

# GEODESIC TRANSFORMATIONS IN ALMOST HERMITIAN GEOMETRY

By

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**Abstract.** We treat partially conformal geodesic transformations with respect to submanifolds in almost Hermitian manifolds. Non-isometric ones only exist when the submanifold is a real hypersurface or reduces to a point. In these two cases, we derive necessary and sufficient conditions for the existence in terms of the Jacobi operator and show how this existence influences the geometry of the hypersurface and that of the ambient space. As an application, we use these transformations to obtain a new characterization of complex space forms.

## 1. Introduction

Local reflections with respect to points or submanifolds of a Riemannian manifold have been studied intensively. The properties of these reflections have been used to obtain several geometric properties and characterizations of special classes of Riemannian manifolds and submanifolds. We refer to [4], [22] and [24] for examples, basic material and further references. Local reflections are maps which preserve tubular hypersurfaces about the point or submanifold  $P$ . This kind of transformations has been generalized to geodesic transformations with respect to  $P$ . These transformations map a tubular hypersurface about  $P$  into another tubular hypersurface by moving points along normal geodesics of  $P$ , but leaving the points of  $P$  invariant. Geodesic transformations were introduced in [18] and studied also in [6]. Recently the authors begun a systematic study of such transformations. (See [10], [11] and [12] for information about divergence-

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preserving and holomorphic geodesic transformations and [4], [5] for holomorphic and symplectic geodesic reflections).

Homotheties and inversions with respect to spheres form a class of remarkable transformations in Euclidean geometry. They are fundamental conformal transformations. In [9], we use conformal transformations with respect to points and geodesic spheres to characterize real space forms, where non-Euclidean similarities and inversions are introduced. The existence of conformal geodesic transformations was investigated further in [13]. These studies show that conformality is a strong condition and this fact motivated the study, in [9], of the closely related notion of a partially conformal geodesic transformation.

In this paper, we focus on *partially conformal* geodesic transformations with respect to submanifolds in almost Hermitian manifolds. In Section 2, we consider the analytic description of these transformations by using Fermi coordinates and derive the first results. We study the influence of the existence of a partially conformal transformation with respect to a submanifold on the extrinsic geometry of the submanifold and show that for codimension greater than one, the local reflections are the only partially conformal geodesic transformations. This restricts the study of partially conformal geodesic transformations to the case of points and real hypersurfaces.

Section 3 is devoted to the study of partially conformal geodesic transformations with respect to points. We derive the necessary and sufficient conditions for the existence of such transformations. It turns out that such conditions can be expressed in terms of the Jacobi operator and its derivatives. As a consequence, we obtain a characterization of complex space forms as well as a description of all the possible partially conformal geodesic transformations. In Section 4, we make a similar study for partially conformal geodesic transformations with respect to real hypersurfaces.

Manifolds are assumed to be connected and analytic, although  $C^\infty$  is sometimes sufficient.

## 2. Partially Conformal Geodesic Transformations. First Results

Let  $(M, g, J)$  be an almost Hermitian manifold of real dimension  $n > 2$ ,  $\nabla$  its Levi Civita connection and  $R$  the associated Riemann curvature tensor taken with the sign convention  $R_{XY} = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y]$  for all smooth vector fields  $X, Y$ . Moreover, put  $R_{XYZW} = R(X, Y, Z, W) = g(R(X, Y)Z, W)$ .

Let  $B$  be a topologically embedded submanifold with  $\dim B = q$  and let  $\exp_\nu$  denote the exponential map of the normal bundle  $\nu$  of  $B$ .

DEFINITION 2.1. A *geodesic transformation*  $\varphi_B$  with respect to  $B$  is a map defined by

$$(2.1) \quad \varphi_B : p = \exp_v(ru) \rightarrow \varphi_B(p) = \exp_v(s(r)u)$$

which leaves  $B$  invariant. Here  $u$  is an arbitrary unit normal vector of  $B$  and  $r$  and  $s$  are supposed to be sufficiently small such that  $\varphi_B$  is a local diffeomorphism.

In all what follows, the function  $r \mapsto s(r)$  is supposed to be analytic in a neighborhood of  $r = 0$ .

Conformal geodesic transformations have been investigated by the authors in [9], [13]. The existence of such transformations is closely related to the constancy of the sectional curvature. In the present paper, we shall investigate the weaker notion of partial conformality. The following observation is a key fact motivating the definition of partially conformal geodesic transformations. Let  $N$  denote the gradient of the normal distance function. The almost complex structure  $J$  gives rise to the locally defined vector field  $JN$ . The properties of this distinguished vector field strongly influence the geometry of the manifold. For example, when  $(M, g, J)$  is a nearly Kähler manifold (that is,  $(\nabla_X J)X = 0$  for all vector fields  $X$ ) the constancy of the holomorphic sectional curvature of  $(M, g, J)$  is equivalent to the fact that  $JX$  defines a distinguished eigenspace of the Jacobi operator  $R_X$ , that is,  $R(X, JX)X$  is proportional to  $JX$  [21]. When  $B$  is a small geodesic sphere,  $JN$  is a vector field tangent to  $B$  and it defines a distinguished eigenspace of either the shape or the Ricci operator for all sufficiently small geodesic spheres if and only if the holomorphic sectional curvature is constant, provided that  $(M, g, J)$  is a nearly Kähler manifold [7], [23].

Next, let  $\eta$  be the one-form induced by the metric and the vector field  $JN$  and defined by  $\eta(X) = g(X, JN)$ .

DEFINITION 2.2. A geodesic transformation  $\varphi_B$  with respect to a submanifold  $B$  is said to be *partially conformal* if and only if

$$(2.2) \quad \varphi_B^*g = e^{2\sigma}g + f(\eta \otimes \eta)$$

for some function  $f$  depending only on the normal distance function.

The function  $f$  in (2.2) is assumed to be analytic although at some places this condition can be weakened.

REMARK 2.1. A partially conformal transformation is conformal if and only if the function  $f$  vanishes. Note that our notion of partial conformality tallies with

that of “special partial conformality” introduced by Tanno [20]. Such transformations  $\varphi$  are defined by

$$(\varphi^*g)(X, Y) = e^{2\sigma}g(X, Y)$$

for all vector fields  $X, Y$  on  $M$ , where at least one of them is tangent to the distribution  $D = \text{Ker } \eta$ .

Note that the function  $f$  in (2.2) is defined by  $(\varphi^*g)(JN, JN) = (e^{2\sigma} + f)g(JN, JN)$ .

To describe analytically a partially conformal geodesic transformation, we introduce a system of Fermi coordinates adapted to the submanifold. See [16], [22] for more detailed information. Let  $m \in B$  and let  $\{E_1, \dots, E_n\}$  be a local orthonormal frame field of  $(M, g)$  defined along  $B$  in a neighborhood of  $m$ . Furthermore, we specialize this moving frame such that  $\{E_1, \dots, E_q\}$  are tangent vector fields and  $\{E_{q+1}, \dots, E_n\}$  are normal vector fields of  $B$ . Let  $(y^1, \dots, y^q)$  be a system of coordinates in a neighborhood of  $m$  in  $B$  such that

$$\frac{\partial}{\partial y^i}(m) = E_i(m), \quad i = 1, \dots, q$$

and define the *Fermi coordinates*  $(x^1, \dots, x^n)$  with respect to  $m$ ,  $(y^1, \dots, y^q)$  and  $\{E_{q+1}, \dots, E_n\}$  by

$$\begin{aligned} x^i \left( \exp_v \left( \sum_{q+1}^n t^\alpha E_\alpha \right) \right) &= y^i, \quad i = 1, \dots, q, \\ x^a \left( \exp_v \left( \sum_{q+1}^n t^\alpha E_\alpha \right) \right) &= t^a, \quad a = q+1, \dots, n \end{aligned}$$

in a neighborhood of the zero section of  $B$  in  $v$ , taken sufficiently small such that  $\exp_v$  is a diffeomorphism.

Next, we derive an expression for the components of the metric tensor  $g$ ,

$$g_{ij} = g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right), \quad g_{ia} = g \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^a} \right), \quad g_{ab} = g \left( \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b} \right).$$

Let  $u$  be a normal unit vector,  $u \in T_m^\perp B$ , and  $\gamma(r) = \exp_m(ru)$  a normal geodesic with  $\gamma(0) = m$ ,  $\gamma'(0) = u$ . We specialize the frame field  $\{E_1, \dots, E_n\}$  in such a way that

$$\gamma'(0) = u = E_n(m), \quad J\gamma'(0) = Ju = (-dE_q + cE_{q+1})(m),$$

for real numbers  $c, d$  with  $c^2 + d^2 = 1$ . Now consider the frame field  $\{F_1, \dots, F_n\}$  along  $\gamma$  obtained by parallel translating the basis  $\{E_1(m), \dots, E_n(m)\}$ , and denote by  $Y_\alpha(r)$ ,  $\alpha = 1, \dots, n-1$ , the Jacobi fields along  $\gamma(r)$  with initial conditions

$$(2.3) \quad \begin{aligned} Y_i(0) &= E_i(0), & Y'_i(0) &= \nabla_u \frac{\partial}{\partial x^i}, \\ Y_a(0) &= 0, & Y'_a(0) &= E_a(m) \end{aligned}$$

where the prime denotes covariant differentiation along  $\gamma$ . These fields are related to the Fermi coordinate vector fields by

$$(2.4) \quad Y_i(r) = \frac{\partial}{\partial x^i}(\gamma(r)), \quad Y_a(r) = r \frac{\partial}{\partial x^a}(\gamma(r)).$$

Using the parallel basis  $\{F_1, \dots, F_n\}$ , we identify the tangent spaces  $\{\gamma'(r)\}^\perp$  and write  $Y_\alpha(r) = D_u(r)F_\alpha$  for  $\alpha = 1, \dots, n-1$ , where  $D_u(r)$  is an endomorphism-valued function. Then, the Jacobi equation yields

$$(2.5) \quad D_u''(r) + (R \circ D_u)(r) = 0$$

where  $R(r)X = R_{\gamma'(r)X}\gamma'(r)$ . To derive the initial values for  $D_u(r)$ , we shall use the Gauss and Weingarten equations for the submanifold  $B$  [2]:

$$\begin{aligned} \nabla_X Y &= \tilde{\nabla}_X Y + T_X Y, \\ \nabla_X \xi &= T(\xi)X + \nabla_X^\perp \xi \end{aligned}$$

where  $X, Y$  are tangent to  $B$  and  $\xi$  is a unit normal vector to  $B$ .  $\tilde{\nabla}$  denotes the induced metric connection on  $B$ ,  $T_X Y$  is the second fundamental form,  $T(\xi)$  the shape operator with respect to  $\xi$  and  $\nabla^\perp$  the normal connection along  $B$ . Furthermore,  $T_X Y$  and  $T(\xi)$  are related by

$$g(T(\xi)X, Y) = -g(T_X Y, \xi).$$

Now, using the initial conditions (2.3) for  $Y_\alpha$ , we obtain the following initial values in matrix form with respect to the basis  $\{E_1(m), \dots, E_{n-1}(m)\}$  of  $u^\perp \subset T_m M$ :

$$D_u(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad D'_u(0) = \begin{pmatrix} T(u) & 0 \\ -{}^t\perp(u) & I \end{pmatrix}$$

where

$$\begin{aligned} T(u)_{ij} &= g(T(u)E_i, E_j)(m), \\ \perp(u)_{ia} &= g({}^t\perp_{E_i} E_a, E_n)(m), \end{aligned}$$

$\perp$  being an operator defined in [16]. It satisfies  $\perp_X \xi = \nabla_X^\perp \xi$ . Using the generalized Gauss Lemma [16]

$$(2.6) \quad g_{mn}(p) = 1, \quad g_{\alpha n}(p) = 0, \quad \alpha = 1, \dots, n-1,$$

and (2.4), we obtain at the point  $p = \exp_m(ru)$ :

$$(2.7) \quad \begin{aligned} g_{ij}(p) &= ({}^t D_u D_u)_{ij}(r), \\ g_{ia}(p) &= \frac{1}{r} ({}^t D_u D_u)_{ia}(r), \\ g_{ab}(p) &= \frac{1}{r^2} ({}^t D_u D_u)_{ab}(r). \end{aligned}$$

In order to describe analytically a geodesic transformation with respect to  $B$ , we consider an adapted system of Fermi coordinates about  $B$  as described before, and put  $s(r) = \rho(r)r$  in (2.1). Then one obtains the following analytic description of the geodesic transformation:

$$\varphi_B : (x^1, \dots, x^q, x^{q+1}, \dots, x^n) \mapsto (x^1, \dots, x^q, \rho(r)x^{q+1}, \dots, \rho(r)x^n)$$

where  $r$  denotes the normal distance function. Note that  $r^2 = \sum_{a=q+1}^n (x^a)^2$ . Hence, we have

**LEMMA 2.1.** *A geodesic transformation  $\varphi_B$  with respect to a submanifold  $B$  is partially conformal if and only if the following conditions are satisfied:*

$$\begin{aligned} g_{ij}(\varphi_B(p)) &= e^{2\sigma} g_{ij}(p) + f(r)(\eta \otimes \eta)_{ij}(p), \\ \rho g_{ia}(\varphi_B(p)) &= e^{2\sigma} g_{ia}(p) + f(r)(\eta \otimes \eta)_{ia}(p), \\ \rho^2 g_{ab}(\varphi_B(p)) &= e^{2\sigma} g_{ab}(p) + f(r)(\eta \otimes \eta)_{ab}(p), \\ e^{2\sigma} &= \left( \frac{ds}{dr} \right)^2 \end{aligned}$$

for each point  $p = \exp_m(ru)$ , where  $i, j = 1, \dots, q$  and  $a, b = q+1, \dots, n-1$ .

**PROOF.** Considering the previous expression of  $\varphi_B$  with respect to an adapted system of coordinates, we have

$$(2.8) \quad \begin{aligned} (\varphi_B)_* \frac{\partial}{\partial x^i} &= \frac{\partial}{\partial x^i}, \quad i = 1, \dots, q, \\ (\varphi_B)_* \frac{\partial}{\partial x^a} &= \rho \frac{\partial}{\partial x^a} + \sum_{k=q+1}^n \rho' \frac{\partial \rho}{\partial x^a} x^k \frac{\partial}{\partial x^k}, \quad a = q+1, \dots, n. \end{aligned}$$

Hence, along any normal geodesic  $\gamma(r) = \exp_m(ru)$ , we get

$$\begin{aligned} (\varphi_B^*g)_{ij}(\gamma(r)) &= g_{ij}(\gamma(s)), & (\varphi_B^*g)_{ia}(\gamma(r)) &= \rho(r)g_{ia}(\gamma(s)), \\ (\varphi_B^*g)_{ab}(\gamma(r)) &= \rho(r)^2g_{ab}(\gamma(s)), & (\varphi_B^*g)_{nn}(\gamma(r)) &= (\rho'(r)r + \rho(r))^2g_{nn}(\gamma(s)), \end{aligned}$$

and the result follows from (2.6) and the partial conformality of  $\varphi_B$ .  $\square$

Since the submanifold  $B$  remains fixed under the geodesic transformation, the function  $s(r)$  in (2.1) satisfies  $s(0) = 0$  and further, using the derived expressions for the components of the metric tensor, we get the following

**LEMMA 2.2.** *For any partially conformal transformation  $\varphi_B$  with respect to a submanifold  $B$ , we have  $f(0) = 0$ . Moreover, if  $\dim B \geq 1$ , then  $s'(0)^2 = 1$ .*

**PROOF.** Let  $u \in T_m^\perp B$  and consider an adapted system of Fermi coordinates. From the conditions in Lemma 2.1 we obtain

$$(2.9) \quad \rho g_{qq+1}(s) = e^{2\sigma} g_{qq+1}(r) + f(r)(\eta \otimes \eta)_{qq+1}(r)$$

and taking limits for  $r \rightarrow 0$ , we get

$$s'(0)\delta_{qq+1} = s'(0)^2\delta_{qq+1} - f(0)dc,$$

which shows that  $f(0) = 0$  unless  $cd = 0$ . If  $d = 0$ , from Lemma 2.1 we get

$$(2.10) \quad \rho^2 g_{q+1q+1}(s) = e^{2\sigma} g_{q+1q+1}(r) + f(r)(\eta \otimes \eta)_{q+1q+1}(r)$$

and taking limits for  $r \rightarrow 0$ , we obtain  $s'(0)^2 = s'(0)^2 + f(0)$  which shows that  $f(0) = 0$ . (Note that this case shows also that  $f(0) = 0$  when  $B$  reduces to a single point.) Next, suppose  $c = 0$ , that is,  $Ju$  is tangent to  $B$ . Then we have

$$(2.11) \quad g_{ij}(s) = e^{2\sigma} g_{ij}(r) + f(r)(\eta \otimes \eta)_{ij}(r).$$

Once again, taking limits for  $r \rightarrow 0$ , one obtains

$$\delta_{ij} = s'(0)^2\delta_{ij} + f(0)\delta_{iq}\delta_{jq}$$

and for  $\dim B \geq 2$ , we must have  $s'(0)^2 = 1$  and hence,  $f(0) = 0$ .

Next, we show that  $s'(0)^2 = 1$  for  $\dim B \geq 1$ . First, we obtain  $f(0) = 0$  since  $\dim B \geq 2$  or  $\dim B = 1$  and  $\dim M > 2$ . Then, from this, (2.11) and by taking limits for  $r \rightarrow 0$ , it follows that  $\delta_{qq} = s'(0)^2\delta_{qq}$ , which shows the desired result.  $\square$

In what follows, we put

$$(2.12) \quad g_{\delta\theta}(\exp_m(ru)) = \sum_{k \geq 0} \alpha_k(m, u, \delta, \theta) r^k$$

for the power series expansions of the components  $g_{\delta\theta}$  of the metric tensor  $g$  along the normal geodesic  $\gamma(r) = \exp_m(ru)$ , where  $\delta, \theta \in \{1, 2, \dots, n-1\}$ . Then, from the Jacobi equation (2.5), the initial conditions (2.3) and the expressions (2.7), one gets the following explicit description of the first few terms appearing in (2.12):

$$(2.13) \quad \begin{aligned} g_{ij}(\exp_m(ru)) &= g(E_i, E_j)(m) + 2rg(T(u)E_i, E_j)(m) + O(r^2), \\ g_{ia}(\exp_m(ru)) &= -rg({}^t\perp(u)E_i, E_a)(m) \\ &\quad - \frac{2}{3}r^2g(R(u)E_i, E_a)(m) + O(r^3), \\ g_{ab}(\exp_m(ru)) &= g(E_a, E_b)(m) - \frac{1}{3}r^2g(R(u)E_a, E_b)(m) + O(r^3). \end{aligned}$$

Also, for the one-form  $\eta$  we shall write the formal power series expansion

$$(2.14) \quad \eta\left(\frac{\partial}{\partial x^\delta}\right)(\exp_m(ru)) = \sum_{k \geq 0} \eta_k(m, u, \delta) r^k.$$

Finally, let

$$(2.15) \quad s(r) = \sum_{k \geq 1} \beta_k r^k, \quad f(r) = \sum_{k \geq 1} \frac{1}{k!} f^{(k)}(0) r^k$$

be the power series expansions of the functions  $s(r)$  and  $f(r)$  along the geodesic  $\gamma$ , where  $\beta_k = (1/k!)s^{(k)}(0)$ .

It is clear that the identity transformation satisfies  $s'(0) = 1$ . We shall show that the identity is the only partially conformal geodesic transformation such that  $s'(0) = 1$  holds. To prove this, we consider the following two cases:  $\dim B \geq 1$  (see Theorem 2.1 below) or  $B$  is a single point (see Theorem 3.1).

**THEOREM 2.1.** *Let  $\varphi_B$  be a partially conformal geodesic transformation with respect to a submanifold  $B$  with  $\dim B \geq 1$ . Then  $s'(0) = -1$  unless  $\varphi_B$  is the identity transformation.*

**PROOF.** From the previous lemma we have  $s'(0)^2 = 1$ . Now we show that for  $s'(0) = 1$ ,  $\varphi_B$  is the identity transformation. We proceed by induction. First, we prove that  $s''(0) = 0$ ,  $f'(0) = 0$ .



Since

$$(2.16) \quad \rho^2 g_{ab}(s) = e^{2\sigma} g_{ab}(r) + f(r)(\eta \otimes \eta)_{ab}(r)$$

and  $\alpha_1(m, u, a, b) = 0$  (see (2.13)), we obtain the expansion

$$(2.17) \quad \begin{aligned} \delta_{ab} + s''(0)\delta_{ab}r + O(r^2) \\ = \delta_{ab} + (2s''(0)\delta_{ab} + c^2 f'(0)\delta_{aq+1}\delta_{bq+1})r + O(r^2). \end{aligned}$$

Also, from Lemma 2.1 we have

$$(2.18) \quad g_{ij}(s) = e^{2\sigma} g_{ij}(r) + f(r)(\eta \otimes \eta)_{ij}(r)$$

and hence,

$$(2.19) \quad \begin{aligned} \delta_{ij} + \alpha_1(m, u, i, j)r + O(r^2) \\ = \delta_{ij} + (2s''(0)\delta_{ij} + \alpha_1(m, u, i, j) + d^2 f'(0)\delta_{iq}\delta_{jq})r + O(r^2). \end{aligned}$$

Similarly, from

$$(2.20) \quad \rho g_{ia}(s) = e^{2\sigma} g_{ia}(r) + f(r)(\eta \otimes \eta)_{ia}(r)$$

we get the expansion

$$(2.21) \quad \alpha_1(m, u, i, a)r + O(r^2) = (\alpha_1(m, u, i, a) - cd f'(0)\delta_{iq}\delta_{aq+1})r + O(r^2).$$

Now, if  $Ju$  has tangential and normal component, it follows from (2.21) that  $f'(0) = 0$  and from (2.19) we then get  $s''(0) = 0$ . Next, suppose that  $Ju$  is normal to  $B$ . Then from (2.19) we obtain  $s''(0) = 0$  and it follows then from (2.17) that  $f'(0)$  also vanishes. Finally, suppose that  $Ju$  is tangent to  $B$ . From (2.17) it follows that  $s''(0)\delta_{ab} = 2s''(0)\delta_{ab}$  and hence, if  $\text{codim } B \geq 2$ ,  $s''(0) = 0$ . In this case,  $f'(0) = 0$  as a consequence of (2.19).

Next, consider the case of a real hypersurface  $B$ . From (2.19) we obtain  $2s''(0)\delta_{ij} = -f'(0)\delta_{iq}\delta_{jq}$ . Since  $\dim B \geq 3$ , it follows that  $s''(0) = 0$  and  $f'(0) = 0$ .

Now, we proceed by induction. We suppose that the coefficients in the power series expansions of the functions  $s(r)$  and  $f(r)$  satisfy

$$\begin{aligned} \beta_2 = \dots = \beta_{k-1} = 0, \\ f'(0) = \dots = f^{(k-2)}(0) = 0 \end{aligned}$$

and prove that  $\beta_k = 0$  and  $f^{(k-1)}(0) = 0$ . From (2.16), (2.18) and (2.20), using the induction hypothesis, we get the expansions

$$\begin{aligned}
(2.22) \quad & \delta_{ab} \sum_{l=1}^{k-2} \alpha_l(m, u, a, b) r^l + (2k\beta_k \delta_{ab} + \alpha_{k-1}(m, u, a, b)) \\
& + \frac{c^2}{(k-1)!} f^{(k-1)}(0) \delta_{aq+1} \delta_{bq+1} r^{k-1} + O(r^k) \\
& = \delta_{ab} + \sum_{l=1}^{k-2} \alpha_l(m, u, a, b) r^l + (2\beta_k \delta_{ab} + \alpha_{k-1}(m, u, a, b)) r^{k-1} + O(r^k),
\end{aligned}$$

$$\begin{aligned}
(2.23) \quad & \delta_{ij} + \sum_{l=1}^{k-2} \alpha_l(m, u, i, j) r^l + (2k\beta_k \delta_{ij} + \alpha_{k-1}(m, u, i, j)) \\
& + \frac{d^2}{(k-1)!} f^{(k-1)}(0) \delta_{iq} \delta_{jq} r^{k-1} + O(r^k) \\
& = \delta_{ij} + \sum_{l=1}^{k-2} \alpha_l(m, u, i, j) r^l + \alpha_{k-1}(m, u, i, j) r^{k-1} + O(r^k),
\end{aligned}$$

$$\begin{aligned}
(2.24) \quad & \sum_{l=1}^{k-2} \alpha_l(m, u, i, a) r^l + (\alpha_{k-1}(m, u, i, a)) \\
& - \frac{cd}{(k-1)!} f^{(k-1)}(0) \delta_{iq} \delta_{aq+1} r^{k-1} + O(r^k) \\
& = \sum_{l=1}^{k-2} \alpha_l(m, u, i, a) r^l + \alpha_{k-1}(m, u, i, a) r^{k-1} + O(r^k).
\end{aligned}$$

Considering the terms of degree  $k-1$  in the previous expansions, and proceeding in the same way as before, we obtain that  $\beta_k = 0$  and  $f^{(k-1)}(0) = 0$ . Hence, from the analyticity assumption of  $s$ , it follows that  $s(r) = r$  and hence  $\varphi_B$  is the identity.  $\square$

In the rest of this paper,  $\varphi_B$  will always denote a *non-trivial* geodesic transformation, that is,  $\varphi_B$  is not the identity map.

In what follows we shall show that the study of partially conformal geodesic transformations with respect to a submanifold  $B$  is reduced to that of isometric geodesic reflections, provided that  $0 < \dim B < \dim M - 1$ .

Isometric geodesic reflections with respect to submanifolds are studied in [4] where it is shown that submanifolds admitting such geodesic transformations are necessarily totally geodesic. Now we shall determine some necessary conditions

for the existence of partially conformal geodesic transformations relating to the extrinsic geometry of the submanifold  $B$  in  $M$ .

**THEOREM 2.2.** *Let  $(M, g, J)$  be an almost Hermitian manifold and  $B$  an arbitrary submanifold with  $\dim B \geq 1$ . If there exists a non-trivial partially conformal geodesic transformation with respect to  $B$ , then*

- (i)  $B$  is a totally geodesic submanifold, or
- (ii)  $B$  is a totally umbilical real hypersurface, or
- (iii)  $B$  is a real hypersurface with two distinct constant principal curvatures, where that with multiplicity one corresponds to the principal direction  $JN$ .

**PROOF.** Since the geodesic transformation is non-trivial, we have  $s'(0) = -1$ . Also, from the conditions in Lemma 2.1 we have

$$g_{ij}(s) = e^{2\sigma} g_{ij}(r) + f(r)(\eta \otimes \eta)_{ij}(r).$$

Using the power series expansions of the components of the metric tensor (2.13), we have

$$\delta_{ij} - 2rT_{ij}(m) + O(r^2) = \delta_{ij} + (2T_{ij}(m) - 2s''(0)\delta_{ij} + d^2 f'(0)\delta_{iq}\delta_{jq})r + O(r^2)$$

and hence, the shape operator satisfies

$$(2.25) \quad T(u) = \frac{1}{2} \left( s''(0)I - \frac{1}{2} f'(0)d^2 \eta \otimes Ju \right).$$

If  $B$  is a real hypersurface, then  $d = 1$  and it is totally umbilical or it has two distinct constant principal curvatures, namely  $k_1 = (1/2)s''(0)$  and  $k_2 = (1/2)(s''(0) - (1/2)f'(0))$ , the latter with multiplicity one and corresponding to the principal direction  $Ju$ .

Next, show that  $B$  is a totally geodesic submanifold provided that  $\text{codim } B > 1$ . Using the relation

$$\rho^2 g_{ab}(s) = e^{2\sigma} g_{ab}(r) + f(r)(\eta \otimes \eta)_{ab}(r)$$

of Lemma 2.1, we obtain

$$\delta_{ab} - s''(0)\delta_{ab}r + O(r^2) = \delta_{ab} - (2s''(0)\delta_{ab} - c^2 f'(0)\delta_{aq+1}\delta_{bq+1})r + O(r^2),$$

from which we get

$$(2.26) \quad s''(0) = c^2 f'(0)\delta_{aq+1}.$$

First we show that  $B$  is totally geodesic if  $\text{codim } B > 2$ . In this case, it is

possible to choose  $E_a \in T_m^\perp B$  such that  $g(E_a, u) = 0$ ,  $g(E_a, Ju) = 0$  and hence, it follows from (2.26) that  $s''(0) = 0$ . Also, from (2.26) and if  $c \neq 0$ , it follows that  $f'(0) = 0$  and hence, (2.25) shows that  $B$  is totally geodesic. Furthermore, suppose that  $c = 0$  for each normal vector  $u \in T_m^\perp B$ , that is, for each normal vector  $u$ ,  $Ju$  is tangent to  $B$ . Hence, (2.25) becomes

$$(2.27) \quad T(u)X = -\frac{1}{4}f'(0)g(X, Ju)Ju$$

for each vector  $X \in T_m B$ . Then take orthogonal unit vectors  $u, v \in T_m^\perp B$ . Since  $Ju, Jv$  are tangent to  $B$ , it follows from (2.27) that

$$T\left(\frac{1}{\sqrt{2}}(u+v)\right)J\left(\frac{1}{\sqrt{2}}(u+v)\right) = -\frac{1}{4}f'(0)\frac{1}{\sqrt{2}}J(u+v).$$

Expanding the left-hand side of this equation and using again (2.27) gives

$$T\left(\frac{1}{\sqrt{2}}(u+v)\right)J\left(\frac{1}{\sqrt{2}}(u+v)\right) = -\frac{1}{8}f'(0)J(u+v).$$

Comparing both expressions above,  $f'(0) = 0$  follows at once and hence,  $B$  is totally geodesic.

For the remaining case  $\text{codim } B = 2$  we show that  $B$  is totally geodesic too. From Lemma 2.1, we have

$$\rho g_{ia}(s) = e^{2\sigma} g_{ia}(r) + f(r)(\eta \otimes \eta)_{ia}(r)$$

and hence, the expansion

$$-g({}^t \perp(u)E_i, E_a)r + O(r^2) = -(g({}^t \perp(u)E_i, E_a) + dc f'(0)\delta_{iq}\delta_{aq+1})r + O(r^2).$$

Considering the terms of degree one, we obtain

$$(2.28) \quad cd f'(0) = 0.$$

If  $B$  is a holomorphic submanifold, it follows from (2.25) that  $B$  must be totally umbilical with shape operator  $T(u) = (1/2)s''(0)Id$ . Proceeding as in [13, Theorem 3.1], it follows that  $B$  is totally geodesic.

Next, suppose that  $B$  is not a holomorphic submanifold and take an orthonormal basis  $\{u, v\}$  of  $T_m^\perp B$ . Let  $c_u$  (resp.  $c_v$ ) and  $d_u$  (resp.  $d_v$ ) be the norm of the normal and tangential components of  $Ju$  (resp.  $Jv$ ). Since  $B$  is not holomorphic,  $d_u$  and  $d_v$  cannot both be zero. Put  $d_u \neq 0$ .

If  $c_u = 0$ , it follows from (2.26) that  $s''(0) = 0$ . Now, if  $c_v = 0$ , then both  $Ju, Jv$  are tangent vectors to  $B$  and then, in the same way as for the case of

$\text{codim } B > 2$ , we get  $f'(0) = 0$ . So, suppose  $c_v \neq 0$ . If  $d_v \neq 0$ , then from (2.28) it follows that  $f'(0) = 0$  and  $B$  is totally geodesic. If  $d_v = 0$ , then  $Jv$  is normal to  $B$ . Since we are assuming that  $\text{codim } B = 2$ ,  $Jv$  must be in the direction of  $u$ . This shows that  $Ju$  cannot have a tangential component to  $B$ , which contradicts  $d_u \neq 0$ .

Finally, if  $c_u \neq 0$ , (2.28) yields  $f'(0) = 0$ . Then (2.25) implies that  $B$  is totally umbilical with shape operator  $T(u) = (1/2)s''(0)Id$ , and in the same way as in [13, Theorem 3.1] we get that  $B$  is totally geodesic. This finishes the proof.  $\square$

In this context we recall the following definition.

**DEFINITION 2.3.** Let  $B$  be a real hypersurface in an almost Hermitian manifold  $(M, g, J)$ .  $B$  is said to be a *Hopf hypersurface* if  $JN$  is a principal direction of  $B$ ,  $N$  being a unit normal vector.

Hopf hypersurface form a nice class of real hypersurfaces in almost Hermitian spaces. Indeed, they are the only real hypersurfaces with two distinct constant principal curvatures in non-flat complex space forms. We refer to Takagi [19] for a classification of such hypersurfaces in the complex projective space and to Montiel [17] for the hyperbolic case.

Now, we state the main theorem of this section. It shows that only the partially conformal geodesic transformations with respect to points and real hypersurfaces are essential.

**THEOREM 2.3.** *Let  $B$  be a  $q$ -dimensional submanifold in an almost Hermitian manifold  $(M^n, g, J)$ . If  $\varphi_B$  is a partially conformal geodesic transformation with respect to  $B$ , then it must be the identity or the geodesic reflection provided  $q$  satisfies  $0 < q < n - 1$ .*

**PROOF.** Let  $\varphi_B$  be a non-trivial geodesic transformation with respect to  $B$ . Since  $\text{codim } B \geq 2$ , it follows from the previous theorem that  $B$  must be a totally geodesic submanifold, and moreover from (2.25) it follows that  $s''(0) = 0$ ,  $f'(0) = 0$ . Next, we proceed by induction. We suppose that

$$s''(0) = \dots = s^{(k-1)}(0) = 0,$$

$$f'(0) = \dots = f^{(k-2)}(0) = 0$$

and prove that  $s^{(k)}(0) = 0$ ,  $f^{(k-1)}(0) = 0$ . Then, from the analyticity conditions, it will follow that  $s(r) = -r$  and hence,  $\varphi_B$  must be the geodesic reflection. Note also that this condition shows that the function  $f(r)$  vanishes identically. Hence, the geodesic transformation  $\varphi_B$  must be conformal and from [13, Theorem 3.2],  $\varphi_B$  is an isometric transformation.

Using the induction hypothesis, from (2.16) we have the expansion

$$\begin{aligned}
 (2.29) \quad \delta_{ab} &+ \sum_{l=1}^{k-2} \alpha_l(m, u, a, b) r^l + (\alpha_{k-1}(m, u, a, b) - 2k\beta_k \delta_{ab} \\
 &+ \frac{c^2}{(k-1)!} f^{(k-1)}(0) \delta_{aq+1} \delta_{bq+1}) r^{k-1} + O(r^k) \\
 &= \delta_{ab} + \sum_{l=1}^{k-2} \alpha_l(m, u, a, b) (-1)^l r^l \\
 &+ ((-1)^{k-1} \alpha_{k-1}(m, u, a, b) - 2\beta_k \delta_{ab}) r^{k-1} + O(r^k)
 \end{aligned}$$

and considering the terms of degree  $(k-1)$ , it follows that

$$\begin{aligned}
 (2.30) \quad 2(k-1)\beta_k \delta_{ab} &= (1 - (-1)^{k-1}) \alpha_{k-1}(m, u, a, b) \\
 &+ \frac{c^2}{(k-1)!} f^{(k-1)}(0) \delta_{aq+1} \delta_{bq+1}.
 \end{aligned}$$

Also, if  $\varphi_B$  is partially conformal, (2.18) must hold and hence,

$$\begin{aligned}
 (2.31) \quad \delta_{ij} &+ \sum_{l=1}^{k-2} \alpha_l(m, u, i, j) r^l + \left( \alpha_{k-1}(i, j) - 2k\beta_k \delta_{ij} \right. \\
 &+ \left. \frac{d^2}{(k-1)!} f^{(k-1)}(0) \delta_{iq} \delta_{jq} \right) r^{k-1} + O(r^k) \\
 &= \delta_{ij} + \sum_{l=1}^{k-2} \alpha_l(m, u, a, b) (-1)^l r^l \\
 &+ (-1)^{k-1} \alpha_{k-1}(m, u, i, j) r^{k-1} + O(r^k)
 \end{aligned}$$

and considering the terms of degree  $(k-1)$ , it follows that

$$(2.32) \quad 2k\beta_k \delta_{ij} = (1 - (-1)^{k-1}) \alpha_{k-1}(m, u, i, j) + \frac{d^2}{(k-1)!} f^{(k-1)}(0) \delta_{iq} \delta_{jq}.$$

Furthermore, from (2.20) and using the induction hypothesis, we also have

$$\begin{aligned}
 (2.33) \quad & \sum_{l=1}^{k-2} \alpha_l(m, u, i, a) r^l + (\alpha_{k-1}(m, u, i, a) \\
 & - \frac{cd}{(k-1)!} f^{(k-1)}(0) \delta_{iq} \delta_{aq+1}) r^{k-1} + O(r^k) \\
 & = \sum_{l=1}^{k-2} \alpha_l(m, u, i, a) (-1)^{l+1} r^l + (-1)^k \alpha_{k-1}(m, u, i, a) r^{k-1} + O(r^k)
 \end{aligned}$$

and hence,

$$(2.34) \quad (1 - (-1)^k) \alpha_{k-1}(m, u, i, a) = \frac{cd}{(k-1)!} f^{(k-1)}(0) \delta_{iq} \delta_{aq+1}.$$

In order to show that  $\beta_k = 0$ , we consider the following possibilities:

*Case 1.* There exists a unit vector  $u \in T_m^\perp B$  such that  $Ju \in T_m^\perp B$ .

Considering an adapted system of Fermi coordinates and since  $d_u = 0$ , we obtain from (2.32)

$$2k\beta_k = (1 - (-1)^{k-1}) \alpha_{k-1}(m, u, i, i).$$

Hence, if  $k$  is an odd number, say  $k = 2l + 1$ , we must have  $\beta_k = 0$ . Furthermore, suppose  $k = 2l$ . Then one gets

$$(2.35) \quad \beta_{2l} = \frac{1}{2l} \alpha_{2l-1}(m, u, i, i).$$

Consider the one-parameter family of unit normals  $\omega_\lambda = u \cos \lambda + Ju \sin \lambda$ . Since  $J(Ju)$  is normal,  $d_{Ju} = 0$  and condition (2.35) remains valid for any normal unit-speed geodesic  $\gamma_\lambda(r) = \exp_m(r\omega_\lambda)$ . So,  $\beta_{2l} = (1/2l) \alpha_{2l-1}(m, \omega_\lambda, i, i)$ . Taking the limit for  $\lambda \rightarrow \pi$ , it follows that

$$\beta_{2l} = \frac{1}{2l} \alpha_{2l-1}(m, u, i, i) = \frac{1}{2l} \alpha_{2l-1}(m, -u, i, i).$$

Since  $\alpha_{2l-1}(m, -u, i, i) = -\alpha_{2l-1}(m, u, i, i)$ , we get  $\alpha_{2l-1}(m, u, i, i) = 0$  and hence,  $\beta_{2l} = 0$ .

Next, we show that  $f^{(k-1)}(0) = 0$ . Since  $d_u = 0$ , we have  $c_u = 1$  and (2.30) yields

$$(2.36) \quad \frac{1}{(k-1)!} f^{(k-1)}(0) = ((-1)^{k-1} - 1) \alpha_{k-1}(m, u, q+1, q+1).$$

Then, it is clear that  $f^{(k-1)}(0) = 0$  for odd  $k$ . For even  $k = 2l$ , one gets

$$\frac{-1}{2(2l-1)!} f^{(2l-1)}(0) = \alpha_{2l-1}(m, u, q+1, q+1).$$

Now, since  $Ju$  is also normal to  $B$ , for each normal  $z_{\lambda\mu} = \lambda u + \mu Ju$ , it follows that

$$\frac{-1}{2(2l-1)!} (\nabla_{z_{\lambda\mu}, \dots, z_{\lambda\mu}}^{(2l-1)} f)(0) = \alpha_{2l-1}(m, z_{\lambda\mu}, Jz_{\lambda\mu}, Jz_{\lambda\mu}).$$

Both sides are polynomials in  $\lambda$  and  $\mu$ . Comparing coefficients yields  $\alpha_{2l-1}(m, u, Ju, Ju) = 0$  and hence,  $f^{(2l-1)}(0)$  vanishes.

*Case 2.* For each normal vector  $u \in T_m^\perp B$ ,  $d_u \neq 0$ .

Consider the following possibilities:

**(2.a)** for each  $u \in T_m^\perp B$ ,  $Ju$  is tangent to  $B$ , that is,  $c_u = 0$ .

Since  $\text{codim } B \geq 2$ , take orthogonal unit vectors  $u, v \in T_m^\perp B$  and denote  $E_{q(u)} = Ju$ ,  $E_{q(v)} = Jv$ . Considering an adapted system of Fermi coordinates, it follows from (2.32) that

$$2k\beta_k = (1 - (-1)^{k-1})\alpha_{k-1}(m, u, E_{q(v)}, E_{q(v)}).$$

This shows that  $\beta_k = 0$  for  $k = 2l + 1$ . Also, if  $k = 2l$ , one gets  $\beta_{2l} = (1/2l)\alpha_{2l-1}(m, u, E_{q(v)}, E_{q(v)})$ . Consider now the one-parameter family of unit normals

$$\omega_\lambda = u \cos \lambda + v \sin \lambda, \quad z_\lambda = -u \sin \lambda + v \cos \lambda.$$

For each value of  $\lambda$ ,  $\omega_\lambda$  and  $z_\lambda$  are orthogonal unit vectors in  $T_m^\perp B$  and for the normal geodesic  $\gamma_\lambda(r) = \exp_m(r\omega_\lambda)$  we have

$$\beta_{2l} = \frac{1}{2l} \alpha_{2l-1}(m, \omega_\lambda, E_{q(z_\lambda)}, E_{q(z_\lambda)}).$$

Taking the limit for  $\lambda \rightarrow \pi$ , we obtain  $\beta_{2l} = (1/2l)\alpha_{2l-1}(m, -u, -E_{q(v)}, -E_{q(v)})$ , and this shows that  $\alpha_{2l-1}(m, u, E_{q(v)}, E_{q(v)}) = \alpha_{2l-1}(m, -u, E_{q(v)}, E_{q(v)})$ . Hence,  $\alpha_{2l-1}(m, u, E_{q(v)}, E_{q(v)}) = 0$  and so,  $\beta_{2l} = 0$ .

To show that  $f^{(k-1)}(0) = 0$ , we use (2.32) to get

$$(2.37) \quad \frac{1}{(k-1)!} f^{(k-1)}(0) = ((-1)^{k-1} - 1)\alpha_{k-1}(m, u, E_{q(u)}, E_{q(u)}),$$

and the result follows proceeding as in the previous case.



(2.b) There exists a unit normal  $u \in T_m^\perp B$  such that  $d_u \neq 0$ ,  $c_u \neq 0$ .

Considering an adapted system of Fermi coordinates, from (2.34) we get

$$(2.38) \quad \frac{c_u d_u}{(k-1)!} f^{(k-1)}(0) = (1 - (-1)^k) \alpha_{k-1}(m, u, E_{q(u)}, E_{q+1(u)}).$$

This shows that  $f^{(k-1)}(0)$  vanishes for even  $k$ . If  $k$  is odd, say  $k = 2l + 1$ , we obtain from (2.30) and (2.32)

$$4l\beta_{2l+1}\delta_{ab} = \frac{c_u^2}{(2l)!} f^{(2l)}(0) \delta_{aq+1} \delta_{bq+1}$$

and

$$2(2l+1)\beta_{2l+1}\delta_{ij} = \frac{d_u^2}{(2l)!} f^{(2l)}(0) \delta_{iq} \delta_{jq}.$$

Since  $\text{codim } B \geq 2$  and  $\text{dim } M \geq 4$ , it follows that  $\beta_{2l+1} = 0$  and hence,  $f^{(2l)}(0) = 0$ . This shows that  $f^{(k-1)}(0)$  vanishes. Also, since we have shown that  $\beta_{2l+1} = 0$ , we only have to prove that  $\beta_k = 0$  for even  $k = 2l$ . From (2.30) and (2.32), we have

$$(2l-1)\beta_{2l} = \alpha_{2l-1}(m, v, a, a), \quad 2l\beta_{2l} = \alpha_{2l-1}(m, v, i, i)$$

for all normal  $v \in T_m^\perp B$  and hence, the result follows as in the previous cases.  $\square$

In [13], it is shown that if a geodesic transformation with respect to a submanifold is an isometry, then it must be the identity or the geodesic reflection. Moreover, it is shown that the geodesic reflection is conformal if and only if it is isometric. In what remains in this section, we show a similar result for partially conformal geodesic reflections.

**PROPOSITION 2.1.** *Let  $B^q$  be a submanifold in an almost Hermitian manifold  $(M^n, g, J)$  with  $0 < q < n - 1$ . Then the geodesic reflection with respect to  $B$  is partially conformal if and only if it is an isometry.*

**PROOF.** As a consequence of the induction process in the proof of the previous theorem, the function  $f(r)$  vanishes identically, and this shows that the geodesic reflection is partially conformal if and only if it is conformal. Hence the result follows from [13, Theorem 3.2].  $\square$

### 3. Transformations with Respect to Points

In this section, we shall derive the necessary and sufficient conditions for a geodesic transformation  $\varphi_m$  with respect to a point  $m \in M$  to be partially conformal. Note that, if the submanifold  $B$  reduces to  $B = \{m\}$ , the system of Fermi coordinates in Section 2 becomes a system of normal coordinates in a normal neighborhood of  $m$ .

Unlike for the case of higher dimensional submanifolds, for partially conformal geodesic transformations with respect to points we do not have a fixed initial condition  $s'(0)^2 = 1$ . In fact, the following theorem shows that such condition occurs only for isometric transformations.

**THEOREM 3.1.** *Let  $\varphi_m$  be a partially conformal geodesic transformation with respect to a point  $m \in M$ . Then  $\varphi_m$  is an isometry if and only if  $s'(0)^2 = 1$  and this occurs if  $\varphi_m$  is the identity ( $s'(0) = 1$ ) or the geodesic reflection ( $s'(0) = -1$ ).*

**PROOF.** If  $s'(0)^2 = 1$ , then  $s'(0) = 1$  or  $s'(0) = -1$ . Clearly, the identity transformation satisfies  $s'(0) = 1$ , and moreover, proceeding as for Theorem 2.1, it follows that the identity is the only partially conformal geodesic transformation satisfying  $s'(0) = 1$ .

Next, assume  $s'(0) = -1$ . We show that  $\varphi_m$  is the geodesic reflection. From the conditions in Lemma 2.1 and using the fact that  $\alpha_1(m, u, a, b) = 0$  (see (2.13)), one gets  $s''(0)\delta_{ab} = f'(0)\delta_{a1}\delta_{b1}$  and hence,  $s''(0) = 0$ ,  $f'(0) = 0$ .

Now we proceed by induction and assume that

$$\beta_2 = \dots = \beta_{k-1} = 0, \quad f'(0) = \dots = f^{(k-2)}(0) = 0.$$

We shall prove that  $\beta_k = 0$ ,  $f^{(k-1)}(0) = 0$ . Proceeding as in Theorem 2.3, it follows that (see (2.30))

$$2(k-1)\beta_k\delta_{ab} = (1 - (-1)^{k-1})\alpha_{k-1}(m, u, a, b) + \frac{1}{(k-1)!}f^{(k-1)}\delta_{a1}\delta_{b1}.$$

This yields that  $\beta_k = 0$ ,  $f^{(k-1)}(0) = 0$  for odd  $k$  and, if we suppose  $k$  to be even, say  $k = 2l$ , we get

$$2(2l-1)\beta_k = 2\alpha_{2l-1}(m, u, a, a) + \frac{1}{(k-1)!}f^{(k-1)}(0)\delta_{a1}\delta_{b1}.$$

Proceeding further as for Theorem 2.3, it follows that  $\beta_k = 0$  and  $f^{(k-1)}(0) = 0$ , and then we obtain  $s(r) = -r$ . So,  $\varphi_m$  is the geodesic reflection. Also, it follows that  $f(r)$  vanishes, and hence, the geodesic reflection is a conformal trans-

formation. Thus, it must be an isometry as a consequence of the results in [13]. □

The geometrical significance of the existence of a partially conformal geodesic transformation with respect to a point is expressed by the mutual existing relations between the coefficients in the power series expansions of the components  $g_{ab}$  of the metric tensor and those of the one-form  $\eta$ , jointly with those of the functions  $s(r)$  and  $f(r)$ . The next lemma expresses such relation by means of a recursion formula. Its proof is obtained directly from the conditions in Lemma 2.1, using the power series expansions (2.12), (2.14) and (2.15).

**LEMMA 3.1.** *Let  $\varphi_m$  be a geodesic transformation with respect to a point  $m \in M$ . Then  $\varphi_m$  is partially conformal if and only if the coefficients in the power series expansion of the function  $s(r)$  satisfy the following recurrence formula:*

$$\begin{aligned} & \beta_1^2(1 - \beta_1^k)\alpha_k(m, u, a, b) \\ &= \delta_{ab} \left( \sum_{p+q=k+2} (1 - pq)\beta_p\beta_q \right) + \beta_1^2 \sum_{l=1}^{k-1} \alpha_l(m, u, a, b) \left( \sum_{p_1+\dots+p_l=k} \beta_{p_1} \cdots \beta_{p_l} \right) \\ & \quad - \sum_{l=1}^{k-1} \alpha_{k-l}(m, u, a, b) \left( \sum_{p+q=l+2} pq\beta_p\beta_q \right) \\ & \quad + \sum_{l=1}^{k-1} \left( \sum_{p+q=l+2} \beta_p\beta_q \right) \left( \sum_{v \geq 1} \alpha_v(m, u, a, b) \left( \sum_{p_1+\dots+p_v=k-l} \beta_{p_1} \cdots \beta_{p_v} \right) \right) \\ & \quad + \sum_{l+t+v=k} \left( \frac{1}{l!} f^{(l)}(0)\eta_t(m, u, a)\eta_v(m, u, b) \right) \end{aligned}$$

for all  $a, b \in \{1, 2, \dots, n - 1\}$ .

As a direct consequence of the previous expression, we have the following necessary and sufficient conditions for the existence of partially conformal geodesic transformations, expressed in terms of the Jacobi operators and their derivatives.

**THEOREM 3.2.** *Let  $(M, g, J)$  be an almost Hermitian manifold such that there exists a non-isometric partially conformal geodesic transformation with respect to a point  $m \in M$ . Then the derivatives of the Jacobi operator satisfy*

$$(3.1) \quad \begin{cases} R^{(2k+1)}(m) = 0, \\ R^{(2k)}(m) = \begin{pmatrix} c_1(m, k) & 0 \\ 0 & c_2(m, k)I_{n-2} \end{pmatrix} \end{cases}$$

for some real numbers  $c_1, c_2$  depending only on the base point  $m \in M$  and the order  $k \geq 0$ . Moreover, if  $\varphi_m$  is a conformal transformation, then  $c_1(m, k) = c_2(m, k)$  for all  $k \geq 0$ . Conversely,

(i) if (3.1) holds with  $c_1(m, k) = c_2(m, k)$  and for all  $k \geq 0$ , then there exist infinitely many geodesic conformal transformations with respect to  $m$ ;

(ii) if (3.1) holds and  $c_1(m, k) \neq c_2(m, k)$  for some  $k$ , then there exist infinitely many non-conformal partially conformal geodesic transformations with respect to  $m$ , provided that  $(M, g, J)$  is a nearly Kähler manifold.

**PROOF.** First we show that (3.1) are necessary conditions for the existence of a non-isometric partially conformal geodesic transformation with respect to  $m$ . If  $\varphi_m$  is conformal, the result is shown in [13, Theorem 4.1]. So, we suppose that  $\varphi_m$  is a non-conformal partially conformal geodesic transformation with respect to  $m$ . This occurs if and only if the function  $f$  in (2.2) does not vanish identically. Hence, assume  $f^{(k_0)}(0)$  to be the first non-vanishing derivative of  $f(r)$  at the point  $r = 0$ .

As a first step, we show that the coefficients in the power series expansion of the components of the metric tensor are independent of the direction  $u \in T_m M$ , and furthermore, that they satisfy

$$(3.2) \quad \begin{cases} \alpha_k(m, u, a, b) = 0, & a, b \in \{1, \dots, n-1\}, \quad a \neq b, \\ \alpha_k(m, u, a, a) = \alpha_k(m, u, b, b), & a, b \in \{2, \dots, n-1\} \end{cases}$$

for all  $k \geq 0$ . (Note that  $E_1 = Ju$ ,  $E_n = u$ .)

So, let  $f^{(k_0)}(0)$  be the first non-vanishing derivative of  $f(r)$ . Then the expression in Lemma 3.1 yields

$$\begin{aligned} & \beta_1^2(1 - \beta_1^k)\alpha_k(m, u, a, b) \\ &= \delta_{ab} \left( \sum_{p+q=k+2} (1 - pq)\beta_p\beta_q \right) \\ &+ \beta_1^2 \sum_{l=1}^{k-1} \alpha_l(m, u, a, b) \left( \sum_{p_1+\dots+p_l=k} \beta_{p_1} \cdots \beta_{p_l} \right) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{l=1}^{k-1} \alpha_{k-l}(m, u, a, b) \left( \sum_{p+q=l+2} pq\beta_p\beta_q \right) \\
 & + \sum_{l=1}^{k-1} \left( \sum_{p+q=l+2} \beta_p\beta_q \right) \left( \sum_{v \geq 1} \alpha_v(m, u, a, b) \left( \sum_{p_1+\dots+p_v=k-l} \beta_{p_1} \cdots \beta_{p_v} \right) \right)
 \end{aligned}$$

for all  $k < k_0$ . Hence, in the same way as in the proof of Theorem 4.1 in [13], we obtain

$$(3.3) \quad \begin{cases} \alpha_k(m, u, a, b) = 0, & a, b \in \{1, \dots, n-1\}, \quad a \neq b, \\ \alpha_k(m, u, a, a) = \alpha_k(m, u, b, b), & a, b \in \{2, \dots, n-1\} \end{cases}$$

for all  $k < k_0$ , and furthermore, such coefficients are independent of the direction  $u \in T_m M$ .

Also, since  $\eta_0(m, u, a) = g(E_a, Ju)(m) = 0$ , from the expression in Lemma 3.1, it follows that (3.2) also holds for  $k = k_0$ . Now, we proceed by induction. Suppose (3.2) holds for  $k = 0, \dots, t+k_0$  and also that  $\eta_0(m, u, a) = \dots = \eta_t(m, u, a) = 0$  for all  $a \in \{2, \dots, n-1\}$ . We prove that

$$\begin{cases} \alpha_{t+k_0+1}(m, u, a, b) = 0, & a, b \in \{1, 2, \dots, n-1\}, \\ \alpha_{t+k_0+1}(m, u, a, a) = \alpha_{t+k_0+1}(m, u, b, b), & a, b \in \{2, \dots, n-1\}, \\ \eta_{t+1}(m, u, a) = 0, & a \in \{2, \dots, n-1\} \end{cases}$$

and that they are independent of the direction  $u \in T_m M$ .

From the expression in Lemma 3.1 it follows that (3.2) holds for  $k = n + k_0 + 1$  and  $a, b \in \{2, \dots, n-1\}$ . Hence, we have to show that  $\eta_{t+1}(m, u, a) = 0$ . To do this, we consider the expression in Lemma 3.1 and, using the induction hypothesis, it follows that

$$(3.4) \quad \beta_1^2 (\beta_1^{t+k_0+1} - 1) \alpha_{t+k_0+1}(m, u, a, 1) = \frac{1}{k_0!} f^{(k_0)}(0) \eta_{t+1}(m, u, a).$$

Consider the unit vectors  $z_{\lambda\mu} = \lambda u + \mu Ju$ ,  $\lambda^2 + \mu^2 = 1$ . Since  $E_a$  remains orthogonal to both  $z_{\lambda\mu}$  and  $Jz_{\lambda\mu}$ , it follows that

$$\beta_1^2 (\beta_1^{t+k_0+1} - 1) \alpha_{t+k_0+1}(m, z_{\lambda\mu}, a, Jz_{\lambda\mu}) = (\nabla_{z_{\lambda\mu}, \dots, z_{\lambda\mu}}^{(k_0)} f)(0) \eta_{t+1}(m, z_{\lambda\mu}, E_a).$$

The usual procedure then yields  $\alpha_{t+k_0+1}(m, u, a, Ju) = 0$ .

Next, we will use (3.2) to show the necessary conditions (3.1). Since the components  $g_{ab}$  of the metric tensor are given by (2.7), it follows that the

coefficients in the power series expansion of  $g_{ab}(\gamma(r))$  satisfy

$$(3.5) \quad \alpha_{k+4}(m, u, a, b) = \frac{1}{(k+2)!} ({}^t D_u D_u)_{ab}^{(k+2)}(0).$$

It follows from the power series expansion (2.13) that  $(1/3)g(R(u)E_a, E_b)(m) = -\alpha_2(m, u, a, b)$ , and using the recursion formula in Lemma 3.1,  $R(u)$  is a diagonal matrix with two constant eigenvalues independent of the direction  $u \in T_m M$ ,

$$\beta_1^2(1 - \beta_1^2)R_{uaub}(m) = 2 \left( 6\beta_1\beta_3\delta_{ab} + \frac{1}{4}f''(0)\delta_{a1}\delta_{b1} \right).$$

(Note that  $2\beta_1\beta_2\delta_{ab} = -f'(0)\delta_{a1}\delta_{b1}$  and thus  $\beta_2 = f'(0) = 0$  provided  $\dim M > 2$ .) Since the endomorphism-valued function  $D_u(r)$  is a solution of the Jacobi equation (2.5) with initial conditions  $D_u(0) = 0$ ,  $D'_u(0) = I$ , it follows that  $D'''_u(0) = -R(m)$ , and hence, it is diagonal with at most two distinct eigenvalues, one with multiplicity one corresponding to the eigenvector  $Ju$ . We now use induction. Suppose that the matrices

$$D'''_u(0), \dots, D_u^{(k+1)}(0), \quad R(m), \dots, R^{(k-2)}(m)$$

are diagonal with at most two distinct eigenvalues, one having  $Ju$  as corresponding eigenvector, and show that the same holds for  $D_u^{(k+2)}(0)$  and  $R^{(k-1)}(m)$ . Since

$$(3.6) \quad D_u^{(k+2)}(0) = - \sum_{l=0}^k C_k^l R^{(k-l)}(m) D_u^{(l)}(0),$$

it follows from the hypothesis of induction that  $D_u^{(k+2)}(0)$  is a symmetric matrix and hence, (3.5) shows that it is diagonal with two eigenvalues, one corresponding to the distinguished eigenvector  $Ju$ . Coming back to (3.6), the corresponding result holds for  $R^{(k-1)}(m)$ . Moreover, since those eigenvalues are independent of the direction, it follows that the odd derivatives of the Jacobi operator vanish [14], which shows the necessity of (3.1).

Next we prove the converse. If  $c_1(m, k) = c_2(m, k)$  for all  $k \geq 0$ , (3.1) shows that the Jacobi operator and its higher order derivatives are diagonal with only one constant eigenvalue. Then the result follows from [13, Theorem 4.1]. Next, we suppose that  $c_1(m, k) \neq c_2(m, k)$  for some  $k \geq 0$  and assume  $(M, g, J)$  to be a nearly Kähler manifold. Then it follows that  $Ju$  is also parallel along the geodesic  $\gamma(r) = \exp_m(ru)$ . Using (3.6), it follows that the endomorphism-valued function  $D_u(r)$  can be diagonalized with respect to an orthonormal parallel basis  $\{Ju, E_2, \dots, E_{n-1}\}$ . Moreover, from (3.1) it follows that the eigenvalues are in-

dependent of the direction  $u \in T_mM$ , and it also follows that the coefficients in the power series expansion of the metric tensor satisfy (3.2). This shows that the recursion formula in Lemma 3.1 defines a partially conformal geodesic transformation with respect to  $m$  for each initial value  $\beta_1 = s'(0) \in \mathbf{R} - \{0\}$ .  $\square$

REMARK 3.1. Note, as follows from (3.1), that the existence of a non-isometric partially conformal geodesic transformation with respect to a point is a more restrictive condition than that of an isometric local reflection. If there exists a non-isometric partially conformal geodesic transformation with respect to a single point, then for each value of  $C \in \mathbf{R} - \{0\}$ , there exists a partially conformal geodesic transformation with initial condition  $s'(0) = C$  (in particular, the local reflection for  $C = -1$ ). Moreover, by making use of the curvature conditions in Theorem 3.2, it follows that if there exists a non-conformal, partially conformal geodesic transformation with respect to a point  $m \in M$ , then any conformal geodesic transformation with respect to  $m$  must be isometric. Moreover, if there exists a non-isometric conformal geodesic transformation with respect to  $m$ , then any partially conformal geodesic transformation with respect to  $m$  must be conformal.

As a consequence of the previous theorem, we can now state the following characterization of complex space forms. Note that this result generalizes [9, Theorem 4.5] since we do not assume here that the manifold is Kählerian.

THEOREM 3.3. *Let  $(M, g, J)$  be an almost Hermitian manifold. Then  $M$  is a Kähler manifold of constant holomorphic sectional curvature  $c \neq 0$  if and only if for each point  $m \in M$  there exists a non-conformal partially conformal geodesic transformation.*

PROOF. If  $\varphi_m$  is a non-conformal partially conformal geodesic transformation, the function  $f$  does not vanish identically. Hence, assume  $f^{(k_0)}(0)$  to be the first non-zero derivative of  $f$ , ( $k_0 \geq 1$ ). Considering the recurrence formula in Lemma 3.1 for  $k = k_0 + 1$  and using the induction hypothesis considered in the proof of the previous theorem, it follows that  $f^{(k_0)}(0)\eta_1(m, u, a) = 0$  for all  $a = 2, \dots, n - 1$ . Moreover, the first terms in the power series expansion of  $\eta(\partial/\partial x^a)$  along the geodesic  $\gamma(r)$  are

$$\eta\left(\frac{\partial}{\partial x^a}\right)(\exp_m(ru)) = g(E_a, Ju)(m) + rg(J'u, E_a)(m) + O(r^2).$$

This shows that  $g((\nabla_u J)u, E_a) = 0$  for all  $E_a$  orthogonal to both  $u$  and  $Ju$ . Hence,

$(\nabla_u J)u = 0$  for each unit  $u$ , and this shows that  $(M, g, J)$  is a nearly Kähler manifold. Also, condition (3.1) in Theorem 3.1 for  $k = 0$  yields

$$R(u, Ju)u \sim Ju$$

for all unit vectors  $u \in T_m M$ . Hence the holomorphic sectional curvature of  $M$  is constant at each point  $m \in M$  [21].

In [15], nearly Kähler manifolds of constant holomorphic sectional curvature are classified, showing that they must be complex space forms or locally isometric to the six-dimensional sphere with the nearly Kähler structure induced from the product of the Cayley numbers. Now, since the geodesic transformation  $\varphi_m$  is not conformal, it follows from (3.1) that  $c_1(m, 0) \neq c_2(m, 0)$  and hence, the Jacobi operator  $R(m)$  has two distinct constant eigenvalues. This shows that  $M$  cannot be a space of constant curvature. Hence,  $M$  is a Kähler manifold of constant holomorphic sectional curvature  $c_1 \neq 0$ . (Note that  $c_1(m, 0), c_2(m, 0)$  are constant on each connected component of  $M$ .)

The converse is proved in [9, Theorem 4.1] □

The next theorem classifies the partially conformal geodesic transformations occurring in an almost Hermitian manifold which admits a partially conformal geodesic transformation with respect to each point. Also, it may be viewed as a generalization of the results in [9] and [13].

**THEOREM 3.4.** *Let  $(M, g, J)$  be an almost Hermitian manifold such that there exists a non-trivial partially conformal geodesic transformation with respect to each point  $m \in M$ . Then  $M$  is a locally symmetric space and further we have*

- (i) *the geodesic transformation is the local reflection and hence, an isometry;*
- (ii)  *$M$  is locally flat if and only if there exists a non-isometric homothetic geodesic transformation with respect to some point. Moreover, in this case the transformation must be the Euclidean similarity  $s(r) = Cr$ ,  $C^2 \neq 0, 1$ ;*
- (iii)  *$M$  is a space of constant curvature  $c > 0$  if and only if it there exists a non-homothetic geodesic conformal transformation with respect to some point. In this case, only non-Euclidean similarities*

$$\tan s \frac{\sqrt{c}}{2} = C \tan r \frac{\sqrt{c}}{2}, \quad C^2 \neq 0, 1,$$

*occur;*

- (iv)  *$M$  is a Kähler manifold of constant holomorphic sectional curvature  $c > 0$  if and only if there exists a non-conformal partially conformal geodesic trans-*



formation with respect to some point. In this case, the geodesic transformation must be a non-Euclidean similarity

$$\tan s \frac{\sqrt{c}}{4} = C \tan r \frac{\sqrt{c}}{4}, \quad C^2 \neq 0, 1.$$

**PROOF.** It is clear from Theorem 3.2 that the existence of a partially conformal geodesic transformation with respect to each point implies local symmetry. Furthermore, the existence of a homothetic, conformal or partially conformal geodesic transformation with respect to a single point is equivalent to the condition that the Jacobi operator has only zero constant eigenvalues, or only non-zero equal constant eigenvalues, or two distinct constant eigenvalues (one with multiplicity one), respectively.

Since  $(M, g)$  is locally symmetric, the eigenvalues of the Jacobi operator are constant on  $M$ , and hence, from Theorem 3.2 it follows that there exists a homothetic, conformal or partially conformal geodesic transformation with respect to each point. Hence, the results follow from previous theorems and using those in [9]. The existence of non-Euclidean similarities as before is shown in [9].  $\square$

**REMARK 3.2.** The corresponding cases to (iii) and (iv) in Theorem 3.4 for negative curvature  $c < 0$  are obtained by replacing the trigonometric by hyperbolic functions.

#### 4. Transformations with Respect to Real Hypersurfaces

In this final section we derive the sufficient conditions for the existence of a partially conformal geodesic transformation with respect to a Hopf hypersurface in an almost Hermitian manifold. Proceeding in an analogous way as in the previous section, from the conditions in Lemma 2.1 we obtain the following recursion formula for the coefficients in the power series expansions of the functions  $s(r)$  and  $f(r)$  expressed in terms of the components of the metric tensor  $g$  and the one-form  $\eta$ . Note also that for any partially conformal geodesic transformation with respect to a hypersurface we have  $s'(0) = -1$ .

**LEMMA 4.1.** *Let  $\varphi_B$  be a geodesic transformation with respect to a real hypersurface  $B$  in an almost Hermitian manifold  $(M, g, J)$ . Then  $\varphi_B$  is partially conformal if and only if the coefficients in the power series expansion of the function  $s(r)$  satisfy the recursion formula*

$$\begin{aligned}
 (4.1) \quad & 2(k+1)\beta_{k+1}\delta_{ij} + ((-1)^k - 1)\alpha_k(m, u, i, j) - \frac{1}{k!}f^{(k)}(0)g(E_i, Ju)g(E_j, Ju)(m) \\
 & = \delta_{ij} \sum_{\substack{p+q=k+2 \\ p, q > 1}} pq\beta_p\beta_q + \sum_{\substack{v+l+t=k \\ v < k}} \frac{1}{v!}f^{(v)}(0)\eta_l(m, u, i)\eta_t(m, u, j) \\
 & \quad + \sum_{l=1}^{k-1} \alpha_l(m, u, i, j) \left( \sum_{p+q=k-l+2} pq\beta_p\beta_q - \sum_{p_1+\dots+p_l=k} \beta_{p_1} \cdots \beta_{p_l} \right)
 \end{aligned}$$

for all  $i, j \in \{1, 2, \dots, n-1\}$ .

Using this recursion formula, we shall investigate the sufficient conditions for the existence of a partially conformal geodesic transformation with respect to a real hypersurface  $B$ . We recall (see Theorem 2.2) that  $B$  must be totally umbilical or a Hopf hypersurface with two distinct constant principal curvatures, one with multiplicity one corresponding to the principal direction  $Ju$ . We derive the following sufficient conditions:

**THEOREM 4.1.** *Let  $B$  be a real hypersurface in an almost Hermitian manifold  $(M, g, J)$ . If the normal Jacobi operator satisfies*

$$(4.2) \quad R^{(k)}(m) = \begin{pmatrix} c_1(k, u) & 0 \\ 0 & c_2(k, u)I_{n-2} \end{pmatrix}$$

for all  $k \geq 0, m \in B$ , where  $c_1(k, u), c_2(k, u)$  are constant along  $B$ , then we have

- (1) if  $c_1(k, u) = c_2(k, u)$  for all  $k \geq 0$ , then there exists a geodesic conformal transformation with respect to  $B$ , provided that  $B$  is totally umbilical;
- (2) if  $c_1(k, u) \neq c_2(k, u)$  for some  $k \geq 0$  and  $B$  is a Hopf hypersurface with two distinct constant principal curvatures, then there exists a non-conformal, partially conformal geodesic transformation with respect to  $B$ , provided that  $(M, g, J)$  is nearly Kählerian.

**PROOF.** Part (1) has been proved in [13, Theorem 5.1]. In order to show (2), consider the endomorphism-valued function  $D_u(r)$  satisfying the Jacobi equation with initial values  $D_u(0) = I, D'_u(0) = T(u)$ . It follows that  $D'_u(0)$  is diagonalizable with respect to an orthonormal basis  $\{E_1, \dots, E_{n-2}, Ju\}$  of  $T_mB$ . Proceeding by induction and using the conditions for the normal Jacobi operator, it follows that the higher order derivatives  $D_u^{(k)}(0)$  are diagonal matrices with two distinct eigenvalues, one with multiplicity one having  $Ju$  as corresponding

eigenvector. Note also, that since the principal curvatures of  $B$  are constant and the functions  $c_1(k, u)$ ,  $c_2(k, u)$  are independent of the point  $m \in B$ , those eigenvalues are also constant along  $B$ .

Now, let  $\{F_1, \dots, F_{n-2}\}$  be the parallel translated basis of  $\{E_1, \dots, E_{n-2}\}$  along the normal geodesic  $\gamma(r) = \exp_m(ru)$ . Moreover, since  $(M, g, J)$  is a nearly Kähler manifold,  $Ju$  is also parallel along the geodesic  $\gamma$ . Hence, with respect to the parallel basis  $\{F_1, \dots, F_{n-2}, Ju\}$ , the components of the metric tensor are given by

$$g_{ij}(\gamma(r)) = ({}^t D_u D_u)_{ij}(r)$$

and hence, the coefficients in the power series expansion of the components of the metric tensor are constant along  $B$  and moreover, they satisfy

$$(4.3) \quad \begin{cases} \alpha_k(m, u, i, j) = 0, & i, j \in \{1, \dots, n-1\}, \quad i \neq j, \\ \alpha_k(m, u, i, i) = \alpha_k(m, u, j, j), & i, j \in \{1, \dots, n-2\}. \end{cases}$$

Also, for the one-form  $\eta(X) = g(X, JN)$  along the normal geodesic  $\gamma(r) = \exp_m(ru)$ , it follows that  $\eta(\partial/\partial x^i) = 0$  for all  $i \in \{1, \dots, n-2\}$ . This, together with (4.3), shows that the recursion formula in Lemma 4.1 defines a non-conformal, partially conformal geodesic transformation with respect to  $B$ .  $\square$

For nearly Kähler manifolds, the existence of a partially conformal geodesic transformation with respect to each sufficiently small geodesic sphere characterizes the constancy of the holomorphic sectional curvature, according to the following

**THEOREM 4.2.** *Let  $(M, g, J)$  be a nearly Kähler manifold. Then it is a space of constant holomorphic sectional curvature if and only if there exists a non-isometric partially conformal geodesic transformation with respect to each sufficiently small geodesic sphere, and moreover*

(i)  *$M$  is locally isometric to  $\mathbb{C}^n$  or to the six-dimensional sphere  $S^6$  if and only if for each sufficiently small geodesic sphere there exists a non-isometric geodesic conformal transformation,*

(ii)  *$M$  is locally isometric to a non-flat complex space form if and only if there exists a non-conformal partially conformal geodesic transformation with respect to each sufficiently small geodesic sphere.*

**PROOF.** If there exists a partially conformal geodesic transformation with

respect to each sufficiently small geodesic sphere, then  $Ju$  is an eigenvector of the shape operator  $T(u)$ . From the results in [7] it then follows that the holomorphic sectional curvature is constant. Now, (i) and (ii) follow from the classification of the nearly Kähler manifolds of constant holomorphic sectional curvature [15] together with the results in [13] and Theorems 2.2 and 4.1.  $\square$

Note that the existence of a partially conformal geodesic transformation with respect to any sufficiently small geodesic sphere implies that the mean curvature function of the small geodesic spheres is a radial function, and hence,  $M$  is a harmonic space (see, for example, [3]). Using for example [8] and if  $M$  is locally symmetric, we have the following

**THEOREM 4.3.** *Let  $(M, g, J)$  be a locally symmetric space. If there exists a partially conformal geodesic transformation with respect to each sufficiently small geodesic sphere, then*

- (i)  *$M$  is locally isometric to a space of constant curvature if and only if each geodesic transformation is conformal,*
- (ii)  *$M$  is locally isometric to a non-flat complex space form if and only if there exists a non-conformal, partially conformal geodesic transformation with respect to each sufficiently small geodesic sphere.*

**PROOF.** Any locally symmetric harmonic space is locally isometric to a two-point homogeneous space. Moreover, from [23] it follows that  $(M, g, J)$  must be locally isometric to a complex space form or to a space of constant curvature. Note that in any of these cases,  $R(m)Ju$  is proportional to  $Ju$ , and hence, the result follows from the previous results (Theorem 4.2).  $\square$

As already mentioned, the Hopf hypersurfaces with two distinct constant principal curvatures in a non-flat complex space form of real dimension  $\geq 4$  are completely classified by Takagi [19] for  $CP^{n/2}$  and by Montiel [17] for  $CH^{n/2}$ ,  $n \geq 6$ . Actually, these hypersurfaces are characterized by the existence of a partially conformal geodesic transformation with respect to them. Furthermore, we have

**THEOREM 4.4.** *Let  $(M, g, J)$  be a Kähler manifold of constant holomorphic sectional curvature  $c$  and let  $B$  be a real hypersurface. Then,  $B$  is a Hopf hypersurface with two constant principal curvatures if and only if there exists a partially conformal geodesic transformation with respect to it. Moreover,*

(i)  $B$  is locally isometric to a geodesic sphere of radius  $\alpha$  and the geodesic transformation is the Euclidean inversion  $(s + \alpha)(r + \alpha) = \alpha^2$  if  $c = 0$ ;

(ii)  $B$  is locally isometric to a geodesic sphere of radius  $\alpha$  in the complex projective space of constant holomorphic sectional curvature  $c = 4$  and the geodesic transformation is the non-Euclidean inversion

$$\tan \frac{(s + \alpha)}{2} \tan \frac{(r + \alpha)}{2} = \tan^2 \frac{\alpha}{2};$$

(iii)  $B$  is locally isometric to a geodesic sphere of radius  $\alpha$  in the complex hyperbolic space of constant holomorphic sectional curvature  $c = -4$ , and the geodesic transformation is the non-Euclidean inversion

$$\tanh \frac{(s + \alpha)}{2} \tanh \frac{(r + \alpha)}{2} = \tanh^2 \frac{\alpha}{2};$$

(iv)  $B$  is locally isometric to a tube about  $CH^k$  in the complex hyperbolic space  $CH^n(-4)$  and the geodesic transformation is defined by

$$\arctan \sinh(s + \alpha) + \arctan \sinh(r + \alpha) = 2\arctan \sinh \alpha;$$

(v)  $B$  is locally isometric to a tube of radius  $\log(2 + \sqrt{3})$  about  $RH^{n/2}$  in the complex hyperbolic space  $CH^n(-4)$  and the geodesic transformation is that of the case (iv) above for  $\alpha = \log(2 + \sqrt{3})$ ;

(vi)  $B$  is locally isometric to a horosphere in the complex hyperbolic space  $CH^n(-4)$  and the geodesic transformation is defined by

$$e^{-s} + e^{-r} = 2.$$

PROOF. According to Theorem 2.2,  $B$  must be a totally umbilical hypersurface or a Hopf hypersurface with two distinct constant principal curvatures. Since there are no totally umbilical hypersurfaces in a non-flat complex space form, it follows that a geodesic conformal transformation occurs only when  $M$  is flat. Also, it follows directly from the classification of Hopf hypersurfaces with two distinct constant principal curvatures in the projective and hyperbolic complex spaces that  $B$  must lie in one of the classes above. (See Berndt [1] for a table of the principal curvatures of the hypersurfaces above.) Finally, the result follows proceeding as in [9] (see also [13]).

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## References

- [ 1 ] J. Berndt, Real hypersurfaces with constant principal curvatures in complex hyperbolic space, *J. Reine Angew. Math.* **395** (1989), 132–141.
- [ 2 ] B. Y. Chen, *Geometry of Submanifolds*, Pure Appl. Math. **22**, Marcel Dekker, New York, 1973.
- [ 3 ] B. Y. Chen and L. Vanhecke, Differential geometry of geodesic spheres, *J. Reine Angew. Math.* **325** (1981), 28–67.
- [ 4 ] B. Y. Chen and L. Vanhecke, Isometric, holomorphic and symplectic reflections, *Geom. Dedicata* **29** (1989), 259–277.
- [ 5 ] B. Y. Chen and L. Vanhecke, Symplectic reflections and complex space forms, *Ann. Glob. Anal. Geom.* **3** (1991), 205–210.
- [ 6 ] J. E. D’Atri, Geodesic conformal transformations and symmetric spaces, *Kōdai Math. Sem. Rep.* **26** (1975), 201–203.
- [ 7 ] M. Djorić and L. Vanhecke, Almost Hermitian geometry, geodesic spheres and symmetries, *Math. J. Okayama Univ.* **32** (1990), 187–206.
- [ 8 ] J. H. Eschenburg, A note on symmetric and harmonic spaces, *J. London Math. Soc.* **21** (1980), 541–543.
- [ 9 ] E. García-Río and L. Vanhecke, Geodesic transformations and two-point homogeneous spaces, *Math. J. Toyama Univ.* **20** (1997), 57–97.
- [10] E. García-Río and L. Vanhecke, Geodesic transformations and harmonic spaces, *Rend. Circ. Mat. Palermo* **48** (1999), 101–110.
- [11] E. García-Río and L. Vanhecke, Divergence-preserving geodesic transformations, *Proc. Royal Soc. Edinburgh* **128 A** (1998), 1309–1323.
- [12] E. García-Río and L. Vanhecke, Holomorphic geodesic transformations, *Kodai Math. J.* **21** (1998), 46–60.
- [13] E. García-Río and L. Vanhecke, Conformal geodesic transformations, *Arab J. Math.* **3** (1997), 13–36.
- [14] P. Gilkey, Manifolds whose higher odd order curvature operators have constant eigenvalues at the basepoint, *J. Geom. Anal.* **2** (1992), 151–156.
- [15] A. Gray, Classification des variétés approximativement kählériennes de courbure sectionnelle holomorphe constante, *C. R. Acad. Sci. Paris* **279** (1974), 797–800.
- [16] A. Gray, *Tubes*, Addison-Wesley, Redwood, 1990.
- [17] S. Montiel, Real hypersurfaces of a complex hyperbolic space, *J. Math. Soc. Japan* **37** (1985), 515–535.
- [18] S. Tachibana, On Riemannian spaces admitting geodesic conformal transformations, *Tensor N.S.* **25** (1972), 323–331.
- [19] R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures, *J. Math. Soc. Japan* **27** (1975), 43–53.
- [20] S. Tanno, Partially conformal transformations with respect to  $(m - 1)$ -dimensional distributions of  $m$ -dimensional Riemannian manifolds, *Tōhoku Math. J.* **17** (1965), 358–409.
- [21] S. Tanno, Constancy of holomorphic sectional curvature in almost Hermitian manifolds, *Kōdai Math. Sem. Rep.* **25** (1973), 190–201.
- [22] L. Vanhecke, Geometry in normal and tubular neighborhoods, *Rend. Sem. Fac. Sci. Univ. Cagliari*, supplemento al vol. **58** (1988), 73–176.
- [23] L. Vanhecke and T. J. Willmore, Jacobi fields and geodesic spheres, *Proc. Royal Soc. Edinburgh* **82 A** (1979), 233–240.
- [24] L. Vanhecke and T. J. Willmore, Interaction of tubes and spheres, *Math. Ann.* **263** (1983), 31–42.

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