TSUKUBA J. MATH.
Vol. 23 No. 2 (1999), 253–277

WELL-POSEDNESS OF CAUCHY PROBLEMS FOR LINEAR EVOLUTION OPERATORS WITH TIME DEPENDENT COEFFICIENTS

By

Reiko SAKAMOTO

Introduction

Let

$$A(t, D_t, D_x) = D_t^m + a_1(t, D_x)D_t^{m-1} + \cdots + a_m(t, D_x)$$

be a linear partial differential operator, where $a_j(t, \xi)$ is a $\mathscr{B}^\infty(0, T)$-function of t with a parameter $\xi (\in R^n)$ satisfying

$$\sup_{0<t<T}|D_t^r a_j(t, \xi)| \leqq C_r(1 + |\xi|)^{p(r)} \quad (r = 0, 1, 2, \ldots, \quad \xi \in R^n),$$

where $C_r > 0$ and $p(r) \geqq 0$, and consider the Cauchy problem (P):

$$\begin{cases} A(t, D_t, D_x)u = f(t, x) & \text{in } \{0 < t < T, x \in R^n\}, \\ D_t^j u = g_j(x) \ (j = 0, 1, \ldots, m-1) & \text{on } \{t = 0, x \in R^n\}. \end{cases}$$

Under what conditions is (P) well-posed in $H^\infty(R^n)$? The answer must be thought very easy, because the problem (P) can be reduced to the Cauchy problem of ordinary differential equations $(P^\wedge)$:

$$\begin{cases} A(t, D_t, \xi)u^\wedge = f^\wedge(t, \xi) & \text{in } \{0 < t < T\}, \\ D_t^j u^\wedge = g_j^\wedge(\xi) \ (j = 0, 1, \ldots, m-1) & \text{on } \{t = 0\}, \end{cases}$$

where $f^\wedge$ is the Fourier transform of f with respect to x. But it is not so easy, because we have not explicit solutions in general. Of course, if $a_j(t, \xi)$ is independent of t, it is well known that (P) is well-posed in $H^\infty(R^n)$ iff there exists $C > 0$ such that

$$\operatorname{Im} \tau_j(\xi) \geqq -C \log|\xi| \quad (|\xi| \geqq 2) \ (j = 1, 2, \ldots, m)$$

Received May 11, 1998.
Revised December 25, 1998.

holds, where

$$A(\tau,\xi)=(\tau-\tau_1(\xi))\cdots(\tau-\tau_m(\xi)).$$

On the other hand, if $a_j(t,\xi)$ depends on t, the structure of well-posedness is not so clear, although we have many sufficient conditions for well-posedness, e.g. parabolic, hyperbolic, Schrödinger, etc ...

Recently, Y. Shiozaki [1] considered the Cauchy problem:

$$\begin{cases} \{\partial_t^2+t^k|D_x|^m+\alpha t^\ell|D_x|^n\}u=0 & \text{in } \{0<t<T, x\in R^n\}, \\ \partial_t^j u=g_j(x) \ (j=0,1) & \text{on } \{t=0, x\in R^n\}, \end{cases}$$

where $2\geqq m>n>0$, $k\geqq 0$, $\ell\geqq 0$, $\alpha\in C$, and he obtained the following results.

I. In case when $\operatorname{Im}\alpha\neq 0$, it is well-posed iff

$$(*)\qquad n\leqq m/2$$

and

$$(**)\qquad (\ell+2)/n\geqq(k+2)/m.$$

II. In case when $\operatorname{Im}\alpha=0$ and $\alpha\geqq 0$, it is well-posed.

III. In case when $\operatorname{Im}\alpha=0$ and $\alpha<0$, it is well-posed iff

$$(**)\qquad (\ell+2)/n\geqq(k+2)/m.$$

Similarly, S. Tarama [2] considered the Cauchy problem:

$$\begin{cases} \{\partial_t^2-\exp(-2t^{-h})\partial_x^2-i\alpha t^\ell\exp(-t^{-h})\partial_x\}u=f(t,x) & \text{in } \{0<t<T, x\in R^n\}, \\ \partial_t^j u=g_j(x) \ (j=0,1) & \text{on } \{t=0, x\in R^n\}, \end{cases}$$

where $h>0$, $\ell\in R$, $\alpha\in C$, and he obtained the following results.

I. In case when $\operatorname{Im}\alpha\neq 0$, it is well-posed iff

$$(☆)\qquad \ell\geqq -h-1.$$

II. In case when $\operatorname{Im}\alpha=0$ and $\alpha\neq 0$, it is well-posed iff

$$(☆☆)\qquad \ell\geqq -2h-1.$$

Their interest is concentrated on weakly hyperbolic equations. But similar situations are widely seen in the field of evolution equations. There are many studies on Schrödinger equations with variable coefficients, which depend on x (: space variable), but do not depend on t (: time variable) ([3], [4], ..). This paper is an attempt to study evolution equations with coefficients depending on t, by generalizing Tarama-Shiozaki's method.

§ 1. Assumptions and Results

Let us state assumptions on

$$A(t,\tau,\xi)=\tau^m+a_1(t,\xi)\tau^{m-1}+\cdots+a_m(t,\xi)$$

in the followings. The interval $(0,T)$ is split into a finite number of sub-intervals $I_h=(T_{h-1}(\xi),T_h(\xi))$ for $|\xi|>K_0$ $(h=0,1,\dots,N,K_0>0)$, where

$$0=T_{-1}<T_0(\xi)<T_1(\xi)<\cdots<T_{N-1}(\xi)<T_N=T$$

such that $A(t,\tau,\xi)$ satisfies Condition(A) in I_0 and $A(t,\tau,\xi)$ satisfies Condition(B) in I_h $(h=1,\dots,N)$. We say that $A(t,\tau,\xi)$ satisfies Condition(A) in I_0 iff it holds

(A) $$t^j|a_j(t,\xi)|\leqq C\ \ (j=1,\dots,m,\ \ (t,\xi)\in I_0^{\sim}),$$

where we use the notation

$$I^{\sim}=\{(t,\xi)\,|\,t\in I,|\xi|>K_0\}.$$

We say that $A(t,\tau,\xi)$ satisfies Condition(B) in an interval $I=(a(\xi),b(\xi))$ iff it holds that there exists a weight function $\phi(t,\xi)$ $(>c>0)$ defined in $I^{\sim}$ such that

(Φ) $$\int_{a(\xi)}^{b(\xi)}\phi(t,\xi)dt\leqq K_1\log|\xi|$$

and

$$\begin{aligned}A(t,\tau,\xi)&=P(t,\tau,\xi)+Q(t,\tau,\xi),\\ P(t,\tau,\xi)&=\tau^m+p_1(t,\xi)\tau^{m-1}+\cdots+p_m(t,\xi)\\ &=(\tau-\tau_1(t,\xi))\cdots(\tau-\tau_m(t,\xi)),\\ Q(t,\tau,\xi)&=q_1(t,\xi)\tau^{m-1}+q_2(t,\xi)\tau^{m-2}+\cdots+q_m(t,\xi),\end{aligned}$$

where P satisfies (B-1), (B-2), $(\text{B-3})_\phi$, $(\text{B-4})_\phi$, and Q satisfies $(\text{B-5})_\phi$ as follows.

(B-1) $\{\tau_j(t,\xi)\,|\,j=1,\dots,m\}$ are distinct for $(t,\xi)\in I^{\sim}$ and satisfy

$$|\tau_1(t,\xi)|\geqq|\tau_2(t,\xi)|\geqq\cdots\geqq|\tau_m(t,\xi)|,$$

(B-2) $$|\tau_j(t,\xi)|/|\tau_j(t,\xi)-\tau_k(t,\xi)|\leqq C\quad(j<k,\ j,k=1,\dots,m,\ (t,\xi)\in I^{\sim}),$$

$(\text{B-3})_\phi$ $$\operatorname{Im}\tau_j(t,\xi)\geqq-C\phi(t,\xi)\quad(j=1,\dots,m,\ (t,\xi)\in I^{\sim}),$$

$(\text{B-4})_\phi$ $$|D_t\tau_j(t,\xi)|/|\tau_j(t,\xi)|\leqq C\phi(t,\xi)\quad(j=1,\dots,m,\ (t,\xi)\in I^{\sim}),$$

(B-5)$_\phi$ $$|q_j(t,\xi)| \leqq C\phi(t,\xi)|\tau_1(t,\xi)|\,|\tau_2(t,\xi)|\cdots|\tau_{j-1}(t,\xi)|$$
$$(j=2,3,\dots,m,\ \ (t,\xi)\in I^{\sim}),$$
$$|q_1(t,\xi)| \leqq C\phi(t,\xi), \quad (t,\xi)\in I^{\sim}.$$

REMARK. ϕ in (B-3)$_\phi$, (B-4)$_\phi$, (B-5)$_\phi$ may be replaced by ϕ_1, ϕ_2, ϕ_3 satisfying (Φ). In fact, if (B-3)$_{\phi_1}$, (B-4)$_{\phi_2}$, (B-5)$_{\phi_3}$ are satisfied, then (B-3)$_\phi$, (B-4)$_\phi$, (B-5)$_\phi$ are satisfied, where
$$\phi = \max(\phi_1, \phi_2, \phi_3).$$

THEOREM. *For*
$$A(t,\tau,\xi) = \tau^m + a_1(t,\xi)\tau^{m-1} + \cdots + a_m(t,\xi),$$
the interval $(0,T)$ *is assumed to be split into a finite number of sub-intervals* $I_h = (T_{h-1}(\xi), T_h(\xi))$ $(h=0,1,\dots,N)$ *for* $|\xi| > K_0$ $(K_0 > 0)$, *where*
$$0 = T_{-1} < T_0(\xi) < T_1(\xi) < \cdots < T_{N-1}(\xi) < T_N = T,$$
such that $A(t,\tau,\xi)$ *satisfies Condition*(A) *in* I_0 *and* $A(t,\tau,\xi)$ *satisfies Condition*(B) *in* I_h $(h=1,\dots,N)$. *Then the Cauchy problem*(P):
$$\begin{cases} A(t,D_t,D_x)u = f(t,x) & \text{in } \{0<t<T, x\in R^n\} \\ D_t^j u = g_j(x)\ \ (j=0,1,\dots,m-1) & \text{on } \{t=0, x\in R^n\} \end{cases}$$
is well-posed in H^∞. *Namely, for any data*
$$f \in H^\infty((0,T)\times R^n), \quad g_j \in H^\infty(R^n) \quad (j=0,1,\dots,m-1),$$
there exists a unique solution
$$u \in H^\infty((0,T)\times R^n)$$
satisfying (P).

This theorem will be proved in §3–§6.

§2. Examples

EXAMPLE 1. Let us consider Shiozaki's operator
$$A = \tau^2 - t^k|\xi|^m - \alpha t^\ell|\xi|^n \quad (m > n \geqq 0, k \geqq 0, \ell \geqq 0),$$
and see the relations between $\{(*), (**)\}$ and {Condition(A), Condition(B)}.

Since

$$\text{(A)} \qquad t^{k+2}|\xi|^m \leqq C, \quad t^{\ell+2}|\xi|^n \leqq C,$$

is equivalent to

$$\text{(A)} \qquad t^K|\xi| \leqq C, \quad K = \min\{(k+2)/m, (\ell+2)/n\},$$

Condition(A) is satisfied in $(0, T_0(\xi))$ with

$$T_0(\xi) = \delta|\xi|^{-1/K},$$

where δ is any positive constant. Now, let us consider Condition(B) in $(T_0(\xi), T)$ in the following cases I)–III).

I) (The case when $\operatorname{Im}\alpha \neq 0$.) Set

$$P = \tau^2 - t^k|\xi|^m, \quad Q = -\alpha t^\ell|\xi|^n.$$

Since

$$P = (\tau - \tau_1)(\tau - \tau_2), \quad \tau_1 = t^{k/2}|\xi|^{m/2}, \quad \tau_2 = -t^{k/2}|\xi|^{m/2},$$

we have

$$\tau_1 \neq \tau_2, \quad |\tau_1| = |\tau_2|, \quad |\tau_j|/|\tau_1 - \tau_2| = 1/2, \quad \operatorname{Im}\tau_j = 0,$$

therefore, (B-1), (B-2), $(\text{B-3})_0$ are satisfied. Moreover, since

$$|D_t\tau_j|/|\tau_j| = (k/2)t^{-1},$$

$(\text{B-4})_\phi$ is satisfied with $\phi(t) = t^{-1}$, where we remark

$$\int_{T_0(\xi)}^T t^{-1}dt = \log T - \log\delta + K^{-1}\log|\xi|.$$

On the other hand, $(\text{B-5})_\phi$ with $\phi(t) = t^{-1}$ is that

$$|Q|/|\tau_j| = |\alpha|t^{\ell-k/2}|\xi|^{n-m/2} \leqq Ct^{-1}$$

holds for any $t \in (T_0(\xi), T) = (\delta|\xi|^{-1/k}, T)$, that is,

$$t^{\ell-k/2+1}|\xi|^{n-m/2} = t^{\ell-k/2+1+(m/2-n)K}(t^K|\xi|)^{-(m/2-n)} \leqq C$$

holds for any $t \in (T_0(\xi), T) = (\delta|\xi|^{-1/K}, T)$. For this, it is neccesary and sufficient that

$$m/2 - n \geqq 0, \quad (\ell - k/2 + 1) + (m/2 - n)K \geqq 0,$$

that is,

$$(*) \qquad n \leqq m/2,$$

$$(**) \qquad (\ell+2)/n \geqq (k+2)/m.$$

II) (The case when $\operatorname{Im}\alpha = 0$, $\alpha > 0$.) Set

$$P = A = \tau^2 - t^k|\xi|^m - \alpha t^\ell |\xi|^n = (\tau - \tau_1)(\tau - \tau_2), \quad Q = 0.$$

(B-1), (B-2), $(\text{B-3})_0$ are satisfied, because

$$\tau_1 \neq \tau_2, \quad |\tau_1| = |\tau_2|, \quad |\tau_j|/|\tau_1 - \tau_2| = 1/2, \quad \operatorname{Im}\tau_j = 0$$

for $0 < t < T$. Moreover, $(\text{B-4})_\phi$ with $\phi(t) = t^{-1}$ is satisfied, because

$$|D_t\tau_j|/|\tau_j| \leqq ((k+\ell)/2)t^{-1}.$$

III) (The case when $\operatorname{Im}\alpha = 0$, $\alpha < 0$.) Set $P = A$ and $Q = 0$. Let us remark that

$$2(|\alpha| t^\ell |\xi|^n) < t^k |\xi|^m$$

holds iff

$$t^{K'}|\xi| > (2|\alpha|)^{1/(m-n)}, \quad K' = (k-\ell)/(m-n)$$

holds. Therefore, set

$$T_*(\xi) = (2|\alpha|)^{1/(k-\ell)}|\xi|^{-1/K'},$$

then we have

$$\tau_1 \neq \tau_2, \quad |\tau_1| = |\tau_2|, \quad |\tau_j|/|\tau_1 - \tau_2| = 1/2, \quad \operatorname{Im}\tau_j = 0, \quad |D_t\tau_j|/|\tau_j| \leqq 2kt^{-1}$$

in $(T_*(\xi), T)$, which means that (B-1), (B-2), $(\text{B-3})_0$, $(\text{B-4})_\phi$ with $\phi(t) = t^{-1}$ are satisfied in $(T_*(\xi), T)$. Set

$$\delta = (2|\alpha|)^{1/(k-\ell)},$$

that is, set

$$T_0(\xi) = \delta|\xi|^{-1/K} = (2|\alpha|)^{1/(k-\ell)}|\xi|^{-1/K},$$

then we have $T_*(\xi) \leqq T_0(\xi)$, if

$$K = \min((k+2)/m, (\ell+2)) \leqq (k-\ell)/(m-n) = K',$$

that is, if

$$(**) \qquad (\ell+2)/n \geqq (k+2)/m.$$

Hence (B-1), (B-2), $(B\text{-}3)_0$, $(B\text{-}4)_\phi$ with $\phi(t) = t^{-1}$ are satisfied in $(T_0(\xi), T)$, if $(**)$ holds.

Our Theorem maintains the followings.

I. In case when $\operatorname{Im}\alpha \neq 0$, it is well-posed if

$$(*) \qquad n \leqq m/2$$

and

$$(**) \qquad (\ell + 2)/n \geqq (k+2)/m.$$

II. In case when $\operatorname{Im}\alpha = 0$ and $\alpha \geqq 0$, it is well-posed.

III. In case when $\operatorname{Im}\alpha = 0$ and $\alpha < 0$, it is well-posed if

$$(**) \qquad (\ell + 2)/n \geqq (k+2)/m.$$

EXAMPLE 2. Let us consider a version of Tarama's operator

$$A = \tau^2 - t^k \exp(-mt^{-h})|\xi|^m - \alpha t^\ell \exp(-nt^{-h})|\xi|^n \quad (m > n > 0,\ h > 0,\ k, l \in \boldsymbol{R}).$$

If $m = 2$, $n = 1$, $k = 0$, then A is nearly equal to Tarama's operator.

Remark that

$$(A) \qquad t^{k+2} \exp(-mt^{-h})|\xi|^m \leqq C, \quad t^{\ell+2} \exp(-nt^{-h})|\xi|^n \leqq C,$$

is equivalent to

$$(A) \qquad t^K \exp(-t^{-h}) \leqq \delta|\xi|^{-1}, \quad K = \min((k+2)/m, (\ell+2)/n).$$

Since $t^K \exp(-t^{-h})$ is strictly incresing in $(0, T)$ (T : small), there exists unique $t = T_0(\xi)$ such that

$$t^K \exp(-t^{-h}) = \delta|\xi|^{-1},$$

satisfying

$$(1/2)\log|\xi| \leqq T_0(\xi)^{-h} \leqq 2\log|\xi|$$

for $|\xi| > K_0$ (K_0 : large). Then, Condition(A) is satisfied in $(0, T_0(\xi))$. Now we consider of Condition(B) in $(T_0(\xi), T)$ in the following cases I)–III).

I) (The case when $\operatorname{Im}\alpha \neq 0$.) Set

$$P = \tau^2 - t^k \exp(-mt^{-h})|\xi|^m, \quad Q = -\alpha t^\ell \exp(-nt^{-h})|\xi|^n,$$

then we have

$$P = (\tau - \tau_1)(\tau - \tau_2),$$

$$\tau_1 = t^{k/2}\exp(-(m/2)t^{-h})|\xi|^{m/2},$$

$$\tau_2 = -t^{k/2}\exp(-(m/2)t^{-h})|\xi|^{m/2},$$

and

$$\tau_1 \neq \tau_2, \quad |\tau_1| = |\tau_2|, \quad |\tau_j|/|\tau_1 - \tau_2| = 1/2, \quad \operatorname{Im}\tau_j = 0,$$

therefore, (B-1), (B-2), $(\text{B-3})_0$ are satisfied. Moreover, since

$$|D_t\tau_j|/|\tau_j| = (k/2)t^{-1} + (m/2)ht^{-h-1},$$

$(\text{B-4})_\phi$ is satisfied with $\phi(t) = t^{-h-1}$ in $(T_0(\xi), T)$, where we remark that $\phi(t) = t^{-h-1}$ is a weight functin satisfying (Φ) throughout in $(T_0(\xi), T)$, because

$$\int_{T_0(\xi)}^{T} t^{-h-1}dt = h^{-1}(T_0(\xi)^{-h} - T^{-h}) \leqq 2h^{-1}\log|\xi|.$$

On the other hand, $(\text{B-5})_\phi$ with $\phi(t) = t^{-h-1}$ in $(T_0(\xi), T)$ is that

$$\begin{aligned}|Q|/|\tau_j| &= |\alpha|t^{\ell-k/2}(\exp(-t^{-h})|\xi|)^{-(m/2-n)}\\ &= |\alpha|t^{\ell-k/2+(m/2-n)K}(t^K\exp(-t^{-h})|\xi|)^{-(m/2-n)}\\ &\leqq Ct^{-h-1}\end{aligned}$$

holds for any $t \in (T_0(\xi), T)$. Therefore, $(\text{B-5})_\phi$ with $\phi(t) = t^{-h-1}$ in $(T_0(\xi), T)$ is satisfied if

$$m/2 - n \geqq 0, \quad (\ell - k/2 + h + 1) + (m/2 - n)K \geqq 0,$$

that is, if

$$(*) \qquad n \leqq m/2,$$

$$(**)' \qquad \{(\ell + 2)/n - (k + 2)/m\} + 2h/m \geqq 0.$$

II) (The case when $\operatorname{Im}\alpha = 0$, $\alpha > 0$.) Set

$$P = A = \tau^2 - t^k\exp(-mt^{-h})|\xi|^m - \alpha t^\ell\exp(-nt^{-h})|\xi|^n, \quad Q = 0,$$

then Condition(B) is satisfied with $\phi(t) = t^{-h-1}$ in $(T_0(\xi), T)$.

III) (The case when $\operatorname{Im}\alpha = 0$, $\alpha < 0$.) Set

$$F = t^k\exp(-mt^{-h})|\xi|^m, \quad G = \alpha t^\ell\exp(-nt^{-h})|\xi|^n,$$

then $|G| < F$ holds iff

$$t^{K'} \exp(-t^{-h})|\xi| > |\alpha|^{1/(m-n)}, \quad K' = (k-\ell)/(m-n)$$

holds, that is, iff $T_1(\xi) < t < T$, where $t = T_1(\xi)$ is defined by

$$t^{K'} \exp(-t^{-h})|\xi| = |\alpha|^{1/(m-n)}.$$

In the same way, $2|G| < F$ holds iff $T_2(\xi) < t < T$, where $t = T_2(\xi)$ is defined by

$$t^{K'} \exp(-t^{-h})|\xi| = (2|\alpha|)^{1/(m-n)}.$$

Moreover, from the definition of $\{T_j(\xi)\}_{j=0,1,2}$, we have

$$T_1(\xi) < T_2(\xi), \quad (1/2)\log|\xi| \leqq T_j(\xi)^{-h} \leqq 2\log|\xi| \quad (|\xi| : \text{large}).$$

III-a) When $T_2(\xi) < t < T$, set

$$P = \tau^2 - F - G, \quad Q = 0,$$

then we have

$$P = (\tau - \tau_1)(\tau - \tau_2), \quad \tau_1 = (F+G)^{1/2}, \quad \tau_2 = -(F+G)^{1/2}.$$

Since $F > 2|G|$, (B-1), (B-2), $(\text{B-3})_0$ are satisfied. Moreover, $(\text{B-4})_\phi$ is satisfied with $\phi(t) = t^{-h-1}$, because

$$|D_t\tau_j/\tau_j| = |D_t(F+G)^{1/2}/(F+G)^{1/2}| = (1/2)|D_t(F+G)/(F+G)|$$
$$\leqq |D_t(F+G)|/F \leqq Ct^{-h-1}.$$

Here we remark that $K \geqq K'$ holds iff

$$(**) \qquad (\ell+2)/n \geqq (k+2)/m$$

holds. Therefore, if $(**)$ is satisfied, set

$$\delta = (2|\alpha|)^{1/(m-n)},$$

then we have

$$T_2(\xi) \leqq T_0(\xi) \quad (|\xi| : \text{large}).$$

Hence, Condition(B) is satisfied in $(T_0(\xi), T)$, if $(**)$ is satisfied. If $(**)$ is not valid, that is, if $K < K'$, then we have

$$T_0(\xi) < T_1(\xi) < T_2(\xi) \quad (|\xi| : \text{large}),$$

whose case is considerd in the following III-b) and III-c).

III-b) When $T_1(\xi) < t < T_2(\xi)$, set

$$P = \tau^2 - F, \quad Q = -G,$$

then we have

$$P = (\tau - \tau_1)(\tau - \tau_2), \quad \tau_1 = F^{1/2}, \quad \tau_2 = -F^{1/2},$$

therefore we have (B-1), (B-2), $(\text{B-3})_0$, $(\text{B-4})_\phi$ with $\phi = t^{-h-1}$. On the other hand, since

$$|Q/\tau_j| = |G|F^{-1/2} \leqq F^{1/2},$$

$(\text{B-5})_\psi$ is satisfied with $\psi = F^{1/2}$. Since

$$\begin{aligned}\int_{T_1}^{T_2} F^{1/2}\,dt &= \int_{T_1}^{T_2} t^{k/2}\exp(-(m/2)t^{-h})\,dt|\xi|^{m/2} \\ &\leqq C\max_{j=1,2}\{T_j^{k/2+h+1}\exp(-(m/2)T_j^{-h})\}|\xi|^{m/2} \\ &\leqq C'\max_{j=1,2}\{T_j^{k/2+h+1-K'm/2}\} \\ &\leqq C''(\log|\xi|)^{-\{k/2+h+1-K'm/2\}/h},\end{aligned}$$

$\psi = F^{1/2}$ satisfies (Φ) if

$$-(k/2 + h + 1 - K'm/2)/h \leqq 1,$$

that is, if

$$(**)'' \qquad \{(\ell + 2)/n - (k + 2)/m\} + 4h(1/n - 1/m) \geqq 0.$$

III-c) When $T_0(\xi) < t < T_1(\xi)$, set

$$P = \tau^2 - G = \tau^2 + |G|, \quad Q = -F,$$

then we have

$$P = (\tau - \tau_1)(\tau - \tau_2), \quad \tau_1 = i|G|^{1/2}, \quad \tau_2 = -i|G|^{1/2},$$

therefore

$$\tau_1 \neq \tau_2, \quad |\tau_1| = |\tau_2|, \quad |\tau_j|/|\tau_1 - \tau_2| = 1/2$$

and (B-1), (B-2) are satisfied. Moreover, since

$$|D_t\tau_j/\tau_j| = |D_t|G|^{1/2}/|G|^{1/2}| = (1/2)|D_tG/G| \leqq Ct^{-h-1},$$

$(\text{B-4})_\phi$ is saisfied with $\phi = t^{-h-1}$. On the other hand, since

$$|\operatorname{Im} \tau_j| = |G|^{1/2},$$

$$|Q/\tau_j| = F/|G| \leqq |G|^{1/2},$$

$(\text{B-3})_\psi$, $(\text{B-5})_\psi$ are satisfied with $\psi = |G|^{1/2}$. Since

$$\begin{aligned}\int_{T_0}^{T_1} |G|^{1/2}\, dt &= \int_{T_0}^{T_1} t^{\ell/2} \exp(-(n/2)t^{-h})\, dt|\xi|^{n/2} \\ &\leqq C \max_{j=0,1}\{T_j^{\ell/2+h+1} \exp(-(n/2)T_j^{-h})\}|\xi|^{n/2} \\ &\leqq C' \max\{T_0^{\ell/2+h+1-Kn/2}, T_1^{\ell/2+h+1-K'n/2}\} \\ &\leqq C''(\log|\xi|)^{-\{\ell/2+h+1-K'n/2\}/h},\end{aligned}$$

$\psi = |G|^{1/2}$ satisfies (Φ) if

$$-(\ell/2 + h + 1 - K'n/2)/h \leqq 1,$$

that is, if

$$K' \leqq (\ell/2 + 2h + 1)/(n/2),$$

that is, if

$$(**)'' \qquad \{(\ell + 2)/n - (k + 2)/m\} + 4h(1/n - 1/m) \geqq 0.$$

From III-a), III-b) and III-c), Condition(B) is satisfied in $(T_0(\xi), T)$ if $(**)''$ holds.

Our Theorem maintains the followings.

I. In case when $\operatorname{Im} \alpha \neq 0$, it is well-posed if

$$(*) \qquad n \leqq m/2$$

and

$$(**)' \qquad \{(\ell + 2)/n - (k + 2)/m\} + 2h/m \geqq 0.$$

II. In case when $\operatorname{Im} \alpha = 0$ and $\alpha \geqq 0$, it is well-posed.

III. In case when $\operatorname{Im} \alpha = 0$ and $\alpha < 0$, it is well-posed if

$$(**)'' \qquad \{(\ell + 2)/n - (k + 2)/m\} + 4h(1/n - 1/m) \geqq 0.$$

Remark. Let $m = 2$, $n = 1$, $k = 0$ in Example 2, then $(*)$ is trivial, $(**)'$ becomes (☆) and $(**)''$ becomes (☆☆).

§ 3. Standard Systems of Equations

Let us consider a system of equations (S):

$$(D_t - \tau_j(t,\xi))u_j + \Sigma_{1 \leqq k \leqq m} \beta_{jk}(t,\xi)u_k = f_j \quad (j = 1, 2, \dots, m).$$

Let us say that (S) is (A)-standard in I_0 iff it holds that

$$\operatorname{Im} \tau_j(t,\xi) \geqq -Ct^{-1} \quad (j = 1, \dots, m, \ (t,\xi) \in I_0^{\sim}),$$

$$|\beta_{jk}(t,\xi)| \leqq Ct^{-1} \quad (j,k = 1, \dots, m, \ (t,\xi) \in I_0^{\sim}).$$

Let us say that (S) is (B)-standard in $I = (a(\xi), b(\xi))$ iff it holds that

$$\operatorname{Im} \tau_j(t,\xi) \geqq -C\phi(t,\xi) \quad (j = 1, \dots, m, \ (t,\xi) \in I^{\sim}),$$

$$|\beta_{jk}(t,\xi)| \leqq C\phi(t,\xi) \quad (j,k = 1, \dots, m, \ (t,\xi) \in I^{\sim}),$$

where ϕ is a weight function satisfying (Φ).

LEMMA 3.1. *Assume that $\{u_j\}$ satisfy a (A)-standard system of equations*

$$(D_t - \tau_j(t,\xi))u_j + \Sigma_{1 \leqq k \leqq m} \beta_{jk}(t,\xi)u_k = f_j \quad (j = 1, 2, \dots, m)$$

in $I_0^{\sim}$, then there exists $c > 0$ such that

$$\Sigma_{1 \leqq j \leqq m} |u_j(t_2)|^2 \leqq t_2^c \left\{ t_1^{-c} \Sigma_{1 \leqq j \leqq m} |u_j(t_1)|^2 + \int_{t_1}^{t_2} s^{-c} \Sigma_{1 \leqq j \leqq m} |f_j(s)|^2 \, ds \right\},$$

$$(t_1,\xi), (t_2,\xi) \in I_0^{\sim} \ (0 < t_1 < t_2).$$

PROOF. Since

$$f_j\bar{u}_j - u_j\bar{f}_j = \{(D_t - \tau_j)u_j\bar{u}_j - u_j\overline{(D_t - \tau_j)u_j}\} + \{\Sigma_k \beta_{jk} u_k \bar{u}_j - u_j \overline{\Sigma_k \beta_{jk} u_k}\}$$

$$= -i\{\partial_t |u_j|^2 + 2 \operatorname{Im} \tau_j |u_j|^2\} + \Sigma_k \{\beta_{jk} u_k \bar{u}_j - \overline{\beta_{jk}} u_j \bar{u}_k\},$$

and since

$$\operatorname{Im} \tau_j(t,\xi) \geqq -Ct^{-1} \quad (j = 1, \dots, m),$$

$$|\beta_{jk}(t,\xi)| \leqq Ct^{-1} \quad (j,k = 1, \dots, m),$$

we have

$$\partial_t \Sigma_j |u_j|^2 \leqq C_1 t^{-1} \Sigma_j |u_j|^2 + \Sigma_j |f_j|^2,$$

that is,

$$\partial_t E \leqq C_1 t^{-1} E + F,$$

where

$$E = \Sigma_j |u_j|^2, \quad F = \Sigma_j |f_j|^2.$$

Hence we have

$$\partial_t \{t^{-c} E\} \leqq t^{-c} F \quad (c = C_1),$$

that is,

$$E(t_2) \leqq t_2^c \left\{ t_1^{-c} E(t_1) + \int_{t_1}^{t_2} s^{-c} F(s)\, ds \right\}. \qquad \square$$

LEMMA 3.2. *Assume that* $\{u_j\}$ *satisfy a* (B)*-standard system of equations*

$$(D_t - \tau_j(t,\xi))u_j + \Sigma_{1 \leqq k \leqq m} \beta_{jk}(t,\xi) u_k = f_j \quad (j = 1, 2, \dots, m)$$

in $I = (a(\xi), b(\xi))$, *then there exists* $c > 0$ *such that*

$$\Sigma_{1 \leqq j \leqq m} |u_j(t_2)|^2 \leqq |\xi|^c \left\{ \Sigma_{1 \leqq j \leqq m} |u_j(t_1)|^2 + \int_{t_1}^{t_2} \Sigma_{1 \leqq j \leqq m} |f_j(s)|^2\, ds \right\},$$

$$(t_1, \xi), (t_2, \xi) \in I^{\sim} (0 < t_1 < t_2).$$

PROOF. Since

$$\begin{aligned} f_j \bar{u}_j - u_j \bar{f}_j &= \{(D_t - \tau_j) u_j \bar{u}_j - u_j \overline{(D_t - \tau_j) u_j}\} + \{\Sigma_k \beta_{jk} u_k \bar{u}_j - u_j \overline{\Sigma_k \beta_{jk} u_k}\} \\ &= -i\{\partial_t |u_j|^2 + 2 \operatorname{Im} \tau_j |u_j|^2\} + \Sigma_k \{\beta_{jk} u_k \bar{u}_j - \overline{\beta_{jk}} u_j \bar{u}_k\}, \end{aligned}$$

and since

$$\operatorname{Im} \tau_j(t,\xi) \geqq -C\phi(t,\xi) \quad (j = 1, \dots, m),$$
$$|\beta_{jk}(t,\xi)| \leqq C\phi(t,\xi) \quad (j,k = 1, \dots, m),$$

we have

$$\partial_t \Sigma_j |u_j|^2 \leqq C_1 \phi \Sigma_j |u_j|^2 + \Sigma_j |f_j|^2,$$

that is,

$$\partial_t E \leqq C_1 \phi E + F,$$

where

$$E = \Sigma_j |u_j|^2, \quad F = \Sigma_j |f_j|^2.$$

Set

$$\Phi(t, \xi) = \int_{a(\xi)}^{t} \phi(s, \xi)\, ds,$$

then we have

$$\partial_t \{\exp(-C_1 \Phi(t, \xi)) E\} \leqq \exp(-C_1 \Phi(t, \xi)) F,$$

therefore,

$$E(t_2) \leqq \exp(C_1 \Phi(t_2, \xi)) \left\{ E(t_1) + \int_{t_1}^{t_2} \exp(-C_1 \Phi(s, \xi)) F(s)\, ds \right\}.$$

Since

$$1 \leqq \exp(C_1 \Phi(t, \xi)) \leqq |\xi|^c, \quad c = C_1 K_1$$

from (Φ), we have

$$E(t_2) \leqq |\xi|^c \left\{ E(t_1) + \int_{t_1}^{t_2} F(s)\, ds \right\}. \qquad \square$$

§4. Energy Inequalities in $(0, T_0(\xi))$

Set

$$A(t, D_t, \xi) = D_t^m + a_{01}(t, \xi) D_t^{m-1} + \cdots + a_{0m}(t, \xi), \quad a_{0k} = a_k,$$

then we have

$$D_t^j A(t, D_t, \xi) = D_t^{m+j} + a_{j1}(t, \xi) D_t^{m+j-1} + a_{j2}(t, \xi) D_t^{m+j-2} + \cdots + a_{j\ m+j}(t, \xi)$$

$$|a_{jk}(t, \xi)| \leqq C_j (1 + |\xi|)^{p(j)} \quad (k = 1, 2, \ldots, m + j, \ \ j = 0, 1, 2, \ldots),$$

where

$$a_{j1}(t,\xi) = a_{j-1\ 1}(t,\xi),$$
$$a_{j2}(t,\xi) = (D_t a_{j-1\ 1})(t,\xi) + a_{j-1\ 2}(t,\xi),$$
$$a_{j3}(t,\xi) = (D_t a_{j-1\ 2})(t,\xi) + a_{j-1\ 3}(t,\xi),$$
$$\cdots\cdots$$
$$a_{j\ m+j-1}(t,\xi) = (D_t a_{j-1\ m+j-2})(t,\xi) + a_{j-1\ m+j-1}(t,\xi),$$
$$a_{j\ m+j}(t,\xi) = (D_t a_{j-1\ m+j-1})(t,\xi).$$

Let us define

$$g_{m+j} = (D_t^j f)(0) - \{a_{j1}(0,\xi)g_{m+j-1} + a_{j2}(0,\xi)g_{m+j-2} + \cdots + a_{j\ m+j}(0,\xi)g_0\}$$
$$(j = 0, 1, 2, \ldots),$$

then the solution u of the Cauchy problem$(P^\wedge)$:

$$\begin{cases} A(t,D_t,\xi)u(t) = f(t) & (0 < t < T_0(\xi)), \\ D_t^j u(0) = g_j & (j = 0, 1, \ldots, m-1), \end{cases}$$

satisfies

$$D_t^{m+j}u(0) = g_{m+j} \quad (j = 0, 1, 2, \ldots).$$

The following lemma is clear by the definition of $\{g_j\}_{j \geqq m}$.

Lemma 4.1. *It holds*

$$\Sigma_{0 \leqq j \leqq m+M}|g_j| \leqq C_M(1+|\xi|)^{q(M)}\{\Sigma_{0 \leqq j \leqq M}|D_t^j f(0)| + \Sigma_{0 \leqq j \leqq m-1}|g_j|\}$$
$$(M = 0, 1, 2, \ldots, \quad \xi \in R^n),$$

where $q(M) = p(0) + p(1) + \cdots + p(M)$.

Using Lemma 4.1, we have

Lemma 4.2. *Set*

$$u_0(t) = \Sigma_{0 \leqq j \leqq m-1+M}(g_j/j!)t^j,$$
$$f_1(t) = f(t) - A(t, D_t, \xi)u_0(t),$$

then it holds

$$(D_t^j f_1)(0) = 0 \quad (j = 0, 1, \ldots, M-1),$$

and

$$\Sigma_{0\leqq j\leqq M}|D_t^j f_1(t)| \leqq \Sigma_{0\leqq j\leqq M}|D_t^j f(t)| + C_M(1+|\xi|)^{q(M)}\{\Sigma_{0\leqq j\leqq M-1}|D_t^j f(0)| + \Sigma_{0\leqq j\leqq m-1}|g_j|\}.$$

COROLLARY 4.1.

$$t^{-M}|f_1(t)| \leqq C_M(1+|\xi|)^{q(M)}\{\Sigma_{0\leqq j\leqq M}\sup_{0<s<t}|D_s^j f(s)| + \Sigma_{0\leqq j\leqq m-1}|g_j|\}.$$

PROOF. Since

$$f_1(t) = \{(M-1)!\}^{-1}\int_0^t s^{M-1}(\partial_t^M f_1)(t-s)\,ds,$$

we have

$$|f_1(t)| \leqq (M!)^{-1}t^M \sup_{0<s<t}|(\partial_s^M f_1)(s)|. \qquad \square$$

Now change the unknown u into u_1 by

$$u_1(t) = u(t) - u_0(t),$$

then the Cauchy problem $(P^\wedge)$ is reduced to the problem $(P^\wedge)_M$:

$$\begin{cases} A(t,D_t,\xi)u_1(t) = f_1(t) & (0<t<T_0(\xi)), \\ D_t^j u_1(0) = 0 & (j=0,1,\dots,m-1+M), \\ \text{where } (D_t^j f_1)(0) = 0 & (j=0,1,\dots,M-1). \end{cases}$$

LEMMA 4.3. *Assume that Condition* (A) *is satisfied in* $I_0 = (0, T_0(\xi))$, *and that* u_1 *satisfies*

$$A(t,D_t,\xi)u_1(t) = f_1(t) \quad \text{in } I_0.$$

Then

$$v_j = (tD_t)^j u_1 \quad (j=0,1,\dots,m-1)$$

satisfy

$$\begin{cases} D_t v_0 = t^{-1}v_1, \\ D_t v_1 = t^{-1}v_2, \\ \dots\dots \\ D_t v_{m-1} = t^{-1}\{-b_1(t,\xi)v_{m-1} - \cdots - b_m(t,\xi)v_0\} + t^{m-1}f_1(t,\xi), \end{cases}$$

where

$$|b_j(t,\xi)| \leqq C \quad in\ I_0^{\sim}.$$

Namely, $\{v_j\}$ *satisfy a* (A)*-standard system of equations.*

PROOF. Let us denote

$$\begin{aligned} t^m A(t, D_t, \xi) &= t^m D_t^m + t a_1(t,\xi)(t^{m-1} D_t^{m-1}) + \cdots + t^m a_m(t,\xi) \\ &= (tD_t)^m + b_1(t,\xi)(tD_t)^{m-1} + \cdots + b_m(t,\xi), \end{aligned}$$

then we have

$$|b_j(t,\xi)| \leqq C \quad \text{in } I_0^{\sim}$$

from Condition(A). Set

$$v_j = (tD_t)^j u_1 \quad (j = 0, 1, \ldots, m-1),$$

then we have

$$\begin{cases} D_t v_0 = t^{-1} v_1, \\ D_t v_1 = t^{-1} v_2, \\ \cdots\cdot \\ D_t v_{m-1} = t^{-1}(-b_1 v_{m-1} - \cdots - b_m v_0) + t^{m-1} f_1 \end{cases}$$

from $\mathrm{A}u_1 = f_1$. □

PROPOSITION 4.1. *Assume that Condition(A) is satisfied in* $I_0 = (0, T_0(\xi))$, *then there exists* $c > 0$ *and* $C' > 0$ *such that*

$$\Sigma_{0 \leqq j \leqq m-1} |D_t^j u_1(t)|^2 \leqq C' t^{c-2m+2} \int_0^t s^{-c+2m-2} |f_1(s)|^2 \, ds \quad ((t,\xi) \in I_0^{\sim})$$

holds for u_1 *satisfying* $(P^{\wedge})_M$ *with* $M > c/2 - m$.

PROOF. From Lemma 3.1 and Lemma 4.3, we have

$$\begin{aligned} & t^{-c+2(m-1)} \Sigma_{0 \leqq j \leqq m-1} |D_t^j u_1(t)|^2 \\ & \leqq C' t^{-c} \Sigma_{0 \leqq j \leqq m-1} |(tD_t)^j u_1(t)|^2 \\ & \leqq C' \left\{ \tau^{-c} \Sigma_{0 \leqq j \leqq m-1} |(\tau D_\tau)^j u_1(\tau)|^2 + \int_\tau^t s^{-c+2(m-1)} |f_1(s)|^2 \, ds \right\} \end{aligned}$$

$(0 < \tau < t)$. Since

$$\Sigma_{0 \leqq j \leqq m-1} |(\tau D_\tau)^j u_1(\tau)| = 0(\tau^{m+M}),$$

we have

$$\tau^{-c} \Sigma_{0 \leqq j \leqq m-1} |(\tau D_\tau)^j u_1(\tau)|^2 = 0(\tau^{-c+2(m+M)}) \to 0 \quad (\tau \to 0),$$

if $M > c/2 - m$. Hence we have

$$t^{-c+2(m-1)} \Sigma_{0 \leqq j \leqq m-1} |D_t^j u_1(t)|^2 \leqq C' \int_0^t s^{-c+2(m-1)} |f_1(s)|^2 \, ds \quad (M > c/2 - m). \quad \square$$

Now we come back to the problem($P^\wedge$), then we have

$$t^{-M} |f_1(t)| \leqq C_M (1 + |\xi|)^{q(M)} \{ \Sigma_{0 \leqq j \leqq M} \sup_{0 < s < t} |D_s^j f(s)| + \Sigma_{0 \leqq j \leqq m-1} |g_j| \}$$

from Corollary 4.1. Therefore, we have from Proposition 4.1.

PROPOSITION 4.2. *Assume that Condition(A) is satisfied in* $I_0 = (0, T_0(\xi))$, *then there exist* $C > 0$ *and* $M > 0$ *such that it holds*

$$\Sigma_{0 \leqq j \leqq m-1} \sup_{I_0} |D_t^j u(t)|$$
$$\leqq C(1 + |\xi|)^{q(M)} \{ \Sigma_{0 \leqq j \leqq M} \sup_{I_0} |D_t^j f(t)| + \Sigma_{0 \leqq j \leqq m-1} |g_j| \}, \quad (t, \xi) \in I_0^\sim$$

where u satisfies $(P^\wedge)$ *in* $I_0^\sim$.

§5. Condition(B)

At first, we consider conditions (B-1), (B-2), $(\text{B-3})_\phi$, $(\text{B-4})_\phi$ about

$$P(t, \tau, \xi) = \tau^m + p_1(t, \xi) \tau^{m-1} + \cdots + p_m(t, \xi)$$
$$= (\tau - \tau_1(t, \xi)) \cdots (\tau - \tau_m(t, \xi)).$$

LEMMA 5.1. *Assume that* $P(t, \tau, \xi)$ *satisfies* (B-1) *in* $I^\sim$, *then*

$$P_j(t, \tau, \xi) = \Pi_{k \neq j} (\tau - \tau_k(t, \xi)) \quad (j = 1, 2, \ldots, m)$$

are linear independent polynomials in τ, *and satisfy* $(\text{B-5})_1$ *in* $I^\sim$.

PROOF. Set

$$\begin{aligned} P_j(t,\tau,\xi) &= \Pi_{k\neq j}(\tau-\tau_k(t,\xi)) \\ &= \tau^{m-1} - \Sigma_{k\neq j}\tau_k\tau^{m-2} + \Sigma_{k<h,k,h\neq j}\tau_k\tau_h\tau^{m-3} - \cdots + (-1)^{m-1}\Pi_{k\neq j}\tau_k \\ &= \tau^{m-1} + b_2\tau^{m-2} + \cdots + b_m, \end{aligned}$$

then we have

$$\begin{aligned} |b_2| &\leqq C|\tau_1|, \\ |b_3| &\leqq C|\tau_1|\,|\tau_2|, \\ &\cdots\cdots \\ |b_m| &\leqq C|\tau_1|\,|\tau_2|\cdots|\tau_{m-1}| \end{aligned}$$

from (B-1). □

LEMMA 5.2. *Assume that P satisfies* (B-1), (B-4)$_\phi$ *in* $I^\sim$, *then*

$$P_j'(t,\tau,\xi) = (\partial_t P_j)(t,\tau,\xi)$$

satisfies (B-5)$_\phi$ *in* $I^\sim$.

PROOF. Set

$$\begin{aligned} P_j'(t,\tau,\xi) &= \partial_t\Pi_{k\neq j}(\tau-\tau_k(t,\xi)) \\ &= \Sigma_{\ell\neq j}(-\tau_\ell'(t,\xi))\Pi_{k\neq j,\ell}(\tau-\tau_k(t,\xi)) \\ &= \Sigma_{\ell\neq j}(-\tau_\ell')\{\tau^{m-2} - \Sigma_{k\neq j,\ell}\tau_k\tau^{m-3} + \Sigma_{k<h,k,h\neq j,\ell}\tau_k\tau_h\tau^{m-4} - \cdots \\ &\qquad + (-1)^{m-2}\Pi_{k\neq j,\ell}\tau_k\} \\ &= \Sigma_{\ell\neq j}(-\tau_\ell'/\tau_\ell)\{\tau_\ell\tau^{m-2} - \tau_\ell\Sigma_{k\neq j,\ell}\tau_k\tau^{m-3} + \cdots \\ &\qquad + (-1)^{m-2}\tau_\ell\Pi_{k\neq j,\ell}\tau_k\} \\ &= \Sigma_{\ell\neq j}(-\tau_\ell'/\tau_\ell)\{b_{\ell 2}\tau^{m-2} + b_{\ell 3}\tau^{m-3} + \cdots + b_{\ell m}\}, \end{aligned}$$

then we have

$$\begin{aligned} |b_{\ell 2}| &\leqq C|\tau_1|, \\ |b_{\ell 3}| &\leqq C|\tau_1|\,|\tau_2|, \\ &\cdots\cdots \\ |b_{\ell m}| &\leqq C|\tau_1|\,|\tau_2|\cdots|\tau_{m-1}| \end{aligned}$$

from (B-1). Since

$$|\tau'_\ell/\tau_\ell| \leqq C\phi$$

from $(\text{B-4})_\phi$, P'_j satisfies $(\text{B-5})_\phi$. □

Next, we consider $(\text{B-5})_\phi$ about

$$\begin{aligned} Q(t,\tau,\xi) &= q_1(t,\xi)\tau^{m-1} + \cdots + q_m(t,\xi) \\ &= \Sigma_{1\leqq j\leqq m}\beta_j(t,\xi)P_j(t,\tau,\xi). \end{aligned}$$

LEMMA 5.3. *Assume that P satisfies* (B-1), (B-2) *in $I^\sim$, then Q satisfies* $(\text{B-5})_\phi$ *in $I^\sim$ iff*

$$|\beta_j(t,\xi)| \leqq C\phi(t,\xi) \quad (j = 1,\dots,m,\ (t,\xi)\in I^\sim).$$

PROOF. Since sufficiency is easy from Lemma 5.1, we concider necessity. Since $\{\tau_j\}$ are distinct, we can descrive a closed curve Γ_j, on the complex plane, enclosing only $\tau = \tau_j$, and we have

$$\begin{aligned} \beta_j(t,\xi) &= (2\pi i)^{-1}\int_{\Gamma_j} Q(t,\tau,\xi)/P(t,\tau,\xi)\,d\tau \\ &= (2\pi i)^{-1}\int_{\Gamma_j} Q(t,\tau,\xi)/\{\Pi_{1\leq k\leq m}(\tau-\tau_k)\}\,d\tau \\ &= Q(t,\tau_j,\xi)/\{\Pi_{1\leqq k\leqq m, k\neq j}(\tau_j-\tau_k)\} \end{aligned}$$

Therefore we have

$$\begin{aligned} |\beta_j(t,\xi)| &= |Q(t,\tau_j,\xi)|/\Pi_{k\neq j}|\tau_j-\tau_k| \\ &\leqq \{|q_1|\,|\tau_j|^{m-1} + \cdots + |q_m|\}/\Pi_{k\neq j}|\tau_j-\tau_k| \\ &\leqq C\phi\{|\tau_j|^{m-1} + |\tau_1|\,|\tau_j|^{m-2} + \cdots + |\tau_1|\cdots|\tau_{m-1}|\} \\ &\quad /\{\Pi_{k>j}|\tau_j-\tau_k|\Pi_{k<j}|\tau_j-\tau_k|\} \quad (\because (\text{B-5})_\phi) \\ &\leqq C'\phi\{|\tau_j|^{m-1} + |\tau_1|\,|\tau_j|^{m-2} + \cdots + |\tau_1|\cdots|\tau_{m-1}|\} \\ &\quad /\{|\tau_j|^{m-j}|\tau_1|\cdots|\tau_{j-1}|\} \quad (\because (\text{B-2})) \\ &= C'\phi\{R_j + R'_j\}, \end{aligned}$$

where

$$R_1 = 1$$

$$R_j = |\tau_j|^{j-1}/|\tau_1|\cdots|\tau_{j-1}| + |\tau_j|^{j-2}/|\tau_2|\cdots|\tau_{j-1}| + \cdots + |\tau_j|/|\tau_{j-1}| + 1$$

$$(j = 2, \ldots, m),$$

$$R'_j = 1 + |\tau_{j+1}|/|\tau_j| + |\tau_{j+1}|\,|\tau_{j+2}|/|\tau_j|^2 + \cdots + |\tau_{j+1}|\cdots|\tau_{m-1}|/|\tau_j|^{m-j-1}$$

$$(j = 1, \ldots, m-2),$$

$$R'_{m-1} = 1, \quad R'_m = 0.$$

Hence we have

$$|\beta_j(t,\xi)| \leqq mC'\phi \quad (\because \text{(B-1)}). \qquad \square$$

LEMMA 5.4. *Assume that P satisfies* (B-1), (B-2) *in $I^{\sim}$, then*

$$\pi_j(t,\tau,\xi) = \tau_1(t,\xi)\tau_2(t,\xi)\cdots\tau_{m-j}(t,\xi)\tau^{j-1} \quad (j = 1, \ldots, m-1),$$

$$\pi_m = \tau^{m-1}$$

are linearly independent polynomials in τ in $I^{\sim}$. Set

$$\pi_j(t,\tau,\xi) = \Sigma_{1\leqq k\leqq m}\beta_{jk}(t,\xi)P_k(t,\tau,\xi),$$

$$P_j(t,\tau,\xi) = \Sigma_{1\leqq k\leqq m}\alpha_{jk}(t,\xi)\pi_k(t,\tau,\xi),$$

then it holds that

$$|\alpha_{jk}(t,\xi)| \leqq C, \quad |\beta_{jk}(t,\xi)| \leqq C.$$

PROOF. Since $\pi_j(t,\tau,\xi)$ satisfies (B-5)$_1$, we have

$$\pi_j(t,\tau,\xi) = \Sigma_{1\leqq k\leqq m}\beta_{jk}(t,\xi)P_k(t,\tau,\xi), \quad |\beta_{jk}(t,\xi)| \leqq C$$

from Lemma 5.3. On the other hand, since

$$P_j(t,\tau,\xi) = \tau^{m-1} + b_{j1}(t,\xi)\tau^{m-2} + \cdots + b_{jm}(t,\xi),$$

$$|b_{jk}(t,\xi)| \leqq C|\tau_1(t,\xi)|\,|\tau_2(t,\xi)|\cdots|\tau_k(t,\xi)|,$$

we have

$$P_j(t,\tau,\xi) = \Sigma_{1\leqq k\leqq m}\alpha_{jk}(t,\xi)\pi_k(t,\tau,\xi), \quad |\alpha_{jk}(t,\xi)| \leqq C. \qquad \square$$

§ 6. Energy Inequalities in $(T_{h-1}(\xi), T_h(\xi))$

Assume that

$$A(t,\tau,\xi) = P(t,\tau,\xi) + Q(t,\tau,\xi)$$

satisfies Condition(B) in $I = (T_{h-1}(\xi), T_h(\xi))$, where

$$P(t,\tau,\xi) = (\tau - \tau_1(t,\xi))\cdots(\tau - \tau_m(t,\xi)).$$

Set

$$P_j(t,\tau,\xi) = \Pi_{k\neq j}(\tau - \tau_k(t,\xi)),$$

then we have

$$P(t,D_t,\xi) = (D_t - \tau_j(t,\xi))P_j(t,D_t,\xi) + i(\partial_t P_j)(t,D_t,\xi).$$

Set

$$\begin{aligned} A(t,D_t,\xi) &= (D_t - \tau_j(t,\xi))P_j(t,D_t,\xi) + \{i(\partial_t P_j)(t,D_t,\xi) + Q(t,D_t,\xi)\} \\ &= (D_t - \tau_j(t,\xi))P_j(t,D_t,\xi) + Q_j(t,D_t,\xi), \end{aligned}$$

then Q_j satisfies $(\text{B-5})_\phi$, from Lemma 4.2, therefore

$$Q_j(t,\tau,\xi) = \Sigma_{1\leqq k\leqq m}\beta_{jk}(t,\xi)P_k(t,\tau,\xi),$$

where

$$|\beta_{jk}(t,\xi)| \leqq C\phi(t,\xi) \quad (j = 1,\dots,m,\ \ (t,\xi)\in I^{\sim}),$$

from Lemma 5.3.

LEMMA 6.1. *Assume that Condition(B) in $I = (T_{h-1}(\xi), T_h(\xi))$ and u satisfies*

$$A(t,D_t,\xi)u = f, \quad t\in I.$$

Set

$$u_j = P_j(t,D_t,\xi)u,$$

then $\{u_j\}$ satisfy

$$(D_t - \tau_j(t,\xi))u_j + \Sigma_{1\leqq k\leqq m}\beta_{jk}(t,\xi)u_k = f \quad (j = 1,2,\dots,m),$$

where

$$\operatorname{Im}\tau_j(t,\xi) \geqq -C\phi(t,\xi) \quad (j = 1,\dots,m),$$

$$|\beta_{jk}(t,\xi)| \leqq C\phi(t,\xi) \quad (j,k = 1,\dots,m)$$

in $I^{\sim}$, that is, $\{u_j\}$ satisfy a (B)*-standard system of equations in I.*

From Lemma 3.2 and Lemma 6.1, we have

PROPOSITION 6.1. *Assume that $A(t,\tau,\xi)$ satisfies Condition(B) in $I=(T_{k-1}(\xi),T_k(\xi))$, and that u satisfies*

$$A(t,D_t,\xi)u=f \quad \text{in } I^{\sim}.$$

Then it holds

$$\Sigma_{1\leqq j\leqq m}|P_j(t,D_t,\xi)u(t)|^2$$
$$\leqq |\xi|^c\left\{\Sigma_{1\leqq j\leqq m}|P_j(t,D_t,\xi)u(T_{k-1}(\xi))|^2+\int_{T_{k-1}(\xi)}^t |f(s)^2\,ds\right\} \quad \text{in } I^{\sim}.$$

LEMMA 6.2. *Assume that $A(t,\tau,\xi)$ satisfies Condition(B) in $I=(T_{h-1}(\xi), T_h(\xi))$, then it holds*

$$c_1|D_t^{m-1}u|\leqq \Sigma_{1\leqq j\leqq m}|P_j(t,D_t,\xi)u|\leqq c_2|\xi|^c\Sigma_{0\leqq j\leqq m-1}|D_t^ju|, \quad \text{in } I^{\sim},$$

where $c=mp(0)$.

PROOF. Set

$$\pi_j(t,\tau,\xi)=\tau_1(t,\xi)\tau_2(t,\xi)\cdots\tau_{m-j}(t,\xi)\tau^{j-1} \quad (j=1,\dots,m),$$

then we have

$$\pi_j(t,\tau,\xi)=\Sigma_{1\leqq k\leqq m}\beta_{jk}(t,\xi)P_k(t,\tau,\xi), \quad |\beta_{jk}(t,\xi)|\leqq C$$

from Lemma 5.4, therefore

$$|\pi_j(t,D_t,\xi)u|\leqq C\Sigma_{1\leqq k\leqq m}|P_k(t,D_t,\xi)u|.$$

Since $\pi_m(t,D_t,\xi)=D_t^{m-1}$, we have

$$c_1|D_t^{m-1}u|\leqq \Sigma_{1\leqq j\leqq m}|P_j(t,D_t,\xi)u|.$$

On the other hand, we have

$$P_j(t,\tau,\xi)=\Sigma_{1\leqq k\leqq m}\alpha_{jk}(t,\xi)\pi_k(t,\tau,\xi), \quad |\alpha_{jk}(t,\xi)|\leqq C$$

from Lemma 5.4, therefore

$$|P_j(t,D_t,\xi)u|\leqq C\Sigma_{1\leqq k\leqq m}|\pi_k(t,D_t,\xi)u|.$$

Remarking that

$$|\tau_j| \leqq C|\xi|^{p(0)},$$

we have

$$|\pi_j(t, D_t, \xi)u| \leqq C|\xi|^{(m-j)p(0)}|D_t^{j-1}u|,$$

therefore

$$\Sigma_{1 \leqq j \leqq m}|P_j(t, D_t, \xi)u| \leqq c_2|\xi|^c \Sigma_{0 \leqq j \leqq m-1}|D_t^j u|. \qquad \square$$

PROPOSITION 6.2. *Assume that* $A(t, \tau, \xi)$ *satisfies Condition*(B) *in* $I = (T_{h-1}(\xi), T_h(\xi))$, *and that* u *satisfies*

$$A(t, D_t, \xi)u = f \quad \text{in } I^\sim.$$

Then it holds

$$\Sigma_{0 \leqq j \leqq m-1}|D_t^j u(t)|^2$$
$$\leqq C|\xi|^c\left\{\Sigma_{0 \leqq j \leqq m-1}|D_t^j u(T_{h-1}(\xi))|^2 + \int_{T_{h-1}(\xi)}^t |f(s)|^2\, ds\right\} \quad \text{in } I^\sim.$$

PROOF. We have

$$|D_t^{m-1}u(t)|^2 \leqq C|\xi|^c\left\{\Sigma_{0 \leqq j \leqq m-1}|D_t^j u(T_{h-1}(\xi))|^2 + \int_{T_{h-1}(\xi)}^t |f(s)|^2\, ds\right\} \quad \text{in } I^\sim$$

from Proposition 6.1 and Lemma 6.2. On the other hand, since

$$|D_t^j u(t)| \leqq |D_t^j u(T_{h-1}(\xi))| + \int_{T_{h-1}(\xi)}^t |D_t^{j+1}u(s)|\, ds \quad \text{in } I^\sim,$$

we have

$$\Sigma_{0 \leqq j \leqq m-2}|D_t^j u(t)|^2$$
$$\leqq C\left\{\Sigma_{0 \leqq j \leqq m-2}|D_t^j u(T_{h-1}(\xi))|^2 + \int_{T_{h-1}(\xi)}^t |D_t^{m-1}u(s)|^2\, ds\right\} \quad \text{in } I^\sim. \qquad \square$$

The theorem stated in §1 follows directly from

PROPOSITION 6.3. *For*

$$A(t, \tau, \xi) = \tau^m + a_1(t, \xi)\tau^{m-1} + \cdots + a_m(t, \xi),$$

the interval $(0, T)$ *is assumed to split into sub-intervals* $I_h = (T_{h-1}(\xi), T_h(\xi))$ *for* $|\xi| > K_0$ $(h = 0, 1, \dots, N,\ K_0 > 0)$, *where*

$$0 = T_{-1} < T_0(\xi) < T_1(\xi) < \cdots < T_{N-1}(\xi) < T_N = T,$$

and $A(t, \tau, \xi)$ *satisfies Condition*(A) *in* I_0 *and Condition*(B) *in* I_h $(h = 1, \dots, N)$. *Let* u *be a solution of* $(P^\wedge)$, *then for any* μ $(\geqq m-1)$, *there exist positive constants* $C\mu$, $c(\mu)$, $M(\mu)$ *such that it holds*

$$\Sigma_{0 \leqq j \leqq \mu} \sup_I |D_t^j u(t)|^2$$
$$\leqq C_\mu (1 + |\xi|)^{c(\mu)} \{\Sigma_{0 \leqq j \leqq M(\mu)} \sup_I |D_t^j f(t)|^2 + \Sigma_{0 \leqq j \leqq m-1} |g_j|^2\}$$
$$\text{in } (0, T) \times \{|\xi| > K_0\}.$$

PROOF. From Proposition 4.2 and Proposition 6.2, we have

$$\Sigma_{0 \leqq j \leqq m-1} \sup_I |D_t^j u(t)|^2 \leqq C(1 + |\xi|)^c \{\Sigma_{0 \leqq j \leqq M} \sup_I |D_t^j f(t)|^2 + \Sigma_{0 \leqq j \leqq m-1} |g_j|^2\}.$$

Energy estimates for derivatives of higher order than $m-1$ follow from the equation $A(t, D_t, \xi)u = f$. □

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