TSUKUBA J. MATH.
Vol. 23 No. 1 (1999), 113–130

ON CURVATURE PROPERTIES OF CERTAIN GENERALIZED ROBERTSON-WALKER SPACETIMES

Dedicated to the memory of Professor Dr. Georgii Ionovich Kruchkovich

By

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1. Introduction

The warped product $\overline{M} \times_F N$, of a 1-dimensional manifold $(\overline{M}, \bar{g}), \bar{g}_{11} = -1$, with a warping function F and a 3-dimensional Riemannian manifold $(N, \tilde{g})$ is said to be a generalized Robertson-Walker spacetime (cf. [2], [32]). In particular, when the manifold $(N, \tilde{g})$ is a Riemannian space of constant curvature, the warped product $\overline{M} \times_F N$ is called a Robertson-Walker spacetime. In [11] it was shown that at every point of a generalized Robertson-Walker spacetime $\overline{M} \times_F N$ the following condition is satisfied:

$(*)_1$ the tensors $R \cdot R - Q(S, R)$ and $Q(g, C)$ are linearly dependent.

This condition is equivalent to the relation

$$R \cdot R - Q(S, R) = L_1 Q(g, C) \tag{1}$$

on the set $\mathcal{U}_C$ consisting of all points of the manifold $\overline{M} \times_F N$ at which its Weyl tensor C is non-zero, where L_1 is a certain function on $\mathcal{U}_C$. For precise definitions of the symbols used, we refer to the Sections 2 and 3. $(*)_1$ is a curvature condition of pseudosymmetry type. In this paper we will investigate generalized Robertson-Walker spacetimes realizing a condition of pseudosymmetry type introduced in [25]. Namely, semi-Riemannian manifolds (M, g), $n \geq 4$, fulfilling at every point of M the following condition

$(*)$ the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent.

were considered in [25]. This condition is equivalent to the relation

$$R \cdot C = LQ(S, C) \tag{2}$$

Received January 20, 1998
Revised June 2, 1998

on the set $\mathscr{U} = \{x \in M | Q(S, C) \neq 0 \text{ at } x\}$, where L is a certain function on $\mathscr{U}$. We note that every semisymmetric manifold $(R \cdot R = 0)$ as well as every Weyl-semisymmetric manifold $(R \cdot C = 0)$ realizes $(*)$ trivially (see [25]). There exist also non semisymmetric and non Weyl-semisymmetric manifolds realizing $(*)$ ([25]). We mention that warped products realizing curvature conditions of pseudosymmetry type were studied in: [7], [8], [9], [11], [13], [14], [15], [16], [17], [19], [20], [21], [24], [26], [28] and [29].

In Section 2 we present a review of the family of curvature conditions of pseudosymmetry type. In the next section we give results on warped products which we apply in the last two sections. In Section 4 we find necessary and sufficient conditions for a warped product to be a manifold satisfying (2). Finally, in Section 5 we present our main results.

Let (M, g) be a semi-Riemannian manifold satisfying $(*)$. We denote by $\mathscr{U}_L$ the set of all points of the set $\mathscr{U} \subset M$ at which the function L is non-zero. It is clear that the tensors $R \cdot C$ and $Q(S, C)$ are non-zero at every point of the set $\mathscr{U}_L$. Moreover, let (M, g) be a 4-dimensional warped product $\overline{M} \times_F N$, $\dim \overline{M} = 1$. We denote by $\mathscr{U}_F$ the subset of $\mathscr{U}_L$ consisting of all non-critical points of F. Our main result states (see Theorem 5.1) that if the 4-dimensional warped product $\overline{M} \times_F N$, $\dim \overline{M} = 1$, satisfies $(*)$ and the set $\mathscr{U}_F$ is a dense subset of $\mathscr{U}_L$ then the open submanifold U_L of the manifold $\overline{M} \times_F N$ is a pseudosymmetric warped product of the 1-dimensional manifold, with the function F, defined by $F(x^1) = a \exp(bx^1)$, $a = const. > 0$, $b = const. \neq 0$, and a 3-dimensional semi-Riemannian manifold such that its Ricci tensor is of rank one and its scalar curvature vanishes identically. From this statement it follows immediately (see Corollary 5.1) that if a generalized Robertson-Walker spacetime $\overline{M} \times_F N$ realizes above assumptions then at every point of $\overline{M} \times_F N$ at least one of the tensors $R \cdot C$ or $Q(S, C)$ must vanish. Finally, using this fact we prove (see Theorem 5.2) that every Robertson-Walker spacetime satisfying $(*)$ is a pseudosymmetric manifold.

2. Curvature Conditions of Pseudosymmetry Type

Let (M, g) be a connected n-dimensional, semi-Riemannian manifold of class C^∞ and let ∇ be its Levi-Civita connection. We define on M the endomorphisms $X \wedge Y, \mathscr{R}(X, Y)$ and $\mathscr{C}(X, Y)$ by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y, \quad \mathscr{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z,$$

$$\mathscr{C}(X, Y) = \mathscr{R}(X, Y) - \frac{1}{n-2}\Big(X \wedge \mathscr{S}Y + \mathscr{S}X \wedge Y - \frac{\kappa}{n-1} X \wedge Y\Big),$$

respectively, where $X, Y, Z \in \Xi(M), \Xi(M)$ being the Lie algebra of vector fields of M. The Ricci operator $\mathscr{S}$ is defined by $S(X, Y) = g(X, \mathscr{S}Y)$, where S is the Ricci tensor and κ the scalar curvature of (M, g), respectively. Next, we define the tensors U, G, the Riemann-Christoffel curvature tensor R and the Weyl conformal tensor C of (M, g), by

$$\begin{aligned}
U(X_1, X_2, X_3, X_4) &= g(X_1, X_4)S(X_2, X_3) + g(X_2, X_3)S(X_1, X_4) \\
&\quad - g(X_1, X_3)S(X_2, X_4) - g(X_2, X_4)S(X_1, X_3), \\
G(X_1, X_2, X_3, X_4) &= g((X_1 \wedge X_2)X_3, X_4), \\
R(X_1, X_2, X_3, X_4) &= g(\mathscr{R}(X_1, X_2)X_3, X_4), \\
C(X_1, X_2, X_3, X_4) &= g(\mathscr{C}(X_1, X_2)X_3, X_4),
\end{aligned}$$

respectively. Now we can present the Weyl tensor C in the following form

$$C = R - \frac{1}{n-2}U + \frac{\kappa}{(n-2)(n-1)}G. \tag{3}$$

For a $(0, k)$-tensor field $T, k \geq 1$, we define the $(0, k+2)$-tensors $R \cdot T$ and $Q(g, T)$ by

$$\begin{aligned}
(R \cdot T)(X_1, \ldots, X_k; X, Y) &= (\mathscr{R}(X, Y) \cdot T)(X_1, \ldots, X_k) \\
&= -T(\mathscr{R}(X, Y)X_1, X_2, \ldots, X_k) \\
&\quad - \cdots - T(X_1, \ldots, X_{k-1}, \mathscr{R}(X, Y)X_k), \\
Q(g, T)(X_1, \ldots, X_k; X, Y) &= ((X \wedge Y) \cdot T)(X_1, \ldots, X_k) \\
&= -T((X \wedge Y)X_1, X_2, \ldots, X_k) \\
&\quad - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge Y)X_k).
\end{aligned}$$

Putting in the last formulas $T = R, T = S$ or $T = C$, we obtain the tensors $R \cdot R, R \cdot S, R \cdot C, Q(g, R), Q(g, S)$ and $Q(g, C)$, respectively. The tensor $C \cdot C$ we define in the same way as the tensor $R \cdot R$.

Let (M, g) be a Riemannian manifold covered by a system of charts $\{\mathscr{W}; x^r\}$. We denote by $g_{rs}, \Gamma^r_{st}, R_{rstu}, S_{st}, G_{rstu} = g_{ru}g_{st} - g_{rt}g_{su}$ and

$$C_{rstu} = R_{rstu} - \frac{1}{n-2}(g_{ru}S_{st} - g_{rt}S_{su} + g_{st}S_{ru} - g_{su}S_{rt}) + \frac{\kappa}{(n-2)(n-1)}G_{rstu},$$

the local components of the metric g, the Levi-Civita connection ∇, the Riemann-Christoffel curvature tensor R, the Ricci tensor S, the tensor G, and the Weyl conformal curvature tensor C of (M, g), respectively, where $p, g, r, s, t, u, v, w \in \{1, 2, \dots, n\}$. The local components of the tensors $R \cdot R$ and $Q(g, R)$ are given by the following formulas

$$\begin{aligned}(R \cdot R)_{rstuvw} &= \nabla_w \nabla_v R_{rstu} - \nabla_v \nabla_w R_{rstu}\\ &= g^{pq}(R_{pstu} R_{qrvw} - R_{prtu} R_{qsvw} + R_{purs} R_{qtvw} - R_{ptrs} R_{quvw}),\\ Q(g, R)_{rstuvw} &= g_{rv} R_{wstu} + g_{sv} R_{rwtu} + g_{tv} R_{rswu} + g_{uv} R_{rstw}\\ &\quad - g_{rw} R_{vstu} - g_{sw} R_{rvtu} - g_{tw} R_{rsvu} - g_{uw} R_{rstv}.\end{aligned}$$

A semi-Riemannian manifold (M, g), $n \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{\kappa}{n} g \tag{4}$$

holds on M. According to [4] (p. 432), (4) is called the Einstein metric condition. Einstein manifolds form a natural subclass of various classes of semi-Riemannian manifolds determined by a curvature condition imposed on their Ricci tensor ([4], Table, pp. 432–433). For instance, every Einstein manifold belongs to the class of semi-Riemannian manifolds (M, g) realizing the following relation

$$\nabla\left(S - \frac{\kappa}{2(n-1)} g\right)(X, Y; Z) = \nabla\left(S - \frac{\kappa}{2(n-1)} g\right)(X, Z; Y), \tag{5}$$

which means that $S - (\kappa/(2(n-1)))g$ is a Codazzi tensor on M. Manifolds of dimension ≥ 4 fulfilling (5) are called manifolds with harmonic Weyl tensor ([4], p. 440). It is known that every warped product $S^1 \times_F M$ of the sphere S^1, with a positive smooth function F, and an Einstein manifold (M, g), $\dim M \geq 2$, realizes (5) ([4], p. 433). Such warped product is a non-Einstein manifold, in general. We say that (5) is a generalized Einstein metric condition ([4], chapter XVI). On the other hand, such warped product realizes a condition of pseudosymmetry type too. Namely, the warped product $S^1 \times_F M$ of the sphere S^1, with a positive smooth function F, and an Einstein manifold (M, g), $\dim M \geq 2$, is a Ricci-pseudosymmetric manifold ([24], Corollary 3.2). Thus, in particular, the warped product $S^1 \times_F CP^n$ of S^1, with a positive smooth function $\mathscr{F}$, and the complex projective space CP^n (considered with its standard Riemannian locally symmetric metric) is a Ricci-pseudosymmetric manifold.

A semi-Riemannian manifold $(M, g), n \geq 3$, is said to be Ricci-pseudosymmetric ([14], [24]) if at every point of M the following condition is satisfied:

$(*)_2$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

Evidently, any Einstein manifold is Ricci-pseudosymmetric. Thus we see that $(*)_2$ is a generalized Einstein metric condition. The manifold (M, g) is Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S Q(g, S) \tag{6}$$

holds on the set $U_S = \{x \in M \mid S - (\kappa/n)g \neq 0 \text{ at } x\}$, where L_S is some function on U_S. Warped products realizing $(*)_2$ were considered in [14], [17], [24] and [26]. Certain examples of compact and non-Einstein Ricci-pseudosymmetric manifolds were found in [26] and [30]. For instance, in [30] (Theorem 1) it was shown that the Cartan hypersurfaces M in the spheres S^7, S^{17} or S^{25} are non-pseudosymmetric, Ricci-pseudosymmetric manifolds with non-pseudosymmetric Weyl tensor. The Cartan hypersurfaces M in S^4 are non-semisymmetric, pseudosymmetric manifolds. Ricci-pseudosymmetric hypersurfaces immersed isometrically in a semi-Riemannian manifolds of constant curvature were investigated in [10].

A very important subclass of the class of Ricci-pseudosymmetric manifolds form pseudosymmetric manifolds. The semi-Riemannian manifold $(M, g), n \geq 3$, is said to be pseudosymmetric ([21]) if at every point of M the following condition is satisfied:

$(*)_3$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

The manifold (M, g) is pseudosymmetric if and only if

$$R \cdot R = L_R Q(g, R) \tag{7}$$

holds on the set $U_R = \{x \in M \mid R - (\kappa/(n(n-1)))G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. It is clear that any semisymmetric manifold ($R \cdot R = 0$, [36]) is pseudosymmetric. Very recently the theory of Riemannian semisymmetric manifolds has been presented in [6]. The condition $(*)_3$ arose during the study of totally umbilical submanifolds of semisymmetric manifolds ([1]) as well as when we consider geodesic mappings of semisymmetric manifolds ([18], [37]). There exist many examples of pseudosymmetric manifolds which are not semisymmetric ([13], [19], [20], [21], [28]). Among these examples we can distinguish also compact pseudosymmetric manifolds (for instance, see [19], Example 3.1 and Theorem 4.1). Another example of a compact pseudosymmetric manifold is the warped product $S^1 \times_F S^{n-1}$, with a positive smooth function F, as well as n-dimensional tori T^n with a certain metric (see [19], Examples 4.1 and 4.2). It is clear that if at a point x of a manifold (M, g) $(*)_3$ is satisfied then also $(*)_2$ holds at x. The converse statement is not true. E.g. every warped product $M_1 \times_F M_2$, $\dim M_1 = 1$, $\dim M_2 \geq 3$, of a manifold $(M_1, \bar{g})$ and a non-pseudosymmetric, Einstein manifold $(M_2, \tilde{g})$ is a non-pseudosymmetric, Ricci-pseudosymmetric manifold (cf. [24], Remark 3.4 and [21], Theorem 4.1).

It is easy to see that if $(*)_3$ holds on a semi-Riemannian manifold (M, g), $n \geq 4$, then at every point of M the following condition is satisfied:

$(*)_4$ the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

Manifolds fulfilling $(*)_4$ are called Weyl-pseudosymmetric. Weyl-pseudosymmetric manifolds has been studied in [15], [17] and [22]. The manifold (M, g) is a Weyl-pseudosymmetric manifold if and only if the relation $R \cdot C = L_2 Q(g, C)$ holds on the set $\mathscr{U}_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_2 is some function on $\mathscr{U}_C$.

A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be a manifold with pseudosymmetric Weyl tensor ([29]) if at every point of M the following condition is satisfied:

$(*)_5$ the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent.

Thus (M, g) is a manifold with pseudosymmetric Weyl tensor if and only if the relation $C \cdot C = L_3 Q(g, C)$ holds on the set $\mathscr{U}_C$, where L_3 is a certain function on $\mathscr{U}_C$. The condition $(*)_5$ arose during the study of 4-dimensional warped products ([17]). Namely, in [17] (Theorem 2) it was shown that at every point of a warped product $M_1 \times_F M_2$, with $\dim M_1 = \dim M_2 = 2$, $(*)_5$ is fulfilled. Many examples of manifolds satisfying $(*)_5$ are presented in [9]. For instance, the Cartesian product of two manifolds of constant curvature is a manifold realizing $(*)_5$. Warped products satisfying $(*)_5$ were considered in [29]. In [9] it was shown that the classes of manifolds realizing $(*)_3$ and $(*)_5$ do not coincide. However, there exist pseudosymmetric manifolds fulfilling $(*)_5$, e.g. Einsteinian pseudosymmetric manifolds ([9], Theorem 3.1). Curvature properties of pseudosymmetric manifolds with pseudosymmetric Weyl tensor were obtained in [31].

For a $(0, k)$-tensor field $T, k \geq 1$, and a symmetric $(0, 2)$-tensor field A, we define the $(0, k+2)$-tensor $Q(A, T)$ by

$$\begin{aligned} Q(A, T)(X_1, \ldots, X_k; X, Y) &= ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k) \\ &= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) \\ &\quad - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k), \end{aligned}$$

where $X \wedge_A Y$ is the endomorphism defined by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y.$$

In particular, we have $X \wedge_g Y = X \wedge Y$. Putting in the above formula $A = S$ and $T = R, T = C$ or $T = G$, we obtain the tensors $Q(S,R), Q(S,C)$ and $Q(S,G)$, respectively.

A semi-Riemannian manifold (M,g) is said to be Ricci-generalized pseudosymmetric ([7]) if at every point of M the following condition is satisfied:

$(*)_6$ the tensors $R \cdot R$ and $Q(S,R)$ are linearly dependent.

A very important subclass of Ricci-generalized pseudosymmetric manifolds form manifolds fulfilling the following relation $R \cdot R = Q(S,R)$ ([7], [8], [23]). Every 3-manifold (M,g) as well as every hypersurface M immersed isometrically in an $(n+1)$-dimensional semi-Euclidean space E_s^{n+1}, of index $s, n \geq 3$, fulfils the last equality, see [16] (Theorem 3.1) and [27] (Corollary 3.1), respectively.

As it was shown in [27], every hypersurface M in a semi-Riemannian space of constant curvature $M^{n+1}(c), n \geq 4$, fulfils (1). More precisely, we have the following

Remark 2.1 ([27], Proposition 3.1). *Every hypersurface M immersed isometrically in a semi-Riemannian space of constant curvature $M^{n+1}(c)$, $n \geq 4$, satisfies the equality $R \cdot R - Q(S,R) = -(((n-2)\tilde{\kappa})/(n(n+1)))Q(g,C)$, where $\tilde{\kappa}$ is the scalar curvature of $M^{n+1}(c)$ and R, S and C are the curvature tensor, the Ricci tensor and the Weyl tensor of M, respectively.*

Using Theorem 3.1 of [16], which was mentioned above, and the fact that the Weyl tensor of every 3-dimensional semi-Riemannian manifold vanishes identically, we conclude that $(*)_1$ is trivially satisfied on any 3-dimensional semi-Riemannian manifold. Recently, warped products realizing $(*)_1$ were considered in [11].

The relations $(*), (*)_1$–$(*)_6$ are called conditions of pseudosymmetry type. We refer to [12], [18] and [37] as the review papers on semi-Riemannian manifolds satisfying such conditions. A hypersurface fulfilling a curvature conditon of pseudosymmetry type is said to be a hypersurface of pseudosymmetry type ([12]). We finish this section with the following

Lemma 2.1. *Let (M,g), $n = \dim M \geq 3$, be a semi-Riemannian manifold.*

(i) ([13], *Lemma* 1.2; [23], *Lemma* 2) *If the Weyl tensor C of (M,g) vanishes at a point $x \in M$ then at x any of the following three identities is equivalent to each* *other*:

$$R \cdot R = \alpha Q(g, R), \quad R \cdot S = \alpha Q(g, S),$$

$$\left(\frac{\kappa}{n-1} + (n-2)\alpha\right)\left(S - \frac{\kappa}{n} g\right) = S^2 - \frac{1}{n} tr(S^2) g,$$

where $\alpha \in \boldsymbol{R}$.

(ii) ([3], *Lemma* 3.1) *The following identity is fulfilled on* $M : Q(S, G) = -Q(g, U)$.

(iii) ([16], *Theorem* 3.1) *If* $\dim M = 3$ *then* $R \cdot R = Q(S, R)$ *holds on* M.

(iv) *If the following conditions are fulfilled at a point* $x \in M : C = 0$, $rank(S) = 1$ *and* $\kappa = 0$, *then* $R \cdot R = 0$ *holds at* x.

PROOF. (iv) The condition $rank(S) = 1$ we can present in the following form

$$S_{ij} = \beta u_i u_j, \quad u \in T_x^*(M), \quad \beta \in \boldsymbol{R}, \tag{8}$$

where u_i are the local components of u. From (8), by $\kappa = 0$, it follows that $\beta g^{ij} u_i u_j = 0$. Transvecting now (8) with $u^i = g^{ir} u_r$ we get $u^r S_{rj} = 0$. Next, transvecting (8) with S^i_k and using the last relation we get $S^2_{ij} = 0$ which, in view of (i), completes the proof.

3. Warped Products

Let now $(\overline{M}, \bar{g})$ and $(N, \tilde{g}), \dim \overline{M} = p, \dim N = n - p, 1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{\mathscr{U}; x^a\}$ and $\{\mathscr{V}; y^\alpha\}$, respectively. Let F be a positive smooth function on $\overline{M}$. The warped product $\overline{M} \times_F N$ of $(\overline{M}, \bar{g})$ and $(N, \tilde{g})$ ([5], [33]) is the product manifold $\overline{M} \times N$ with the metric $g = \bar{g} \times_F \tilde{g}$ defined by

$$\bar{g} \times_F \tilde{g} = \pi_1^* \bar{g} + (F \circ \pi_1) \pi_2^* \tilde{g},$$

where $\pi_1 : \overline{M} \times N \to \overline{M}$ and $\pi_2 : \overline{M} \times N \to N$ are the natural projections on $\overline{M}$ and N, respectively. Let $\{\mathscr{U} \times \mathscr{V}; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times N$. The local components of the metric $g = \bar{g} \times_F \tilde{g}$ with respect to this chart are the following $g_{rs} = \bar{g}_{ab}$ if $r = a$ and $s = b, g_{rs} = F \tilde{g}_{\alpha\beta}$ if $r = \alpha$ and $s = \beta$, and $g_{rs} = 0$ otherwise, where $a, b, c, \ldots \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \ldots \in \{p+1, \ldots, n\}$ and $r, s, t, \ldots \in \{1, 2, \ldots, n\}$. We will denote by bars (resp., by tildes) tensors formed from $\bar{g}$ (resp., $\tilde{g}$). The local components Γ^r_{st} of the Levi-Civita connection ∇ of $\overline{M} \times_F N$ are the following ([34]):

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc}, \quad \Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma}, \quad \Gamma^a_{\alpha\beta} = -\frac{1}{2}\bar{g}^{ab} F_b \tilde{g}_{\alpha\beta}, \quad \Gamma^\alpha_{a\beta} = \frac{1}{2F} F_a \delta^\alpha_\beta,$$

$$\Gamma^a_{\alpha b} = \Gamma^\alpha_{ab} = 0, \quad F_a = \partial_a F = \frac{\partial F}{\partial x^a}, \quad \partial_a = \frac{\partial}{\partial x^a}.$$

The local components

$$R_{rstu} = g_{rw} R^w_{stu} = g_{rw}(\partial_u \Gamma^w_{st} - \partial_t \Gamma^w_{su} + \Gamma^v_{st}\Gamma^w_{vu} - \Gamma^v_{su}\Gamma^w_{vt}), \quad \partial_u = \frac{\partial}{\partial x^u},$$

of the Riemann-Christoffel curvature tensor R and the local components S_{ts} of the Ricci tensor S of the warped product $\overline{M} \times_F N$ which may not vanish identically are the following:

$$R_{abcd} = \bar{R}_{abcd}, \quad R_{\alpha ab\beta} = -\frac{1}{2} T_{ab} \tilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = F\tilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}\Delta_1 F \tilde{G}_{\alpha\beta\gamma\delta}, \tag{9}$$

$$S_{ab} = \bar{S}_{ab} - \frac{n-p}{2}\frac{1}{F} T_{ab}, \quad S_{\alpha\beta} = \tilde{S}_{\alpha\beta} - \frac{1}{2}\left(tr(T) + \frac{n-p-1}{2F}\Delta_1 F\right)\tilde{g}_{\alpha\beta}, \tag{10}$$

where

$$T_{ab} = \bar{\nabla}_b F_a - \frac{1}{2F} F_a F_b, \quad tr(T) = \bar{g}^{ab} T_{ab}, \quad \Delta_1 F = \Delta_{1\bar{g}} F = \bar{g}^{ab} F_a F_b, \tag{11}$$

and T is the $(0,2)$-tensor with the local components T_{ab}. The scalar curvature κ of $\overline{M} \times_F N$ satisfies the following relation

$$\kappa = \bar{\kappa} + \frac{1}{F}\tilde{\kappa} - \frac{n-p}{F}\left(tr(T) + \frac{n-p-1}{4F}\Delta_1 F\right). \tag{12}$$

From now we assume that $\dim \overline{M} \times_F N = 4$ and $\dim \overline{M} = 1$. Then (9), (10) and (12) turn into

$$R_{\alpha 11\beta} = -\frac{1}{2} T_{11} \tilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = F\tilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{4}\Delta_1 F \tilde{G}_{\alpha\beta\gamma\delta}, \tag{13}$$

$$S_{11} = -\frac{3}{2F} T_{11}, \quad S_{\alpha\beta} = \tilde{S}_{\alpha\beta} - \frac{1}{2}\left(tr(T) + \frac{\Delta_1 F}{F}\right)\tilde{g}_{\alpha\beta}, \tag{14}$$

$$\kappa = \frac{1}{F}\tilde{\kappa} - \frac{3}{F}\left(tr(T) + \frac{1}{2}\frac{\Delta_1 F}{F}\right). \tag{15}$$

respectively. Further, by making use of (13), (14) and (15), we obtain the following relations (see [17], Lemma 6):

$$C_{\alpha 11\beta} = -\frac{1}{2}\bar{g}_{11}\left(\tilde{S}_{\alpha\beta} - \frac{1}{3}\tilde{\kappa}\tilde{g}_{\alpha\beta}\right),$$

$$C_{\alpha\beta\gamma\delta} = \frac{1}{2}F(\tilde{g}_{\alpha\delta}\tilde{S}_{\beta\gamma} + \tilde{g}_{\beta\gamma}\tilde{S}_{\alpha\delta} - \tilde{g}_{\alpha\gamma}\tilde{S}_{\beta\delta} - \tilde{g}_{\beta\delta}\tilde{S}_{\alpha\gamma}) - \frac{1}{3}F\tilde{\kappa}\tilde{G}_{\alpha\beta\gamma\delta}. \tag{16}$$

On the other hand, from (3) it follows that

$$\tilde{C}_{\alpha\beta\gamma\delta} = \tilde{R}_{\alpha\beta\gamma\delta} - \tilde{U}_{\alpha\beta\gamma\delta} + \frac{1}{2}\tilde{\kappa}\tilde{G}_{\alpha\beta\gamma\delta}. \tag{17}$$

Since $\tilde{C}_{\alpha\beta\gamma\delta} = 0$, the last identity reduces to

$$\tilde{U}_{\alpha\beta\gamma\delta} = \tilde{R}_{\alpha\beta\gamma\delta} + \frac{1}{2}\tilde{\kappa}\tilde{G}_{\alpha\beta\gamma\delta}. \tag{18}$$

Now (16) turns into

$$C_{\alpha 11\beta} = -\frac{1}{2}\bar{g}_{11}(\tilde{S}_{\alpha\beta} - \frac{1}{3}\tilde{\kappa}g_{\alpha\beta}), \quad C_{\alpha\beta\gamma\delta} = \frac{1}{2}F\tilde{R}_{\alpha\beta\gamma\delta} - \frac{1}{12}F\tilde{\kappa}\tilde{G}_{\alpha\beta\gamma\delta}. \tag{19}$$

4. Preliminary Results

Let $\overline{M} \times_F N$ be a 4-dimensional warped product with 1-dimensional base manifold $(\overline{M}, \bar{g})$. Using (13), (14), (18) and (19), we can verify that the local components of the tensors $R \cdot C$ and $Q(S, C)$ of the manifold $\overline{M} \times_F N$, which may not vanish identically are the following:

$$(R \cdot C)_{\alpha\beta\gamma\delta\lambda\mu} = \frac{1}{2}F(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} - \frac{1}{8}\Delta_1 F Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}, \tag{20}$$

$$Q(S, C)_{\alpha\beta\gamma\delta\lambda\mu} = -\frac{1}{4}F\left(tr(T) + \frac{\Delta_1 F}{F}\right)Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} + \frac{1}{2}FQ(\tilde{S}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} - \frac{1}{12}F\tilde{\kappa}Q(\tilde{S}, \tilde{G})_{\alpha\beta\gamma\delta\lambda\mu}, \tag{21}$$

$$(R \cdot C)_{1\alpha\beta\gamma 1\delta} = -\frac{1}{2}\frac{1}{F}T_{11}C_{\delta\alpha\beta\gamma} + \frac{1}{12}\tilde{\kappa}T_{11}\tilde{G}_{\delta\alpha\beta\gamma} - \frac{1}{4}T_{11}(\tilde{g}_{\gamma\delta}\tilde{S}_{\alpha\beta} - \tilde{g}_{\beta\delta}\tilde{S}_{\alpha\gamma}), \tag{22}$$

$$\begin{aligned} Q(S, C)_{1\alpha\beta\gamma 1\delta} = &-\frac{3}{2}\frac{1}{F}T_{11}C_{\delta\alpha\beta\gamma} + \frac{1}{2}\bar{g}_{11}\left(\tilde{S}_{\delta\gamma}\tilde{S}_{\alpha\beta} - \tilde{S}_{\delta\beta}\tilde{S}_{\alpha\gamma} - \frac{1}{3}\tilde{\kappa}(\tilde{g}_{\alpha\beta}\tilde{S}_{\delta\gamma} - \tilde{g}_{\alpha\gamma}\tilde{S}_{\delta\beta})\right) \\ &+ \frac{1}{12}\tilde{\kappa}\left(tr(T) + \frac{\Delta_1 F}{F}\right)\bar{g}_{11}\tilde{G}_{\delta\alpha\beta\gamma} \\ &- \frac{1}{4}\bar{g}_{11}\left(tr(T) + \frac{\Delta_1 F}{F}\right)(\tilde{g}_{\delta\gamma}\tilde{S}_{\alpha\beta} - \tilde{g}_{\delta\beta}\tilde{S}_{\alpha\gamma}), \end{aligned} \tag{23}$$

$$(R \cdot C)_{\alpha 11\beta\gamma\delta} = -\frac{1}{2}\frac{1}{F}\bar{g}_{11}\left(F(\tilde{R} \cdot \tilde{S})_{\alpha\beta\gamma\delta} - \frac{1}{4}\Delta_1 F Q(\tilde{g}, \tilde{S})_{\alpha\beta\gamma\delta}\right), \tag{24}$$

$$Q(S, C)_{\alpha 11\beta\gamma\delta} = -\frac{1}{2}\bar{g}_{11}\left(\frac{1}{3}\tilde{\kappa} - \frac{1}{2}\left(tr(T) + \frac{\Delta_1 F}{F}\right)\right)Q(\tilde{g}, \tilde{S})_{\alpha\beta\gamma\delta}. \tag{25}$$

From Lemma 2.1(ii) it follows that $Q(\tilde{S}, \tilde{G})_{\alpha\beta\gamma\delta\lambda\mu} = -Q(\tilde{g}, \tilde{U})_{\alpha\beta\gamma\delta\lambda\mu}$, which by making use of (18), turns into $Q(\tilde{S}, \tilde{G})_{\alpha\beta\gamma\delta\lambda\mu} = -Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}$. Now (21) takes the form

$$Q(S, C)_{\alpha\beta\gamma\delta\lambda\mu} = \frac{1}{2}F\left(Q(\tilde{S}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} + \left(\frac{1}{6}\tilde{\kappa} - \frac{1}{2}\left(tr(T) + \frac{\Delta_1 F}{F}\right)\right)Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}\right). \tag{26}$$

In view of Lemma 2.1(iii) we have also $(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} = Q(\tilde{S}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}$. Substituting this in (26) we obtain

$$Q(S, C)_{\alpha\beta\gamma\delta\lambda\mu} = \frac{1}{2}F(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} + \left(\frac{1}{6}\tilde{\kappa} - \frac{1}{2}\left(tr(T) + \frac{\Delta_1 F}{F}\right)Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}\right),$$

whence

$$Q(S, C)_{\alpha\beta\gamma\delta\lambda\mu} = \frac{1}{2}F(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} - \frac{1}{2}F\tau Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}, \tag{27}$$

$$\tau_1 = \frac{1}{2}\left(-\frac{1}{3}\tilde{\kappa} + tr(T) + \frac{\Delta_1 F}{F}\right). \tag{28}$$

Now, the equality $(R \cdot C)_{\alpha\beta\gamma\delta\lambda\mu} = LQ(S, C)_{\alpha\beta\gamma\delta\lambda\mu}$, in virtue of (20) and (27), gives

$$(1 - L)(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} = \left(\frac{1}{4}\frac{\Delta_1 F}{F} - \tau_1 L\right)Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}. \tag{29}$$

By (22) and (23) the relation $(R \cdot C)_{1\alpha\beta\gamma 1\delta} = LQ(S, C)_{1\alpha\beta\gamma 1\delta}$ is equivalent to

$$\begin{aligned}
&-\frac{1}{2}\frac{1}{F}T_{11}C_{\delta\alpha\beta\gamma} + \frac{1}{12}\tilde{\kappa}T_{11}\tilde{G}_{\delta\alpha\beta\gamma} - \frac{1}{4}T_{11}(\tilde{g}_{\gamma\delta}\tilde{S}_{\alpha\beta} - \tilde{g}_{\beta\delta}\tilde{S}_{\alpha\gamma})\\
&= -\frac{3}{2}\frac{1}{F}LT_{11}C_{\delta\alpha\beta\gamma} + \frac{1}{2}L\bar{g}_{11}(\tilde{S}_{\gamma\delta}\tilde{S}_{\alpha\beta} - \tilde{S}_{\beta\delta}\tilde{S}_{\alpha\gamma}) + \frac{1}{12}\tilde{\kappa}\left(tr(T) + \frac{\Delta_1 F}{F}\right)L\bar{g}_{11}\tilde{G}_{\delta\alpha\beta\gamma}\\
&\quad + \frac{1}{2}L\bar{g}_{11}\left(-\frac{1}{2}\left(tr(T) + \frac{\Delta_1 F}{F}\right)(\tilde{g}_{\delta\gamma}\tilde{S}_{\alpha\beta} - \tilde{g}_{\delta\beta}\tilde{S}_{\alpha\gamma}) - \frac{1}{3}\tilde{\kappa}(\tilde{g}_{\alpha\beta}\tilde{S}_{\delta\gamma} - \tilde{g}_{\alpha\gamma}\tilde{S}_{\delta\beta})\right).
\end{aligned} \tag{30}$$

Further, we can check that the relation $(R \cdot C)_{\alpha 11\beta\gamma\delta} = L(S, C)_{\alpha 11\beta\gamma\delta}$ turns into

$$(\tilde{R} \cdot \tilde{S})_{\alpha\beta\gamma\delta} = \left(\frac{1}{4}\frac{\Delta_1 F}{F} - \tau_1 L\right)Q(\tilde{g}, \tilde{S})_{\alpha\beta\gamma\delta}. \tag{31}$$

This, in view of Lemma 2.1(iii), is equivalent to

$$(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} = \left(\frac{1}{4}\frac{\Delta_1 F}{F} - \tau_1 L\right) Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu}. \tag{32}$$

Thus we have the following

PROPOSITION 4.1. *A 4-dimensional warped product* $\overline{M} \times_F N, \dim \overline{M} = 1$, *satisfies the condition* $R \cdot C = LQ(S, C)$ *if and only if* (29), (30) *and* (32) *hold on* $\mathscr{U}$.

5. Main Results

EXAMPLE 5.1. (i) *We present an example of a 4-dimensional warped product, with 1-dimensional base manifold, realizing* $(*)$ *and* $(*)_3$. *Let* $(N, \tilde{g}), \dim N = 3$, *be a semi-Riemannian manifold such that its Ricci tensor* $\tilde{S}$ *is of rank one and its scalar curvature* $\tilde{\kappa}$ *vanishes identically on N. Then, in view of Lemma* 2.1(iv), $(N, \tilde{g})$ *is a semisymmetric manifold. Furthermore, let F, defined by* $F(x^1) = a\exp(bx^1)$, $a = const. > 0, b = const. \neq 0$, *be a function on a 1-dimensional manifold* $(\overline{M}, g_1)$. *It is easy to check, that* $\overline{M} \times_F N$ *realizes* (29), (30) *and* (32), *with* $L = 1/3$. *Thus, in view of Proposition* 4.1, $\overline{M} \times_F N$ *fulfils* $R \cdot C = (1/3)Q(S, C)$. *From Corollary* 4.2 *of* [21] *it follows that the manifold* $\overline{M} \times_F N$ *is pseudosymmetric too. Next, using* (3.12) *of* [21] *and* (15), *we get* $R \cdot R = (1/12)\kappa Q(g, R)$, *where k is the scalar curvature of* $\overline{M} \times_F N$.

(ii) *We present an example of a 3-dimensional semisymmetric warped product such that the rank of its Ricci tensor is one and its scalar curvature vanishes identically. Let* $M_2 = \{(x^2, x^3) : x^2, x^3 \in \boldsymbol{R}\}$ *be a connected, non-empty, open subset of* $\boldsymbol{R}^2$, *equipped with the metric tensor* g_2 *defined by* $g_{2,22} = g_{2,33} = 0$, $g_{2,23} = g_{2,32} = 1$, *and let* $H = H(x^2)$ *be a smooth function on* M_2. *Moreover, let* (M_3, g_3) *be a 1-dimensional manifold. In* [35] (*see p.* 177) *it was shown that the rank of the Ricci tensor* $\tilde{S}$ *of the warped product* $M_2 \times_H M_3$ *is equal to one and that the scalar curvature of this manifold vanishes identically. Moreover, we have* (*cf.* [35], *p.* 177)

$$\tilde{S}_{22} = -\frac{1}{H}\left(\frac{\partial H_2}{\partial x^2} - \frac{1}{2H}H_2 H_2\right), \quad H_2 = \frac{\partial H}{\partial x^2}, \quad \tilde{S}_{33} = 0, \quad \tilde{S}_{44} = 0.$$

Furthermore, from Lemma 2.1 (iv) *it follows that* $M_2 \times_H M_3$ *is a semi-symmetric manifold.* (iii) *We consider the warped product* $\overline{M} \times_F N$, *where* $\dim \overline{M}$

$= 1$, *the warping function* F *is defined by* $F(x^1) = a\exp(bx^1), a = const. > 0,\ b = const. \neq 0$, *and* $(N, \tilde{g})$ *is a semisymmetric manifold defined in* (ii). *We can verify that the tensor* $S - (\kappa/4)g$ *is of rank one, i.e. the warped product* $\overline{M} \times_F N$ *is a quasi-Einstein manifold.*

In this section we prove, that under certain assumptions every 4-dimensional warped product $\overline{M} \times_F N, \dim \overline{M} = 1$, realizing (*) is the manifold described in Example 5.1(i).

Symmetrizing (30) in α, δ we obtain

$$\left(\frac{1}{2}T_{11} - \tau L\bar{g}_{11}\right) Q(\tilde{g}, \tilde{S})_{\alpha\beta\gamma\delta} = 0, \tag{33}$$

where

$$\tau = \frac{1}{2}\left(-\frac{2}{3}\tilde{\kappa} + tr(T) + \frac{\Delta_1 F}{F}\right). \tag{34}$$

From (19) it follows that the Weyl tensor C of every 4-dimensional warped product $\overline{M} \times_F N, \dim \overline{M} = 1$, vanishes at a point $x \in M_1 \times_F N$ if and only

$$\tilde{S}_{\alpha\beta} = \frac{1}{3}\tilde{\kappa}\tilde{g}_{\alpha\beta}. \tag{35}$$

holds at x. We note also that $Q(\tilde{g}, \tilde{S})$ vanishes at x if and only if (35) is satisfied at x. So, if the tensor C is non-zero at the point $x \in M_1 \times_F N$ then from (33) it follows that

$$\frac{1}{2}T_{11} = \tau L\bar{g}_{11} \tag{36}$$

holds at x. Applying (36) in (30) we obtain

$$\begin{aligned}
&-\frac{\tau}{F}C_{\delta\alpha\beta\gamma} + \frac{\tau}{6}\tilde{\kappa}\tilde{G}_{\delta\alpha\beta\gamma} - \frac{\tau}{2}(\tilde{g}_{\delta\gamma}\tilde{S}_{\alpha\beta} - \tilde{g}_{\delta\beta}\tilde{S}_{\alpha\gamma}) \\
&\quad = -3\frac{\tau}{F}C_{\delta\alpha\beta\gamma} + \frac{1}{2}(\tilde{S}_{\delta\gamma}\tilde{S}_{\alpha\beta} - \tilde{S}_{\delta\beta}\tilde{S}_{\alpha\gamma}) + \frac{1}{12}\tilde{\kappa}\left(tr(T) + \frac{\Delta_1 F}{F}\right)\tilde{G}_{\delta\alpha\beta\gamma} \\
&\quad - \frac{1}{4}\left(tr(T) + \frac{\Delta_1 F}{F}\right)(\tilde{g}_{\delta\gamma}\tilde{S}_{\alpha\beta} - \tilde{g}_{\delta\beta}\tilde{S}_{\alpha\gamma}) - \frac{1}{6}\tilde{\kappa}(\tilde{g}_{\alpha\beta}\tilde{S}_{\gamma\delta} - \tilde{g}_{\alpha\gamma}\tilde{S}_{\delta\beta}).
\end{aligned} \tag{37}$$

If $x \in U_L$ then the last equality reduces to

$$\frac{1}{F}\tau(3L-1)C_{\delta\alpha\beta\gamma}+\frac{1}{6}\tilde{\kappa}\left(\tau-\frac{1}{2}\left(tr(T)+\frac{\Delta_1 F}{F}\right)\right)\tilde{G}_{\delta\alpha\beta\gamma}$$
$$=\frac{1}{2}(\tilde{S}_{\gamma\delta}\tilde{S}_{\alpha\beta}-\tilde{S}_{\delta\beta}\tilde{S}_{\alpha\gamma})-\frac{1}{6}\tilde{\kappa}(\tilde{g}_{\alpha\beta}\tilde{S}_{\delta\gamma}-\tilde{g}_{\alpha\gamma}\tilde{S}_{\beta\delta}+\tilde{g}_{\gamma\delta}\tilde{S}_{\alpha\beta}-\tilde{g}_{\delta\beta}\tilde{S}_{\alpha\gamma}),$$

which, by (34), turns into

$$\frac{2}{F}\tau(3L-1)C_{\delta\alpha\beta\gamma}-\frac{1}{9}\tilde{\kappa}^2\tilde{G}_{\delta\alpha\beta\gamma}=\tilde{S}_{\gamma\delta}\tilde{S}_{\alpha\beta}-\tilde{S}_{\delta\beta}\tilde{S}_{\alpha\gamma}-\frac{1}{3}\tilde{\kappa}\tilde{U}_{\delta\alpha\beta\gamma}. \tag{38}$$

On the other hand (18) and (19) give

$$C_{\delta\alpha\beta\gamma}=\frac{1}{2}F\left(\tilde{U}_{\delta\alpha\beta\gamma}-\frac{2}{3}\tilde{\kappa}\tilde{G}_{\delta\alpha\beta\gamma}\right).$$

Applying this in (38) we obtain

$$\tau(3L-1)\tilde{U}_{\delta\alpha\beta\gamma}-\frac{2}{3}\tau(3L-1)\tilde{\kappa}\tilde{G}_{\delta\alpha\beta\gamma}-\frac{1}{9}\tilde{\kappa}^2\tilde{G}_{\delta\alpha\beta\gamma}=\tilde{S}_{\gamma\delta}\tilde{S}_{\alpha\beta}-\tilde{S}_{\delta\beta}\tilde{S}_{\alpha\gamma}-\frac{1}{3}\tilde{\kappa}\tilde{U}_{\delta\alpha\beta\gamma},$$

whence

$$\tilde{S}_{\gamma\delta}\tilde{S}_{\alpha\beta}-\tilde{S}_{\beta\delta}\tilde{S}_{\alpha\gamma}=\rho\tilde{U}_{\delta\alpha\beta\gamma}+\mu\tilde{G}_{\delta\alpha\beta\gamma}, \tag{39}$$

where

$$\rho=\tau(3L-1)+\frac{1}{3}\tilde{\kappa},\quad \mu=-\frac{1}{3}\tilde{\kappa}\left(2\tau(3L-1)+\frac{1}{3}\tilde{\kappa}\right). \tag{40}$$

We put $\tilde{A}_{\alpha\beta}=\tilde{S}_{\alpha\beta}-\rho\tilde{g}_{\alpha\beta}$. Thus, by (39), we have

$$\begin{aligned}\tilde{A}_{\gamma\delta}\tilde{A}_{\alpha\beta}-\tilde{A}_{\beta\delta}\tilde{A}_{\alpha\gamma}&=\tilde{S}_{\gamma\delta}\tilde{S}_{\alpha\beta}-\tilde{S}_{\beta\delta}\tilde{S}_{\alpha\gamma}+\rho^2\tilde{G}_{\delta\alpha\beta\gamma}\\&\quad-\rho(\tilde{g}_{\gamma\delta}\tilde{S}_{\alpha\beta}+\tilde{g}_{\alpha\beta}\tilde{S}_{\gamma\delta}-\tilde{g}_{\beta\delta}\tilde{S}_{\alpha\gamma}-\tilde{g}_{\alpha\gamma}\tilde{S}_{\beta\delta})\\&=(\rho^2+\mu)\tilde{G}_{\delta\alpha\beta\gamma},\end{aligned} \tag{41}$$

which leads to

$$Q(\tilde{A},A)=(\rho^2+\mu)Q(\tilde{A},\tilde{G}),$$

where the $(0,4)$-tensor A is defined by

$$A_{\alpha\beta\gamma\delta}=\tilde{A}_{\alpha\delta}\tilde{A}_{\beta\gamma}-\tilde{A}_{\alpha\gamma}\tilde{A}_{\beta\delta}.$$

Since the tensor $Q(\tilde{A},A)$ vanishes identically, we have $(\rho^2+\mu)Q(\tilde{A},\tilde{G})=0$, whence we get easily $(\rho^2+\mu)(\tilde{A}-(1/3)tr(\tilde{A})\tilde{g})=0$. Since $\tilde{S}\neq(1/3)\tilde{\kappa}\tilde{g}$ holds at

x, the last relation yields

$$\rho^2 + \mu = 0. \tag{42}$$

Further, using (29) and (32) we deduce that

$$(a)\quad (\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} = 0, \quad (b)\quad \frac{1}{4}\frac{\Delta_1 F}{F} = \tau L, \tag{43}$$

hold at $x \in \mathcal{U}_L$. Further, contracting (39) with $\tilde{g}^{\alpha\beta}$ we obtain

$$\tilde{S}^2_{\gamma\delta} = (\tilde{\kappa} - \rho)\tilde{S}_{\gamma\delta} + (2\mu + \rho\tilde{\kappa})\tilde{g}_{\gamma\delta},$$

which, by making use of (42), can be rewritten in the following form

$$\tilde{S}^2_{\gamma\delta} - \frac{1}{3}tr(\tilde{S}^2)\tilde{g}_{\gamma\delta} = \left(\frac{\tilde{\kappa}}{2} + \alpha\right)\left(\tilde{S}_{\gamma\delta} - \frac{\tilde{\kappa}}{3}\tilde{g}_{\gamma\delta}\right), \quad \alpha = (\tilde{\kappa} - \rho) - \frac{\tilde{\kappa}}{2} = \frac{\tilde{\kappa}}{2} - \rho. \tag{44}$$

From (44), in view of Lemma 2.1(i), it follows that $(\tilde{R} \cdot \tilde{S})_{\alpha\beta\gamma\delta} = \alpha Q(\tilde{g}, \tilde{S})_{\alpha\beta\gamma\delta}$ holds at $x \in \mathcal{U}_L$ and in a consequence, $(\tilde{R} \cdot \tilde{R})_{\alpha\beta\gamma\delta\varepsilon\mu} = \alpha Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\varepsilon\mu}$. The last relation, by (43)(a), implies $\alpha = 0$, i.e. $\rho = \tilde{\kappa}/2$. Applying the last equality and (42) in (40) we find

$$(3L - 1)\tau = \frac{5}{6}\tilde{\kappa}, \quad \frac{2}{3}\tilde{\kappa}(3L - 1)\tau = -\frac{13}{36}\tilde{\kappa}^2, \tag{45}$$

which gives $(5/6)\tilde{\kappa}^2 = -(13/36)\tilde{\kappa}^2$, whence $\tilde{\kappa} = 0$. Now (45) reduces to $(3L-1)\tau = 0$ and in a consequence, from (40) we get $\rho = \mu = 0$. So, (39) reduces to $rank(\tilde{S}) = 1$. Since $\tilde{\kappa} = 0$, (28) and (34) leads to $\tau_1 = \tau = (1/2)(tr(T) + (\Delta_1 F/F))$. Further, we denote by $\mathcal{U}_F$ the set consisting of all points of $\mathcal{U}_L$ at which $F' \neq 0$. We suppose that τ vanishes at $x \in \mathcal{U}_F$. Then (43)(b) implies $F' = 0$, a contradiction. Thus $L = 1/3$ holds on $\mathcal{U}_F$. We note that if $L = 1/3$ then only the functions F, defined by $F(x^1) = a\exp(bx^1), a = \text{const.} > 0, b = \text{const.} \neq 0$, are non-constant solutions of (36) and (43)(b). Thus we have the following.

THEOREM 5.1. *Let the set $\mathcal{U}_F$ be a dense subset of the set $\mathcal{U}_L$ of a 4-dimensional warped product $\overline{M} \times_F N, \dim \overline{M} = 1$. Then the warped product $\overline{M} \times_F N$ satisfies the condition $R \cdot C = LQ(S, C)$ on the set $\mathcal{U}_L \subset \mathcal{U} \subset \overline{M} \times N$ if and only if $L = 1/3, F(x^1) = a\exp(bx^1), a = const. > 0, b = const. \neq 0$, and $(N, \tilde{g})$ is a 3-dimensional semi-Riemannian manifold fulfilling $rank(\tilde{S}) = 1$ and $\tilde{\kappa} = 0$.*

REMARK 5.1. Let $(N, \tilde{g}), \dim N = 3$, be a semisymmetric manifold with vanishing identically on N scalar curvature $\tilde{\kappa}$. Suppose that $\tilde{g}$ is a Riemannian metric. Using this fact we can easily deduce that the condition rank $\tilde{S} \leq 1$ implies $\tilde{S} = 0$. Therefore, if the assumption rank $\tilde{S} = 1$ is fulfilled on $(N, \tilde{g})$ then the metric $\tilde{g}$ must be necessary indefinite, more precisely, $\tilde{g}$ is a Lorentzian metric.

Now from Theorem 5.1, in view of the above remark, follows the following

COROLLARY 5.1. *If a generalized Robertson-Walker spacetime satisfies* $(*)$ *then at every point of this spacetime at least one of the tensors* $R \cdot C$ *or* $Q(S, C)$ *must vanish.*

Let x be a point of a 4-dimensional warped product $\overline{M} \times_F N, \dim \overline{M} = 1$. If at x the conditions: $C \neq 0$ and $R \cdot C = 0$ are satisfied then $R \cdot R = 0$ holds at x ([17], Theorem 3). If at x the conditions: $C \neq 0, S \neq 0$ and $Q(S, C) = 0$ are satisfied then $R \cdot R = (\kappa/3)Q(g, R)$ holds at x ([25], Theorem 3.1). If at x the condition $S = 0$ is satisfied then $C = 0$ holds at x. This statement is an immediate consequence of (14) and (32). Finally, if at x the condition $C = 0$ is satisfied then $R \cdot R = \alpha Q(g, R), \alpha \in \boldsymbol{R}$, holds at x ([13], Lemma 3.1). These facts, together with Corollary 5.1, leads to the following

THEOREM 5.2. *Every generalized Robertson-Walker spacetime satisfying* $(*)$ *is a pseudosymmetric manifold.*

REMARK 5.2. (i) Theorem 2 of [29] implies that the warped product $\overline{M} \times_F N$, of a 1-dimensional base manifold $(\overline{M}, \bar{g})$, a warping function F and a 3-dimensional manifold $(N, \tilde{g})$ with the Ricci tensor $\tilde{S}$ of rank one realizes $(*)_5$, i.e. $\overline{M} \times_F N$ is a manifold with pseudosymmetric Weyl tensor.

(ii) We can also check that the Weyl tensor of the warped product defined above is not of harmonic curvature.

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