

$$\begin{aligned}
&\frac{c}{x-z} f_{r'}J_{r'1}\sum_k (A_{1uk} + f_uJ_{1k})\theta_k \\
&- \frac{c}{x-y} f_{r'}\sum_a J_{ua}\sum_k (A_{ar'k} + f_{r'}J_{ak})\theta_k \\
&+ \frac{c}{y-z} f_{u'}\sum_s J_{us}\sum_k (A_{su'k} + f_sJ_{u'k} + f_{u'}J_{sk})\theta_k \\
&+ \sum_{b,s} J_{bs}J_{us}\phi_b + \sum_{s,v} J_{vs}J_{us}\phi_v - f_{r'}\phi_u + f_u\phi_{r'} = 0.
\end{aligned}$$

Taking account of the coefficient of θ_t $(t \neq r')$ and using (2.3), (4.26), (4.25), (4.27) and (4.28), we have

$$J_{r'1}A_{1tu} = 0.$$

Hence

(4.29) $$A_{1su} = 0 \quad (s \neq r').$$

Similarly, from $dJ_{ru'} = 0$ and $df_v = 0$ $(v \neq u')$, we have

(4.30) $$A_{1rv} = 0 \quad (v \neq u').$$

Now put $i = 1$, $j = s$ $(s \neq r')$ in (3.4). Then, using (3.2), (4.27), (4.28) and (4.29), we have

$$-(x-y)(c+xy) = 0.$$

Hence

(4.31) $$c + xy = 0.$$

Moreover put $i = 1$, $j = v$ $(v \neq u')$ in (3.4). Then, using (3.2), (4.27), (4.28) and

(4.30), we get

$$c + xz = 0. \tag{4.32}$$

Canceling c from (4.31) and (4.32), we get $x = 0$, which contradicts $c \neq 0$.

If $m(x) = 1$, then we can get same equations (4.27), (4.29), (4.30), (4.31) and (4.32) and prove similarly.

Case 2: $m(y) = 1, m(z) \geq 2$. So we can indicate a special index u' and choose an orthonormal frame $\{e_u\}$ so that $f_v = 0$ if $v \neq u'$. Moreover from (2.3) we have

$$J_{ru} = 0. \tag{4.33}$$

Then $\sum_a J_{au}J_{av} = \delta_{uv} - f_u f_v$. This implies that there are $m(z)$ linearly independent $m(x)$-dimensional vectors. Hence, $m(x) \geq m(z) \geq 2$.

Let us take the exterior derivative of $f_a = 0$. Then, using (2.3), (2.4), (3.1) and (3.2), we have

$$\frac{c}{x-z}\sum_u f_u A_{aru} = -\left(\frac{3c}{x-y}f_r^2 + \frac{c}{x-z}\sum_v f_v^2 - y\right)J_{ar}, \tag{4.34}$$

$$\begin{aligned}\frac{c}{x-y}f_r A_{aru} = &-\left(\frac{c}{x-y}f_r^2 + \frac{2c}{x-z}\sum_v f_v^2 - z\right)J_{au} \\ &- \frac{c}{x-z}f_u \sum_v f_v J_{av}.\end{aligned} \tag{4.35}$$

Canceling A_{aru} from (4.34) and (4.35), we get

$$f_r J_{ar}\left\{\frac{3c(x-z)}{x-y}f_r^2 + \frac{3c(x-y)}{x-z}\sum_u f_u^2 - yx - zx + 2yz + c\right\} = 0 \tag{4.36}$$

since $f_r^2 + \sum_u f_u^2 = 1$. We assert $J_{ar} \neq 0$. In fact, we suppose that $J_{ar} = 0$. Then by (2.3) we have

$$0 = \sum_a J_{ar}^2 = 1 - f_r^2 = f_{u'}^2,$$

which contradicts $f_r f_u \neq 0$. Hence it follows from (4.36) and the relation $f_r^2 + \sum_u f_u^2 = 1$ that

$$\frac{3c(y-z)(2x-y-z)}{(x-y)(x-z)}f_r^2 = yx + zx - 2yz - c - \frac{3c(x-y)}{(x-z)}.$$

If $2x-y-z\neq 0$, then f_r^2 is constant. Taking account of the coefficient of θ_a in $df_r=0$, we have

$$(x-y)(x-z)+x(x-y)+x(x-z)-c=0. \tag{4.37}$$

This equation holds if $2x-y-z=0$. From (4.37) and (4.36), we get

$$c(x-z)^2 f_r^2+c(x-y)^2 f_{u'}^2+(x-y)^2(x-z)^2=0. \tag{4.38}$$

Now we choose an orthonormal frame $\{e_a\}$ so that $J_{1u'}\neq 0$, $J_{au'}=0$ if $a\neq 1$ and then, for a special index $v'\in[u]\backslash u'$, $J_{2v'}\neq 0$, $J_{av'}=0$ if $a\neq 1,2$. Then, from (2.3), we have

$$J_{1a}=J_{2a}=J_{1v'}=0 \tag{4.39}$$

$$J_{ar}=0 \quad \text{if } a\neq 1.$$

Put $a=1$ and $u=v'$ in (4.35) to get

$$A_{1rv'}=0. \tag{4.40}$$

Then putting $i=1$, $j=v'$ in (3.4), from (4.39) and (4.40), we have

$$-(x-z)(c+xz)=0$$

Hence

$$c+xz=0. \tag{4.41}$$

Taking account of the coefficient of θ_u in $dJ_{uv}=0$, we have $z^2-xz+2c=0$. From this and (4.41), we get $z^2=-3c$, $3x^2=-c$, $z=3x$. Then, from (4.37), We have $y=0$. On the other hand, puting $a=2$, $u=v'$ in (4.35), we have

$$cf_r A_{2rv'}=-2cf_{u'}^2 J_{2v'}. \tag{4.42}$$

Put $i=2$, $j=v'$ in (3.4). Then, from $z=3x$, $3x^2=-c$, $y=0$ and (2.3), we have

$$\frac{2c^2}{3xf_r^2}\{-3(-2f_{u'}^2+f_r^2)^2+6f_r^2f_{u'}^2+(-2f_{u'}^2-f_r^2)^2-6f_r^2\}=0.$$

Hence

$$-(2f_r^2-1)^2=0.$$

Then $f_r^2=f_{u'}^2=1/2$, which contradicts (4.38).

We can prove similarly for the case $m(y)\geq 2$, $m(z)=1$.

Case 3: $m(y) = m(z) = 1$. Then $m(x) \geq 3$. Moreover $J_{ru} = 0$. Hence $J_{ab} \neq 0$ since rank $J = 2n - 2$. Let us take the exterior derivative of $f_a = 0$, then, using (2.3), (2.4), (3.1) and (3.2), we have

$$\frac{c}{x-y} f_r^2 + \frac{c}{x-z} f_u^2 - x = 0, \tag{4.43}$$

$$\frac{c}{x-z} f_u A_{aru} = -\left(\frac{3c}{x-y} f_r^2 + \frac{c}{x-z} f_u^2 - y\right) J_{ar}, \tag{4.44}$$

$$\frac{c}{x-y} f_r A_{aru} = -\left(\frac{c}{x-y} f_r^2 + \frac{3c}{x-z} f_u^2 - z\right) J_{au}. \tag{4.45}$$

It follows from (4.43) and the relation $f_r^2 + f_u^2 = 1$ that f_r^2 is constant. Taking account of the coefficient of θ_a in $df_r = 0$, from (4.44) and (4.45) we have

$$(x-y)(x-z) + x(x-y) + x(x-z) - c = 0. \tag{4.46}$$

Now we choose an orthonormal frame $\{e_a\}$ so that $J_{1r} \neq 0$, $J_{ar} = 0$ if $a \neq 1$. Then, from (2.3), we have

$$J_{1r}^2 = f_u^2, \quad J_{1u}^2 = f_r^2, \tag{4.47}$$
$$J_{au} = 0, \quad J_{1a} = 0 \quad \text{if } a \neq 1.$$

Hence, from (4.44), we get

$$A_{aru} = 0 \quad \text{if } a \neq 1. \tag{4.48}$$

We put $i = 2$, $j = r$ in (3.4). Then, from (4.47) and (4.48), we have

$$-\frac{2c^2}{x-y} f_r^2 + 3cx f_r^2 - (x-y)(c+xy) = 0. \tag{4.49}$$

Similarly, putting $i = 2$, $j = u$ in (3.4), we get

$$-\frac{2c^2}{x-z} f_u^2 + 2cx f_u^2 - (x-z)(c+xz) = 0. \tag{4.50}$$

Canceling f_r^2 and f_u^2 from (4.49), (4.50) and (4.43), we have

$$(x-y)(x-z)(y+z-3x) = 0$$

by using (4.46). Hence we get

$$3x - y - z = 0. \tag{4.51}$$

And from (4.51), (4.46), (4.43) and $f_r^2 + f_u^2 = 1$, we get

$$c(y-z)f_r^2 = (x-y)^3, \quad c(z-y)f_u^2 = (x-z)^3. \tag{4.52}$$

On the other hand, from (4.44) and (4.49), we have

$$\frac{c}{x-z} f_u A_{1ru} = -\frac{1}{c}\{3cxf_r^2 - xy(x-y)\}J_{1r}. \tag{4.53}$$

Similarly, from (4.45) and (4.50), we have

$$\frac{c}{x-y} f_r A_{1ru} = -\frac{1}{c}\{3cxf_u^2 - xz(x-z)\}J_{1u}. \tag{4.54}$$

Then, from these equations, (4.46) and (4.51), we have

$$\begin{aligned}
&\left(\frac{c}{x-z} - \frac{c}{x-y}\right) f_r f_u A_{1ru} \\
&\quad = -\frac{1}{c}\{3cx - xy(x-y) - xz(x-z)\} f_r J_{1r} \\
&\quad = -x f_r J_{1r}.
\end{aligned}$$

Here, we may set $J_{1r} = f_u$ by (4.47). Then $J_{1u} = -f_r$, and $cA_{1ru} = x(x-y) \cdot (x-z)/(y-z)$. Moreover we obtain

$$A_{1ru} = \frac{(x-z)f_r^2 + (x-y)f_u^2}{y-z}. \tag{4.55}$$

since $x(x-y)(x-z) = c(x-z)f_r^2 + c(x-y)f_u^2$ by (4.43).

Let $a \neq 1$. We take the exterior derivative of $J_{ar} = 0$. Then, using (2.3), (2.4), (4.47) and (4.48), we have

$$f_u \theta_{1a} = \left(\frac{c}{x-y} - x\right) f_r \theta_a. \tag{4.56}$$

Similarly, from $dJ_{au} = 0$, we have

$$-f_r \theta_{1a} = \left(\frac{c}{x-z} - x\right) f_u \theta_a. \tag{4.57}$$

From above two equations and $f_r^2 + f_u^2 = 1$ we get

$$\theta_{1a} = \frac{c(y-z)}{(x-y)(x-z)} f_r f_u \theta_a. \tag{4.58}$$

Let us take the exterior derivative of (4.58). First, using (2.1) and (4.58), we have

$$\begin{aligned} d\theta_{1a} &= -\frac{c(y-z)}{(x-y)(x-z)} f_r f_u \left\{ \sum_b \theta_{ab}\wedge\theta_b + \theta_{ar}\wedge\theta_r + \theta_{au}\wedge\theta_u \right\} \\ &= -\frac{c(y-z)}{(x-y)(x-z)} f_r f_u \\ &\quad \times \left\{ \sum_b \theta_{ab}\wedge\theta_b + \frac{c}{(x-y)}\sum_b f_r J_{ab}\theta_b\wedge\theta_r + \frac{c}{(x-z)}\sum_b f_u J_{ab}\theta_b\wedge\theta_u \right\} \end{aligned}$$

because of (4.47) and (4.48). Next, using (2.6), we obtain

$$\begin{aligned} d\theta_{1a} &= -\sum_b \theta_{1b}\wedge\theta_{ba} + \theta_{1r}\wedge\theta_{ar} + \theta_{1u}\wedge\theta_{au} + \Theta_{1a} \\ &= -\frac{c(y-z)}{(x-y)(x-z)} f_r f_u \sum_b \theta_{ab}\wedge\theta_b \\ &\quad -\sum_b \left\{ \frac{3c^2}{(x-y)^2} f_r^2 + \frac{c^2}{(x-z)^2}(A_{1ru} - f_u^2) + c \right\} f_u J_{ab}\theta_b\wedge\theta_r \\ &\quad -\sum_b \left\{ \frac{c^2}{(x-y)^2}(A_{1ru} + f_r^2) - \frac{3c^2}{(x-z)^2} f_u^2 - c \right\} f_r J_{ab}\theta_b\wedge\theta_u \\ &\quad + (c+x^2)\theta_1\wedge\theta_a \end{aligned}$$

because of (4.47), (4.48) and (4.58). Hence

$$\begin{aligned} (4.59) \qquad & c\sum_b \left\{ \frac{c(3x-y-2z)}{(x-y)^2(x-z)} f_r^2 + \frac{c+(x-z)(y-z)}{(x-z)(y-z)} \right\} f_u J_{ab}\theta_b\wedge\theta_r \\ & + c\sum_b \left\{ -\frac{c(3x-2y-z)}{(x-y)(x-z)^2} f_u^2 + \frac{c-(x-y)(y-z)}{(x-y)(y-z)} \right\} f_r J_{ab}\theta_b\wedge\theta_u \\ & - (c+x^2)\theta_1\wedge\theta_a = 0. \end{aligned}$$

Taking account of the coefficient of $\theta_1\wedge\theta_a$ in (4.59), we have

$$(4.60) \qquad c + x^2 = 0.$$

We can get the same equation if $J_{1r} = -f_u$. From the (4.60), (4.46) and (4.51), we get $(y^2, z^2) = (-c, -4c)$, $(-4c, -c)$. Hence $x = -y$ or $x = -z$, which contradicts (4.46) and (4.51).

Owing to the above result, we get f_r or $f_u = 0$. Hence the proof of Main Theorem is complete.

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