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CONFORMALLY FLAT MANIFOLDS WITH POSITIVE RICCI CURVATURE

By

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Abstract. In this paper by using Ros's method we prove the following result which has been obtained by Tani M. [1]: An n-dimensional ($n \geq 3$) compact conformally flat manifold with positive Ricci curvature and constant scalar curvature must be of constant sectional curvature.

1. Introduction

In [1] Tani M. showed the following

THEOREM. *Let M be an n-dimensional ($n \geq 3$) compact conformally flat manifold with positive Ricci curvature and constant scalar curvature. Then M must be of constant sectional curvature.*

The main aim of the present paper is to give a new proof for above Tani's theorem. Our method is the maximum principle which was first used by Ros A. (c.f. [2]).

2. Preliminaries

Let M be an n-dimensional ($n \geq 3$) Riemannian manifold with metric $\langle , \rangle$. The Riemannian curvature transformation $R(X, Y), X, Y \in T_pM$, where T_pM is the tangent space at $p \in M$, is related by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]},$$

where ∇_X is the operation of covariant differentiation with respect to X. The Riemannian curvature tensor $R(X, Y, Z, W)$ and the Ricci curvature tensor

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$\mathrm{Ric}(X, Y)$ are defined by

$$R(X, Y, Z, W) = \langle R(X, Y)W, Z\rangle,$$

and

$$\mathrm{Ric}(X, Y) = \mathrm{trace}(Z \to R(Z, X)Y)$$

respectively. We can define the first and the second covariant derivatives of Ric by

$$(\nabla\,\mathrm{Ric})(Z, X, Y) = Z(\mathrm{Ric}(X, Y)) - \mathrm{Ric}(\nabla_Z X, Y) - \mathrm{Ric}(X, \nabla_Z Y)$$

and

$$\begin{aligned}(\nabla^2\mathrm{Ric})(W, Z, X, Y) = {} & W((\nabla\,\mathrm{Ric})(Z, X, Y)) - (\nabla\,\mathrm{Ric})(\nabla_W Z, X, Y) \\ & - (\nabla\,\mathrm{Ric})(Z, \nabla_W X, Y) - (\nabla\,\mathrm{Ric})(Z, X, \nabla_W Y)\end{aligned}$$

respectively. Put $S_p = \{(u, v) : u, v \in T_pM, \langle u, v\rangle = 0 \text{ and } \langle u, u\rangle = \langle v, v\rangle = 1\}$ and $S = \bigcup_{p\in M} S_p$. We define a function $f : S \to R$ by $f(u, v) = \mathrm{Ric}(u, v)$ For any $(u, v) \in S$. If M is compact, then so is S, and the function f must attain its maximum at some point in S. So there exists some $(u_0, v_0) \in S_{p_0} \subset S, p_0 \in M$ such that

$$f(u_0, v_0) = \mathrm{Ric}(u_0, v_0) = \max_{(u,v)\in S}\{\mathrm{Ric}(u, v)\}. \tag{2.1}$$

Obviously we have $f(u_0, v_0) \geq 0$. For any w in $T_{p_0}M$, let $\gamma(t)$ be the geodesic in M given by the initial conditions $\gamma(0) = p_0, \gamma'(0) = w$. By parallel translating u_0 and v_0 along $\gamma(t)$ respectively we obtain vector fields $U(t)$ and $V(t)$. Then the function $f(t) = f(U(t), V(t))$ attains its maximum at $t = 0$. Thus we have (c.f. [2])

$$\begin{aligned} 0 &= \left.\frac{df(t)}{dt}\right|_{t=0} = (\nabla\,\mathrm{Ric})(w, u_0, v_0), \\ 0 &\geq \left.\frac{d^2 f(t)}{dt^2}\right|_{t=0} = (\nabla^2\,\mathrm{Ric})(w, w, u_0, v_0). \end{aligned} \tag{2.2}$$

For any unit vector w in $T_{p_0}M$ with $\langle u_0, w\rangle = \langle v_0, w\rangle = 0$, let $\alpha(s)$ be a curve in the sphere $\{u \in T_{p_0}M : \langle u, u\rangle = 1\}$ such that $\alpha(0) = v_0, \alpha'(0) = w$. The function $g(t) = f(u_0, \alpha(t))$ attains its maximum at $t = 0$. So we find

$$0 = \left.\frac{dg(t)}{dt}\right|_{t=0} = \mathrm{Ric}(u_0, w). \tag{2.3}$$

Similarly for the same w we have

$$\operatorname{Ric}(w, v_0) = 0. \tag{2.4}$$

Now we can choose a local field of orthonormal frame $e_1, \dots, e_n$ in M around the point p_0 such that at the point p_0 we have $e_1 = u_0$ and $e_2 = v_0$. In terms of this frame the Riemannian curvature tensor, Ricci curvature tensor and the first and the second covariant derivatives of Ric can be rewritten as $R_{ijkl} = R(e_i, e_j, e_k, e_l), R_{ij} = \operatorname{Ric}(e_i, e_j)$, $R_{ij,k} = (\nabla \operatorname{Ric})(e_k, e_i, e_j)$ and $R_{ij,kl} = (\nabla^2 \operatorname{Ric})(e_l, e_k, e_i, e_j)$ respectively. The following Ricci identity is well-known:

$$R_{ij,kl} - R_{ij,lk} = \sum_m (R_{mj} R_{mikl} + R_{im} R_{mjkl}). \tag{2.5}$$

By (2.2), (2.3) and (2.4) we get

$$0 \geq R_{12,kk}, \tag{2.6}$$

$$R_{1k} = R_{2k} = 0 \quad \text{for } k \geq 3 \tag{2.7}$$

at the point p_0.

From now on we assume that M is a conformally flat manifold with constant scalar curvature. Then we have

$$\begin{aligned} R_{ijkl} = {} & \frac{1}{n-2}(R_{ik}\delta_{jl} + R_{jl}\delta_{ik} - R_{il}\delta_{jk} - R_{jk}\delta_{il}) \\ & + \frac{\rho}{(n-1)(n-2)}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) \end{aligned} \tag{2.8}$$

and

$$R_{ij,k} = R_{ik,j}, \tag{2.9}$$

where $\rho = \sum_i R_{ii}$ is the scalar curvature.

3. The Proof of the Theorem

Now we shall complete the proof of the theorem. Throughout this section all conditions included in the theorem are assumed to be satisfied. We restrict ourselves to the point p_0. Taking sum about k in (2.6) and using (2.5), (2.7) and (2.9) together with the fact that the scalar curvature is constant, we obtain

$$0 \geq \sum_k R_{12,kk} = R_{12}(R_{11} + R_{22} + R_{1212}) + \sum_{m,k \geq 3} R_{km} R_{m12k}. \tag{3.1}$$

Since the $(n-2)\times(n-2)$-matrix $(R_{ij})_{i,j\geq 3}$ is symmetric, we can assume that the orthonormal frame chosen above satisfies

$$R_{ij}=\lambda_i\delta_{ij} \quad \textit{for } i,j\geq 3. \tag{3.2}$$

Introducing (2.8) and (3.2) into (3.1) we have

$$0\geq \frac{nR_{12}}{(n-1)(n-2)}\left((n-2)(R_{11}+R_{22})-\sum_{k\geq 3}\lambda_k\right). \tag{3.3}$$

From (2.7) and (3.2) we see that the vectors $e_3,\ldots,e_n$ are eigenvectors of the matrix $(R_{ij})_{1\leq i,j\leq n}$. Let E_1,E_2 be other two eigenvectors and set $\lambda_1=\mathrm{Ric}(E_1,E_1),\lambda_2=\mathrm{Ric}(E_2,E_2)$. Without loss of generality we may assume that $\lambda_1\geq\lambda_2$. It is easy to see that e_1,e_2 can be linearly represented by E_1,E_2, say $e_1=\cos\theta E_1-\sin\theta E_2$ and $e_2=\sin\theta E_1+\cos\theta E_2$. Thus, we have $R_{12}=\mathrm{Ric}(e_1,e_2)=\cos\theta\sin\theta(\lambda_1-\lambda_2)$. Since by (2.1), $R_{12}=f(u_0,v_0)=\max_{(u,v)\in S}\cdot\{\mathrm{Ric}(u,v)\}$, we must have $\theta=\pi/4$ and $R_{12}=1/2(\lambda_1-\lambda_2)$. Therefore we conclude that $R_{11}=R_{22}=1/2(\lambda_2+\lambda_2)$. For any $k\geq 3$, set $u(\theta)=\cos\theta E_1-\sin\theta e_k$ and $v(\theta)=\sin\theta E_1+\cos\theta e_k$, then we get $\mathrm{Ric}(u(\theta),v(\theta))=\cos\theta\sin\theta(\lambda_1-\lambda_k)$. Also by (2.1) and the fact that $\max_{\theta\in[0,\pi]}\{\mathrm{Ric}(u(\theta),v(\theta))\}=1/2|\lambda_1-\lambda_k|$ we get

$$\frac{1}{2}|\lambda_1-\lambda_k|\leq R_{12}=\frac{1}{2}(\lambda_1-\lambda_2) \quad \text{for } k\geq 3. \tag{3.4}$$

Similarly we have

$$\frac{1}{2}|\lambda_2-\lambda_k|\leq \frac{1}{2}(\lambda_1-\lambda_2) \quad \text{for } k\geq 3. \tag{3.5}$$

Thus, by (3.4) and (3.5) we see that $\lambda_1\geq\lambda_k\geq\lambda_2$ for any $k\geq 3$. Now (3.3) can be rewritten as

$$0\geq R_{12}\left(\sum_{k\geq 3}(\lambda_1-\lambda_k)+(n-2)\lambda_2\right)\geq (n-2)\lambda_2 R_{12}. \tag{3.6}$$

Since M has positive Ricci curvature, so $\lambda_2>0$ and we must have $R_{12}=\max_{(u,v)\in S}\{\mathrm{Ric}(u,v)\}=0$. Thus $\lambda_1=\lambda_2=\cdots=\lambda_n$ everywhere and the conclusion of the theorem now follows easily.

References

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