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THE CAUCHY PROBLEM FOR WEAKLY HYPERBOLIC EQUATIONS OF SECOND ORDER

By

Haruhisa ISHIDA

§1. Introduction

In this article we shall study the problem of local existence of C^{∞} solutions to the following semilinear Cauchy problem on $[0, T] \times \boldsymbol{R}_x^n (T > 0)$

$$L(t, x, \partial_t, \partial_x) u(t, x) = f(t, x, u), \tag{1.1}$$

$$u(0, x) = u_0(x), u_t(0, x) = u_1(x), \tag{1.2}$$

where

$$\begin{aligned} L(t, x, \partial_t, \partial_x) &= \partial_t^2 - a_1(t) \sum_{j,k=1}^{n} a_{jk} \{a_0(x) \partial_{x_j} \partial_{x_k} + (\partial_{x_j} a_0(x)) \partial_{x_k}\} \\ &\quad - \sum_{j=1}^{n} b_j(t, x) \partial_{x_j} - c(t, x) - d(t, x) \partial_t, \\ &= \partial_t^2 - a^{\sharp}(t, x, \partial_x) - b(t, x, \partial_x) - c(t, x) - d(t, x) \partial_t. \end{aligned}$$

Throughout the present article we assume that $0 < C_a \leq a_0(x) \in \mathfrak{B}^{\infty}(\boldsymbol{R}^n)$, $q(\xi) = \sum_{j,k=1}^{n} a_{jk} \xi_j \xi_k \geq 0$ (a_{jk} is a real constant, $a_{jk} = a_{kj}$) for all $\xi \in \boldsymbol{R}^n$ and that $0 \leq a_1(t) \in C^{\infty}([0, T])$ satisfies the condition below:

$$\begin{aligned} N = \operatorname{card}\{[p, q] \subset [0, T]; a_1'([p, q]) \subset \{0\}, a_1'(p - \varepsilon) a_1'(q + \varepsilon) < 0 (0 < \varepsilon \ll 1)\} \\ < \infty, \end{aligned} \tag{1.3}$$

where card X means the cardinality of a set X, that is, the number N of the connected components of the sign-changing zero-set of a_1' on $[0, T]$ is finite. Moreover we impose that $b_j, c, d, f \in \mathfrak{B}^{\infty}$ (with $f([0, T], \boldsymbol{R}^n, 0) \subset \{0\}$) and that

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there is some constant $A_\alpha > 0$ such that

$$|\partial_x^\alpha b(t,x,\xi)|^2 \le A_\alpha(a(t,x,\xi) + |\partial_t a(t,x,\xi)|) \tag{1.4}$$

for every $\alpha \in \boldsymbol{Z}_+^n$ and $(t,x,\xi) \in [0,T] \times \boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n$, where $a(t,x,\xi) = a_0(x)a_1(t)q(\xi)$ (≥ 0) (refer to K. Kajitani [7] for a more general condition on lower order terms; see also K. Kajitani-S. Wakabayashi [8]).

Then we obtain the following result.

THEOREM 1.1. *For any initial data* $u_0, u_1 \in C_0^\infty(\boldsymbol{R}^n)$ *there exists a small constant* $(T \ge)\ T_0 > 0$ *such that the Cauchy problem* (1.1), (1.2) *has a unique solution* $u(t,x) \in C^\infty([0,T_0] \times \boldsymbol{R}_x^n)$.

When $a_0(x)$ is a constant and a_{jk} $(j,k = 1,\dots,n)$ are real-analytic in t, Theorem 1.1 is proved in P. D'Ancona [4]. He solved the corresponding linearized Cauchy problem according to N. Orrù [11] and did the semilinear problem by applying the implicit function theorem of Nash and Moser (see R. S. Hamilton [5]). In our strategy we shall use pseudo-differential operators to handle the linearized problem and employ the successive approximation method to solve the semilinear problem (1.1), (1.2). Then the corresponding linear problem to (1.1), (1.2) is stated in the following Cauchy problem on $[0,T] \times \boldsymbol{R}_x^n$

$$L(t,x,\partial_t,\partial_x)u(t,x) = f(t,x), \tag{1.5}$$

$$u(0,x) = u_0(x), u_t(0,x) = u_1(x). \tag{1.6}$$

As to the linear problem (1.5), (1.6) the next existence and uniqueness theorem is valid.

THEOREM 1.2. *Suppose that* (1.3) *and* (1.4) *hold. Let the initial data* $u_0(x)$, $u_1(x)$ *belong to* $C^\infty(\boldsymbol{R}^n)$ *and* $f(t,x) \in C^\infty([0,T] \times \boldsymbol{R}_x^n)$. *Then the Cauchy problem* (1.5), (1.6) *admits a solution* $u(t,x) \in C^\infty([0,T] \times \boldsymbol{R}_x^n)$. *In particular, if* $\operatorname{supp} u_0$ *and* $\operatorname{supp} u_1$ *are contained in the open ball* $B(r) = \{x \in \boldsymbol{R}^n; |x| < r\}$ *and if* $\operatorname{supp} f(t,\cdot)$ *lies in the ball* $B(r + t\lambda(T))$ *for every* $t \in [0,T]$, *then the unique solution* $u(t,x)$ *enjoys the finite propagation speed property with speed for* $t \in [0,T]$ *not greater than* $\lambda(T)$, *where*

$$\lambda(T) = \sup_{(t,x) \in [0,T] \times \boldsymbol{R}_x^n} \sqrt{\|(a_{jk}a_0(x)a_1(t))\|};$$

so that $\operatorname{supp} u(t,\cdot)$ *is included in the ball* $B(r + t\lambda(T))$. *Further the following estimate is established: for any* $s \ge 0$ *and* $t \in [0,T]$ *there exist some constants* $h > 0$

and $C = C(h) > 0$ *such that*

$$\|e^{\gamma\Lambda}(t,x,D_x)^{-1}u(t,\cdot)\|^2_{H^s(\boldsymbol{R}^n_x)} + \|e^{\gamma\Lambda}(t,x,D_x)^{-1}u_t(t,\cdot)\|^2_{H^s(\boldsymbol{R}^n_x)}$$

$$\leq e^{\int_0^t c(\tau)\,d\tau}\Big(C\|e^{\gamma\Lambda}(0,x,D_x)^{-1}u_0\|^2_{H^{s+1}(\boldsymbol{R}^n_x)} + \|e^{\gamma\Lambda}(0,x,D_x)^{-1}u_1\|^2_{H^s(\boldsymbol{R}^n_x)}$$

$$+ \int_0^t \|e^{\gamma\Lambda}(\tau,x,D_x)^{-1}f(\tau,\cdot)\|^2_{H^s(\boldsymbol{R}^n_x)}\,d\tau\Big), \tag{1.7}$$

where $c(t)$ *is a nonnegative continuous function dependent on the coefficients* $\partial_x^\alpha\partial_t^\beta a_{jk}a_0(x)a_1(t)$, $\partial_x^\beta b_j(t,x)$, $\partial_x^\delta c(t,x)$, $\partial_x^\delta d(t,x)$ $(|\alpha| \leq 3, |\beta| \leq 2, |\delta| \leq 1)$ *and* γ, $\varepsilon_0 > 0$ *which depend only on* $\partial_x^\alpha\partial_t^\beta a_{jk}a_0(x)a_1(t)$ $(|\alpha| \leq 2, \beta = 0, 1)$, *while*

$$\Lambda(t,x,\xi) = \int_0^t \frac{\sqrt{|\partial_\tau a(\tau,x,\xi)|^2 + (\log q_h(\xi))^4}}{a(\tau,x,\xi) + (\log q_h(\xi))^2}\,d\tau$$

$$+ (A_0 + \varepsilon_0 \log q_h(\xi))t - M\log q_h(\xi), \tag{1.8}$$

$$q_h(\xi) = h + q(\xi) \quad (h > 1),$$

($M > 0$ *is taken large enough, independent of* h, *as* $\Lambda(t,x,\xi) \leq 0$ *for all* $(t,x,\xi) \in [0,T] \times \boldsymbol{R}^{2n}_{x,\xi}$) *and* $e^{\gamma\Lambda}(t,x,D_x)^{-1}$ *denotes the inverse of the pseudo-differential operator* $e^{\gamma\Lambda}(t,x,D_x)$ *with symbol* $e^{\gamma\Lambda(t,x,\xi)}$).

We shall perform the proof of Theorem 1.2 via the approximation to the equation (1.5) by strictly hyperbolic equations because the estimate (1.7) for the equation (1.5) after replacing the coefficients $(a_{jk}a_0(x)a_1(t))$ of the principle part by $(a_{jk}a_0(x)a_1(t)) + \varepsilon^2\delta_{jk}$ ($\varepsilon > 0, \delta_{jk}$ is Kronecker's delta) remains valid. Thus our main task is to lead the estimate (1.7). To do so we shall transform $u(t,x)$ into $v(t,x)$ with $u(t,x) = e^{\gamma\Lambda}(t,x,D_x)v(t,x)$ and take advantage of the energy function

$$E(t) = \|(\partial_t + \gamma\Lambda_t(t,x,D_x) + R_1(t,x,D_x))v(t,\cdot)\|^2_{H^s(\boldsymbol{R}^n_x)}$$

$$+ (a^\sharp(t,x,D_x)v(t,\cdot), v(t,\cdot))_{H^s(\boldsymbol{R}^n_x)} + \|(\log q_h(D_x))v(t,\cdot)\|^2_{H^s(\boldsymbol{R}^n_x)},$$

where $R_1(t,x,D_x)$ is given by

$$R_1(t,x,D_x) = e^{\gamma\Lambda}(t,x,D_x)^{-1} \circ \partial_t e^{\gamma\Lambda}(t,x,D_x) - \gamma\Lambda_t(t,x,D_x).$$

Then we shall get the energy inequality

$$E(t) \le C(T)\left\{E(0) + \int_0^t \|e^{\gamma\Lambda}(\tau, x, D_x)^{-1} f(\tau, \cdot)\|_{H^s}^2 d\tau\right\},$$

which implies the estimate (1.7).

We remark that the same conclusion as Theorem 1.2 generally fails to hold for nonnegative $a_1(t) \in C^\infty([0,T])$ without any restrictive condition (see F. Colombini-S. Spagnolo [3] or S. Tarama [12]).

§ 2. Preliminaries

In this section we shall observe the existence of $e^{\gamma\Lambda}(t, x, D_x)^{-1}$ and mention some propositions adopting in the subsequent sections.

LEMMA 2.1. (i) $|\partial_{x_j} a_0(x)| \le C_1 a_0(x)$ $(j = 1, \dots, n)$.

(ii) ((1.7.2) *in O. A. Oleĭnik-E. V. Radkevič* [10]) *If* $p(x) \in C^2(\boldsymbol{R})$ *is nonnegative, then*

$$|\partial_x p(x)|^2 \le 2p(x) \sup |\partial_x^2 p(x)|.$$

(iii) $|\partial_{\xi_j} a(t, x, \xi)| \le C_2\sqrt{a(t, x, \xi)}$ $(j = 1, \dots, n)$.

(iv) $|D_{x_j} a(t, x, \xi)| \le C_3 a(t, x, \xi)$ $(j = 1, \dots, n)$.

(v) $|\partial_{\xi_k}\partial_{\xi_j} a(t, x, \xi)| \le C_4$ $(j, k = 1, \dots, n)$.

(vi) $|D_{x_k} D_{x_j} a(t, x, \xi)| \le C_5 a(t, x, \xi)$ $(j, k = 1, \dots, n)$.

(vii) $|\partial_{\xi_j} b(t, x, \xi)| \le C_6$ $(j = 1, \dots, n)$.

The next proposition describes basic properties about $\Lambda(t, x, \xi)$.

LEMMA 2.2. *Let* $\langle\xi\rangle_h = \sqrt{h + |\xi|^2}$ $(h \ge 1)$.

(i)

$$|\partial_\xi^\alpha D_x^\beta \Lambda(t, x, \xi)| \le \begin{cases} C_{0,0} \log q_h(\xi), & \text{if } |\alpha| + |\beta| = 0, \\ C_{\alpha,\beta}\{q(\xi) + (\log q_h(\xi))^2\}^{-(|\alpha|/2)} \tilde{\Lambda}_h(t, x, \xi), & \text{if } |\alpha| + |\beta| > 0, \end{cases}$$

where

$$\tilde{\Lambda}_h(t, x, \xi) = \int_0^t \frac{\sqrt{|\partial_s a(s, x, \xi)|^2 + (\log q_h(\xi))^4}}{a(s, x, \xi) + (\log q_h(\xi))^2} ds + 1.$$

(ii) $\Lambda(t,x,\xi+\eta) \leq \Lambda(t,x,\xi) + \{M + 6(N+1)\}\log\left(1 + \frac{|\eta|}{\log q_h(\xi)}\right) + t + C_1$.

(iii) $\Lambda(t,x,\xi+\eta) \leq \Lambda(t,x,\xi) + \Lambda(t,x,\eta) + 2(M+N+1)\log q_h(\eta) + t + C_2$.

(iv) $\Lambda(t,x+y,\xi) \leq \Lambda(t,x,\xi) + t + C_3$.

PROOF. We first note that

$$\Lambda_h(t,x,\xi) + 1 \leq \tilde{\Lambda}_h(t,x,\xi) \leq \Lambda_h(t,x,\xi) + t + 1, \tag{2.1}$$

where

$$\Lambda_h(t,x,\xi) = \int_0^t \frac{|\partial_s a(s,x,\xi)|}{a(s,x,\xi) + (\log q_h(\xi))^2}\, ds.$$

It is easy to verify that

$$|\partial_\xi^\alpha \log q_h(\xi)| \leq \begin{cases} \log q_h(\xi), & \text{if } |\alpha| = 0, \\ C_\alpha q_h(\xi)^{-(|\alpha|/2)}, & \text{if } |\alpha| > 0, \end{cases}$$
$$\leq C_\alpha q_h(\xi)^{-(|\alpha|/2)} \log q_h(\xi).$$

Let us show that $\Lambda_h(t,x,\xi) \leq \Lambda_h(T,x,\xi) \leq C\log q_h(\xi)$. For this purpose we decompose $[0,T]$ as

$$[0,T] = \{t \in [0,T]; \partial_t a_1(t) \geq 0\} \cup \{t \in [0,T]; \partial_t a_1(t) \leq 0\}$$
$$= [r_1^+, s_1^+] \cup \cdots \cup [r_{N^+}^+, s_{N^+}^+] \cup [r_1^-, s_1^-] \cup \cdots \cup [r_{N^-}^-, s_{N^-}^-]$$

with $N^+ + N^- \leq N+1$. Here $r_j^+ < s_j^+ \leq r_{j+1}^+$ $(j = 1,\dots,N^+ - 1)$, $r_k^- < s_k^- \leq r_{k+1}^-$ $(k = 1,\dots,N^- - 1)$ either $r_1^+ = 0$ or $r_1^- = 0$ and either $s_{N^+}^+ = T$ or $s_{N^-}^- = T$. Then the following equality is valid:

$$\Lambda_h(T,x,\xi) = \log \prod_{j=1}^{N^+} \frac{a(s_j^+,x,\xi) + (\log q_h(\xi))^2}{a(r_j^+,x,\xi) + (\log q_h(\xi))^2}$$
$$+ \log \prod_{j=1}^{N^-} \frac{a(r_j^-,x,\xi) + (\log q_h(\xi))^2}{a(s_j^-,x,\xi) + (\log q_h(\xi))^2}, \tag{2.2}$$

which deduces that $\Lambda_h(T,x,\xi) \leq C\log q_h(\xi)$. Using (i) and (ii) in Lemma 2.1, we know that

$$|\partial_\xi^\alpha D_x^\beta \tilde{\Lambda}_h(t,x,\xi)| \leq C_{\alpha,\beta}\{q(\xi) + (\log q_h(\xi))^2\}^{-(|\alpha|/2)} \tilde{\Lambda}_h(t,x,\xi).$$

Thus (i) is established. Now, because of (ii) in Lemma 2.1 we get

$$
\begin{aligned}
&a(s,x,\xi+\eta)+(\log q_h(\xi+\eta))^2 \\
&\quad\le a(s,x,\xi)+2(\log q_h(\xi))^2+\sum_{j=1}^{n}\eta_j\partial_{\xi_j}a(s,x,\xi)+a(s,x,\eta)+2(\log 2q_1(\eta))^2 \\
&\quad\le 2(a(s,x,\xi)+(\log q_h(\xi))^2) \\
&\qquad\times\left(1+\frac{C|\eta|}{\sqrt{a(s,x,\xi)+(\log q_h(\xi))^2}}+\frac{C'|\eta|^2}{a(s,x,\xi)+(\log q_h(\xi))^2}\right) \\
&\quad\le C_0(a(s,x,\xi)+(\log q_h(\xi))^2)\left(1+\frac{|\eta|}{\sqrt{a(s,x,\xi)+(\log q_h(\xi))^2}}\right)^2. \qquad (2.3)
\end{aligned}
$$

Equivalently, the next inequality

$$
\begin{aligned}
&\frac{1}{a(s,x,\xi+\eta)+(\log q_h(\xi+\eta))^2} \\
&\quad\le\frac{C_0}{a(s,x,\xi)+(\log q_h(\xi))^2}\left(1+\frac{|\eta|}{\sqrt{a(s,x,\xi+\eta)+(\log q_h(\xi+\eta))^2}}\right)^2 \qquad (2.4)
\end{aligned}
$$

holds. Here

$$
\begin{aligned}
&\frac{1}{a(s,x,\xi+\eta)+(\log q_h(\xi+\eta))^2} \\
&\quad\le\frac{4C_0}{a(s,x,\xi)+(\log q_h(\xi))^2}\left(1+\frac{|\eta|}{\sqrt{a(s,x,\xi)+(\log q_h(\xi))^2}}\right)^2 \qquad (2.5)
\end{aligned}
$$

if $q(\xi+\eta)\ge q(\xi)/4$. In the meantime, when $q(\xi+\eta)\le q(\xi)/4$ and $q(\xi)\ge 1$, by virtue of $1\le C|\eta|/(\log q_h(\xi))^2$, for $h\ge e$

$$
\begin{aligned}
&\frac{1}{a(s,x,\xi+\eta)+(\log q_h(\xi+\eta))^2} \\
&\quad\le\frac{C_0\sqrt{C}}{a(s,x,\xi)+(\log q_h(\xi))^2}\left(1+\frac{|\eta|}{\log q_h(\xi)}\right)^4 \qquad (2.6)
\end{aligned}
$$

is valid. From (2.2), (2.3), (2.5) and (2.6) we obtain

$$\Lambda_h(t,x,\xi+\eta)$$
$$\leq \log C_0^{2N^+}\left(1+\frac{|\eta|}{\log q_h(\xi)}\right)^{6N^+}\prod_{j=1}^{N^+}\frac{a(s_j^+,x,\xi)+(\log q_h(\xi))^2}{a(r_j^+,x,\xi)+(\log q_h(\xi))^2}$$
$$+\log C_0^{2N^-}\left(1+\frac{|\eta|}{\log q_h(\xi)}\right)^{6N^-}\prod_{j=1}^{N^-}\frac{a(r_j^-,x,\xi)+(\log q_h(\xi))^2}{a(s_j^-,x,\xi)+(\log q_h(\xi))^2}. \tag{2.7}$$

Next, recall that $N^+ + N^- \leq N+1$, and check that for any fixed $2(N+1)$ points $r_j^+, s_j^+; r_k^-, s_k^-$ $(j=1,\ldots,N^+; k=1,\ldots,N^-; N^+ + N^- = N+1)$ with $r_j^+ < s_j^+ \leq r_{j+1}^+$, $r_k^- < s_k^- \leq r_{k+1}^-$ and

$$\left(\bigcup_{j=1}^{N^+}[r_j^+,s_j^+]\right)\cup\left(\bigcup_{j=1}^{N^-}[r_j^-,s_j^-]\right)=\bigcup_{j=1}^{N^+ + N^-}[r_j,s_j]\subset[0,t]$$

satisfying $[r_j,s_j]\cap[r_{j+1},s_{j+1}]=\{s_j\}=\{r_{j+1}\}$ $(r_j < s_j \leq r_{j+1})$, then

$$\log\prod_{j=1}^{N^+}\frac{a(s_j^+,x,\xi)+(\log q_h(\xi))^2}{a(r_j^+,x,\xi)+(\log q_h(\xi))^2}+\log\prod_{j=1}^{N^-}\frac{a(r_j^-,x,\xi)+(\log q_h(\xi))^2}{a(s_j^-,x,\xi)+(\log q_h(\xi))^2}$$
$$=\sum_{j=1}^{N^+}\int_{r_j^+}^{s_j^+}\frac{\partial_s a(s,x,\xi)}{a(s,x,\xi)+(\log q_h(\xi))^2}\,ds+\sum_{j=1}^{N^-}\int_{r_j^-}^{s_j^-}\frac{-\partial_s a(s,x,\xi)}{a(s,x,\xi)+(\log q_h(\xi))^2}\,ds$$
$$\leq \Lambda_h(t,x,\xi). \tag{2.8}$$

So (2.7) and (2.8) imply

$$\Lambda_h(t,x,\xi+\eta)\leq\Lambda_h(t,x,\xi)+6(N+1)\log\left(1+\frac{|\eta|}{\log q_h(\xi)}\right)+2(N+1)\log C_0.$$

Taking account of (2.1), we find that

$$\tilde{\Lambda}_h(t,x,\xi+\eta)\leq\tilde{\Lambda}_h(t,x,\xi)+6(N+1)\log\left(1+\frac{|\eta|}{\log q_h(\xi)}\right)+2(N+1)\log C_0+t+2.$$

In addition, since

$$q_h(\xi+\eta)\leq q_h(\xi)\left(1+\frac{C|\eta|}{\sqrt{q_h(\xi)}}\right)^2,$$

we gain (ii) via the analogous manner as (2.3)–(2.6). Further, we have for $h\geq e$

$$a(s,x,\xi+\eta)+(\log q_h(\xi+\eta))^2$$
$$\leq 2(a(s,x,\xi)+a(s,x,\eta)+(\log q_h(\xi))^2+(\log q_h(\eta))^2)$$
$$\leq 2(a(s,x,\xi)+(\log q_h(\xi))^2)(a(s,x,\eta)+(\log q_h(\eta))^2) \tag{2.9}$$

On the other hand, by substituting (ξ,η) for $(\xi+\eta,-\eta)$ in (2.9)

$$\frac{1}{a(s,x,\xi+\eta)+(\log q_h(\xi+\eta))^2} \leq \frac{2(a(s,x,\eta)+(\log q_h(\eta))^2)}{a(s,x,\xi)+(\log q_h(\xi))^2} \tag{2.10}$$

is true. This time, by applying (2.9) and (2.10) instead of (2.3) and (2.4) to (2.2)

$$\Lambda_h(t,x,\xi+\eta) \leq \Lambda_h(t,x,\xi)+\Lambda_h(t,x,\eta)+2(N+1)\log q_h(\eta)+2(N+1)\log C_0'.$$

Hence we see (iii). Also, in aid of (2.2) and $C^{-1}a_0(x) \leq a_0(x+y) \leq Ca_0(x)$ for $C \geq \sup a_0(x)/C_a(\geq 1)$

$$e^{\Lambda_h(t,x+y,\xi)} \leq C^{2(N^++N^-)}e^{\Lambda_h(t,x,\xi)} \leq C^{2(N+1)}e^{\Lambda_h(t,x,\xi)}$$

is valid. Thanks to (2.1),

$$e^{\Lambda(t,x+y,\xi)} \leq C^{2(N+1)}e^{t+2}e^{\Lambda(t,x,\xi)},$$

which therefore means (iv). □

Let $K(t,x,D_x)$ and $\tilde{K}(t,x,D_x)$ be the pseudo-differential operators with symbols $\sigma(K)(t,x,\xi)=e^{\gamma\Lambda(t,x,\xi)}$ and $\sigma(\tilde{K})(t,x,\xi)=e^{-\gamma\Lambda(t,x,\xi)}$ respectively, where $\gamma>0$ will be determined later. Then the symbol of the product of $\tilde{K}(t,x,D_x)$ and $K(t,x,D_x)$ is given by

$$\sigma(\tilde{K}\circ K)(t,x,\xi)=1-\sigma(R)(t,x,\xi),$$

where

$$\sigma(R)(t,x,\xi)=\sum_{|\alpha|=1}\int_0^1 \text{Os-}\iint e^{-iy\cdot\eta}\{D_\xi^\alpha\gamma\Lambda(t,x,\xi+\eta)\}e^{-\gamma\Lambda(t,x,\xi+\eta)}$$
$$\times\{\partial_x^\alpha\gamma\Lambda(t,x+\theta y,\xi)\}e^{\gamma\Lambda(t,x+\theta y,\xi)}\,dy\,\tilde{d}\eta\,d\theta$$

($\tilde{d}\eta=(2\pi)^{-n}\,d\eta$). Here the above oscillatory integral of a symbol $p(x,\xi)$ indicates

$$\text{Os-}\iint e^{-iy\cdot\eta}p(y,\eta)\,dy\,\tilde{d}\eta=\lim_{\varepsilon\to 0}\iint e^{-iy\cdot\eta}\chi(\varepsilon y,\varepsilon\eta)p(y,\eta)\,dy\,\tilde{d}\eta$$

for $\chi\in\mathscr{S}(\boldsymbol{R}_y^n\times\boldsymbol{R}_\eta^n)$ such that $\chi(0,0)=1$ (see H. Kumano-go [9]).

Let $g_{(x,\xi)} = \phi^{-2}(x,\xi)|dx|^2 + \Psi^{-2}(x,\xi)|d\xi|^2$ be a Riemannian metric on $\boldsymbol{R}^{2n}$. Following L. Hörmander [6], we say that g is σ-temperate if there exist positive constants c, C, $\tilde{C}$ and k such that

$$C^{-1}g_{(x,\xi)}(z,\zeta) \leq g_{(x+y,\xi+\eta)}(z,\zeta) \leq Cg_{(x,\xi)}(z,\zeta)$$

when $g_{(x,\xi)}(y,\eta) \leq c$ and such that

$$g_{(y,\eta)}(z,\zeta) \leq \tilde{C}g_{(x,\xi)}(z,\zeta)(1 + g^{\sigma}_{(y,\eta)}(x-y,\xi-\eta))^k$$

for all (x,ξ), (y,η), $(z,\zeta) \in \boldsymbol{R}^{2n}$, where the dual form $g^{\sigma}_{(x,\xi)}(y,\eta)$ of $g_{(x,\xi)}(y,\eta) = \phi^{-2}(x,\xi)|y|^2 + \Psi^{-2}(x,\xi)|\eta|^2$ is presented by

$$g^{\sigma}_{(x,\xi)}(y,\eta) = \Psi^2(x,\xi)|y|^2 + \phi^2(x,\xi)|\eta|^2.$$

A positive real-valued function $m(x,\xi)$ defined on $\boldsymbol{R}^{2n}$ is called g-continuous if there are positive constants c and C such that

$$C^{-1}m(x,\xi) \leq m(x+y,\xi+\eta) \leq Cm(x,\xi)$$

provided $g_{(x,\xi)}(y,\eta) \leq c$. A g-continuous function $m(x,\xi)$ is said to be (σ,g)-temperate if there exist constants $C > 0$ and $k \in \boldsymbol{R}$ such that

$$m(x+y,\xi+\eta) \leq Cm(x,\xi)(1 + g^{\sigma}_{(x,\xi)}(y,\eta))^k$$

for every (x,ξ), $(y,\eta) \in \boldsymbol{R}^{2n}$. For a positive function $m(x,\xi)$ and $g_{(x,\xi)}$, we define the symbol class $S(m,g)$ of pseudo-differential operators by the set of all $p(x,\xi) \in C^{\infty}(\boldsymbol{R}^{2n})$ satisfying

$$|\partial_{\xi}^{\alpha}D_x^{\beta}p(x,\xi)| \leq C_{\alpha,\beta}m(x,\xi)\phi(x,\xi)^{-|\beta|}\Psi(x,\xi)^{-|\alpha|}$$

for $(x,\xi) \in \boldsymbol{R}^{2n}$. Then we have the next claims.

Lemma 2.3.

$$\begin{aligned} g_{0(x,\xi)} &= |dx|^2 + \frac{1}{q(\xi) + (\log q_h(\xi))^2}|d\xi|^2 \\ &= |dx|^2 + \Psi_0^{-2}(\xi)|d\xi|^2 \end{aligned}$$

and

$$\tilde{g}_{0(x,\xi)} = \tilde{\Lambda}_h(t,x,\xi)^2 g_{0(x,\xi)}$$

are σ-temperate.

PROOF. In the same way as (2.3) and (2.7) we can show that

$$\Psi_0^2(\xi+\eta) \leq C_1\Psi_0^2(\xi)\left(1+\frac{|\eta|}{\Psi_0(\xi)}\right)^2$$

and

$$\begin{aligned}\Psi_0^2(\xi) &\leq 2\Psi_0^2(\eta)(1+q(\xi-\eta)+(\log 2q_1(\xi-\eta))^2)\\ &\leq C_2\Psi_0^2(\eta)(1+|\xi-\eta|^2),\end{aligned}$$

respectively, which follow that g_0 is σ-temperate. As well, similarly to (2.3)

$$\begin{aligned}\tilde{\Lambda}_h(t,x,+y,\xi)^2 &\leq C_3\tilde{\Lambda}_h(t,x,\xi)^2(1+|y|)\\ &\leq C_3\tilde{\Lambda}_h(t,x,\xi)^2(1+\tilde{\Lambda}_h(t,x,\xi)|y|),\\ \tilde{\Lambda}_h(t,x,\xi+\eta) &\leq \tilde{\Lambda}_h(t,x,\xi)\left(1+\frac{C|\eta|}{\Psi_0(\xi)}+\frac{C'|\eta|^2}{\Psi_0^2(\xi)}\right)\\ &\leq C_4\tilde{\Lambda}_h(t,x,\xi)\left(1+\frac{\tilde{\Lambda}_h(t,x,\xi)}{\Psi_0(\xi)}|\eta|\right)^2\end{aligned}$$

and $\tilde{g}_0 \leq \tilde{g}_0^\sigma$ imply that $\tilde{g}_0$ is σ-temperate by Proposition 18.5.6 in L. Hörmander [6].

LEMMA 2.4. *$\sigma(K)(t,x,\xi)$ and $\sigma(\tilde{K})(t,x,\xi)$ are $(\sigma,\tilde{g}_0)$-temperate.*

PROOF. Since $\sigma(\tilde{K})(t,x,\xi)=\sigma(K)(t,x,\xi)^{-1}$, it suffices to observe that $\sigma(K)(t,x,\xi)$ is $(\sigma,\tilde{g}_0)$-temperate. This is easily known from (2.3), (2.4), (ii) and (iv) in Lemma 2.2. □

PROPOSITION 2.5. (i) *Let $g=\phi^{-2}(x,\xi)|dx|^2+\Psi^{-2}(x,\xi)|d\xi|^2$ be a σ-temperate Riemannian metric. Suppose that $H(x,\xi)=(\phi\Psi)^{-1}\leq 1$. Let $m_j(x,\xi)$ be (σ,g)-temperate weight functions and $p_j(x,\xi)\in S(m_j,g)$ $(j=1,2)$. Then*

$$\begin{aligned}&\sigma(p_1(x,D_x)p_2(x,D_x))(x,\xi)\\ &\quad-\sum_{|\alpha|<k}\alpha!^{-1}\{\partial_\xi^\alpha p_1(x,\xi)\}\{D_x^\alpha p_2(x,\xi)\}\in S(m_1m_2H^k,g), \forall k\geq 0.\end{aligned}$$

(ii) *Let $g=\phi^{-2}|dx|^2+\Psi^{-2}|d\xi|^2$ be a σ-temperate Riemannian metric with $\phi\Psi\geq 1$ and $p(x,\xi)\in S(q_h(\xi)^{m/2},g)$ $(m\in \boldsymbol{R}, h\geq 1)$. Then for any $s\geq 0$ there are* *some constant* $C = C_{s,m} > 0$ *and an integer* $\ell = \ell(s,m)$ *such that*

$$\|p(x, D_x)u\|_{H^s(\boldsymbol{R}^n)} \leq C|p|_\ell^{(m)} \|\langle D_x\rangle_h^m u\|_{H^s(\boldsymbol{R}^n)}$$

for $u \in \mathscr{S}(\boldsymbol{R}^n)$, *where*

$$|p|_\ell^{(m)} = \max_{|\alpha|+|\beta| \leq \ell} \sup_{(x,\xi) \in \boldsymbol{R}_x^n \times \boldsymbol{R}_\xi^n} |\partial_\xi^\alpha D_x^\beta p(x,\xi)| \phi^{|\beta|} \Psi^{|\alpha|} q_h(\xi)^{-m/2}.$$

(iii) *Let* $g = \phi^{-2}|dx|^2 + \Psi^{-2}|d\xi|^2$ *be a* σ*-temperate Riemannian metric with* $\phi\Psi \geq 1$ *and* $p(x,\xi) \in S(1,g)$. *If* $p(x,D_x)$ *is bijective in* $H^s(\boldsymbol{R}^n)$, *then the inverse* $p(x,D_x)^{-1}$ *of* $p(x,D_x)$ *is also a pseudo-differential operator with symbol in* $S(1,g)$.

(i) and (ii) are special cases of Theorem 18.5.4 and Theorem 18.6.3 respectively in L. Hörmander [6]. (iii) is cited from (g) of Theorem 3.1 in R. Beals [1].

Now it follows from Lemmas 2.2–2.4 and (i) in Proposition 2.5 (with metric $g = \tilde{g}_0$, weight functions $m_1 = \sigma(K)$ and $m_2 = \sigma(\tilde{K})$) that $\sigma(R)(t,x,\xi)$ satisfies

$$|\partial_\xi^\alpha D_x^\beta \sigma(R)(t,x,\xi)| \leq C_{\alpha,\beta,\gamma} \Psi_0^{-1-|\alpha|}(\xi) \tilde{\Lambda}_h(t,x,\xi)^{|\alpha|+|\beta|+2}$$

for $t \in [0,T]$ and $x, \xi \in \boldsymbol{R}^n$. Therefore, by (ii) and (iii) in Proposition 2.5 (with metric $g = \tilde{g}_0$) for each $s \geq 0$, taking $h > 1$ large enough, we see that

$$K(t,x,D_x)^{-1} = (I-R)^{-1} \circ \tilde{K}(t,x,D_x) : H^{s+\ell}(\boldsymbol{R}^n) \to H^s(\boldsymbol{R}^n) \tag{2.11}$$

exists as pseudo-differential operator, where $\ell = \ell(\gamma) > 0$ is some real number independent of h due to (i) in Lemma 2.2.

Corollary 2.6. *Let* $p \in S(m, g_0)$. *Then*

(i) $\sigma(K^{-1}pK)(x,\xi) = p(x,\xi) + r_1(p)(x,\xi)$ *with* $r_1(p) \in S(\Psi_0^{-1}(\xi)\tilde{\Lambda}_h(t,x,\xi)m,\ \tilde{g}_0)$.

(ii) $\sigma(K^{-1}pK)(x,\xi) = p(x,\xi) + \gamma\{p,\Lambda\}(x,\xi) + r_2(p)(x,\xi)$ *with* $r_2(p) \in S(\Psi_0^{-2}(\xi)\tilde{\Lambda}_h(t,x,\xi)^2 m, \tilde{g}_0)$, *where* $\{p,\Lambda\}$ *is the Poisson bracket of* p *and* Λ, *or*

$$\{p,\Lambda\}(x,\xi) = \sum_{j=1}^n \{(\partial_{\xi_j} p(x,\xi))(D_{x_j}\Lambda(x,\xi)) - (\partial_{\xi_j}\Lambda(x,\xi))(D_{x_j} p(x,\xi))\}.$$

(iii) $\sigma(K^{-1}pK)(x,\xi) = p(x,\xi) + \gamma\{p,\Lambda\}(x,\xi) + p_2(x,\xi) + r_3(p)(x,\xi)$ *with* $r_3(p) \in S(\Psi_0^{-3}(\xi)\tilde{\Lambda}_h(t,x,\xi)^3 m, \tilde{g}_0)$.

Finally we quote a fundamental fact on pseudo-differential operators to need the proof of the estimate (1.7).

PROPOSITION 2.7 (Theorem 18.6.8 in L. Hörmander [6]). *If* $p(x,\xi)\in S(\Psi_0^2(\xi),g_0)$ *is nonnegative, then there exists a constant* $C_s>0$ *such that*

$$\mathrm{Re}(p(x,D_x)u,u)_{H^s(\boldsymbol{R}^n)}\geq -C_s\|u\|_{H^s(\boldsymbol{R}^n)}$$

for $u\in\mathscr{S}(\boldsymbol{R}^n)$.

§3. Proof of Theorem 1.2

The crucial stage in the proof of Theorem 1.2 is to derive the energy estimate (1.7). So we shall first devote ourselves to establishing (1.7).

Set $u(t,x)=K(t,x,D_x)v(t,x)$. If we denote

$$P_\varphi(t,x,\partial_t,D_x)v(t,x)=\int e^{ix\cdot\xi+\varphi(t,x,\xi)}p(t,x,\partial_t,\xi)\hat{v}(t,\xi)\,\tilde{d}\xi,$$

$$(1_\varphi)(t,x,D_x)v(t,x)=\int e^{ix\cdot\xi+\varphi(t,x,\xi)}\hat{v}(t,\xi)\,\tilde{d}\xi,$$

then the following equality holds:

$$\begin{aligned} u_t(t,x)&=(\partial_t+\gamma\Lambda_t)_{\gamma\Lambda}(t,x,\partial_t,D_x)v(t,x)\\ &=(1_{\gamma\Lambda})(t,x,D_x)v_t(t,x)+(\gamma\Lambda_t)_{\gamma\Lambda}(t,x,D_x)v(t,x)\\ &=(1_{\gamma\Lambda})\circ[v_t(t,x)+(I-R)^{-1}\circ\{(1_{-\gamma\Lambda})\circ(\gamma\Lambda_t)_{\gamma\Lambda}(t,x,D_x)v(t,x)\}]\\ &=K(t,x,D_x)(\partial_t+\gamma\Lambda_t(t,x,D_x)+R_1(t,x,D_x))v(t,x), \end{aligned}\tag{3.1}$$

where

$$(I-R)^{-1}\circ(1_{-\gamma\Lambda})\circ(\gamma\Lambda_t)_{\gamma\Lambda}(t,x,D_x)=\gamma\Lambda_t(t,x,D_x)+R_1(t,x,D_x).$$

It is read from an asymptotic expansion of the symbol $\sigma((1_{-\gamma\Lambda})\circ(\gamma\Lambda_t)_{\gamma\Lambda})(t,x,\xi)$ that

$$\sigma(R_1)(t,x,\xi)=o(\gamma\Lambda_t(t,x,\xi)),$$

namely

$$|\sigma(R_1)(t,x,\xi)|\leq C_\gamma\Lambda_t(t,x,\xi)\Psi_0^{-1}(\xi)\tilde{\Lambda}_h(t,x,\xi)^2.\tag{3.2}$$

By $u_t(t,x)=K(t,x,D_x)(\partial_t+\gamma\Lambda_t(t,x,D_x)+R_1(t,x,D_x))\ K(t,x,D_x)^{-1}u(t,x)$

$$u_{tt}(t,x) = K(t,x,D_x)(\partial_t + \gamma\Lambda_t(t,x,D_x) + R_1(t,x,D_x))^2 v(t,x) \tag{3.3}$$

is also valid. From (3.1) and (3.3) the equation (1.5) is transformed as below:

$$(\partial_t + \gamma\Lambda_t + R_1)^2 v = -K^{-1}a^\sharp K v + iK^{-1}bKv + K^{-1}cKv$$
$$+ K^{-1}\,dK(\partial_t + \gamma\Lambda_t + R_1)v + K^{-1}f. \tag{3.4}$$

For the sake of brevity we stand for $\partial_1 = \partial_t + \gamma\Lambda_t + R_1$, $\tilde{a}^\sharp = K^{-1}a^\sharp K$, $\tilde{b} = K^{-1}bK$ and so on. Then, by use of Corollary 2.6 the symbols of these pseudo-differential operators are expanded in the following form:

$$\tilde{a}^\sharp(t,x,\xi) = a^\sharp(t,x,\xi) + \gamma\{a^\sharp, \Lambda\}(t,x,\xi) + a_2^\sharp(t,x,\xi) + r_3(a^\sharp)(t,x,\xi), \tag{3.5}$$

$$\tilde{b}(t,x,\xi) = b(t,x,\xi) + \gamma\{b, \Lambda\}(t,x,\xi) + r_2(b)(t,x,\xi), \tag{3.6}$$

$$\tilde{c}(t,x,\xi) = c(t,x) + r_1(c)(t,x,\xi), \tag{3.7}$$

$$\tilde{d}(t,x,\xi) = d(t,x) + r_1(d)(t,x,\xi), \tag{3.8}$$

where

$$|a_2^\sharp(t,x,\xi)| \le C(\log q_h(\xi))^2,$$
$$|r_3(a^\sharp)(t,x,\xi)|, |r_2(b)(t,x,\xi)|, |r_1(c)(t,x,\xi)|, |r_1(d)(t,x,\xi)| \le C'.$$

Now we define the energy function

$$E(t) = \|\partial_1 v(t,\cdot)\|^2_{H^s(\mathbf{R}^n)} + (a^\sharp(t,x,D_x)v(t,\cdot), v(t,\cdot))_{H^s(\mathbf{R}^n)}$$
$$+ \|(\log q_h(D_x))v(t,\cdot)\|^2_{H^s(\mathbf{R}^n)}$$

for $t \in [0,T]$. Differentiating $E(t)$, we obtain

$$\frac{d}{dt}E(t) = 2\,\mathrm{Re}(\partial_1^2 v(t), \partial_1 v(t))_{H^s} - 2\,\mathrm{Re}((\gamma\Lambda_t + R_1)\partial_1 v(t), \partial_1 v(t))_{H^s}$$
$$+ ((\partial_t a^\sharp)v(t), v(t))_{H^s} + 2\,\mathrm{Re}(a^\sharp v(t), \partial_1 v(t))_{H^s}$$
$$- 2\,\mathrm{Re}((a^\sharp + (\log q_h(D_x))^2)(\gamma\Lambda_t + R_1)v(t), v(t))_{H^s}$$
$$- 2\,\mathrm{Re}((\log q_h(D_x))\partial_1 v(t), (\log q_h(D_x))v(t))_{H^s}. \tag{3.9}$$

And in view of (3.4)–(3.6) the first term of the right hand side in (3.9) can express

$$\begin{aligned}\mathrm{Re}(\partial_1^2 v(t), \partial_1 v(t))_{H^s} = &-\mathrm{Re}(a^\sharp v(t), \partial_1 v(t))_{H^s} - \gamma\,\mathrm{Re}(\{a^\sharp, \Lambda\} v(t), \partial_1 v(t))_{H^s} \\ &- \mathrm{Re}(a_2^\sharp v(t), \partial_1 v(t))_{H^s} - \mathrm{Re}(r_3(a^\sharp) v(t), \partial_1 v(t))_{H^s} \\ &- \mathrm{Im}(bv(t), \partial_1 v(t))_{H^s} - \gamma\,\mathrm{Im}(\{b, \Lambda\} v(t), \partial_1 v(t))_{H^s} \\ &- \mathrm{Im}(r_2(b) v(t), \partial_1 v(t))_{H^s} + \mathrm{Re}(\tilde{c} v(t), \partial_1 v(t))_{H^s} \\ &+ \mathrm{Re}(\tilde{d} v(t), \partial_1 v(t))_{H^s} + \mathrm{Re}(K^{-1} f(t), \partial_1 v(t))_{H^s}. \end{aligned} \tag{3.10}$$

Henceforth we shall estimate the individual terms in the right hand side in (3.9) with the help of (3.10). Then we shall make use of (i) in Lemma 2.2 without notice. $C(t)$, $C_1(t)$, $C_2(t), \ldots, C_{15}(t)$ and $C_s(t)$ appeared below exhibit suitable nonnegative continuous functions on $[0, T]$ independent of h and γ.

First of all, from (3.5)

$$\begin{aligned}-\gamma\,\mathrm{Re}(\{a^\sharp, \Lambda\} v(t), \partial_1 v(t))_{H^s} &= -\sum_{j=1}^n \mathrm{Re}((a_1^{\sharp j} - a_{1j}^\sharp) v(t), \gamma \partial_1 v(t))_{H^s} \\ &\leq \frac{1}{2} n \Big\{ \max_{j=1,\ldots,n} (\|\tilde{\Lambda}_h^{-(1/2)} a_1^{\sharp j} v(t)\|_{H^s}^2 \\ &\quad + \|\tilde{\Lambda}_h^{-(1/2)} a_{1j}^\sharp v(t)\|_{H^s}^2) + 2\gamma^2 \|\tilde{\Lambda}_h^{1/2} \partial_1 v(t)\|_{H^s}^2 \Big\},\end{aligned}$$

where $a_1^{\sharp j}(t, x, D_x)$ and $a_{1j}^\sharp(t, x, D_x)$ denote the pseudo-differential operators with symbols $\{\partial_{\xi_j} a^\sharp(t, x, \xi)\}\{D_{x_j} \Lambda(t, x, \xi)\}$ and $\{\partial_{\xi_j} \Lambda(t, x, \xi)\}\{D_{x_j} a^\sharp(t, x, \xi)\}$ respectively. Here, because $0 \leq a(t, x, \xi) \in S(\Psi_0^2(\xi), g_0)$, by Proposition 2.7

$$\mathrm{Re}(a^\sharp \Lambda_t v(t), v(t))_{H^s} \geq -C_s(t) \|\Lambda_t^{1/2} v(t)\|_{H^s}^2,$$

so that

$$\mathrm{Re}((a^\sharp + (\log q_h)^2) \Lambda_t v(t), v(t))_{H^s} + C_s(t) \|(\log q_h) \Lambda_t^{1/2} v(t)\|_{H^s}^2 \geq 0. \tag{3.11}$$

Now, paying attention to (iii) in Lemma 2.1, we find that

$$\begin{aligned}&\|\tilde{\Lambda}_h^{-(1/2)} a_1^{\sharp j} v(t)\|_{H^s}^2 \\ &\quad \leq C_1(t) \{\mathrm{Re}((a^\sharp + (\log q_h)^2) \Lambda_t v(t), v(t))_{H^s} + C_s(t) \|(\log q_h) \Lambda_t^{1/2} v(t)\|_{H^s}^2\}.\end{aligned}$$

Likewise, by virtue of (iv) in Lemma 2.1

$$\begin{aligned}&\|\tilde{\Lambda}_h^{-(1/2)} a_{1j}^\sharp v(t)\|_{H^s}^2 \\ &\quad \leq C_2(t) \{\mathrm{Re}((a^\sharp + (\log q_h)^2) \Lambda_t v(t), v(t))_{H^s} + C_s(t) \|(\log q_h) \Lambda_t^{1/2} v(t)\|_{H^s}^2\}.\end{aligned}$$

In addition, from $|\tilde{\Lambda}_h(t,x,\xi)/\Lambda_t(t,x,\xi)| \le C/\varepsilon_0$

$$n\gamma^2\|\tilde{\Lambda}_h^{1/2}\partial_1 v(t)\|_{H^s}^2 \le n\gamma^2\varepsilon_0^{-1}C_3(t)\|\Lambda_t^{1/2}\partial_1 v(t)\|_{H^s}^2.$$

Thus

$$\begin{aligned}&-\gamma\,\mathrm{Re}(\{a^\sharp,\Lambda\}v(t),\partial_1 v(t))_{H^s} - n\gamma^2\varepsilon_0^{-1}C_3(t)\|\Lambda_t^{1/2}\partial_1 v(t)\|_{H^s}^2\\ &\le nC_4(t)\{\mathrm{Re}((a^\sharp+(\log q_h)^2)\Lambda_t v(t),v(t))_{H^s}\\ &\quad + C_s(t)\|(\log q_h)\Lambda_t^{1/2}v(t)\|_{H^s}^2\}.\end{aligned}\tag{3.12}$$

Also, in the same manner as above, together with $|\tilde{\Lambda}_h/\Lambda_t| \le C/\varepsilon_0$

$$\begin{aligned}&-\mathrm{Re}(a_2^\sharp v(t),\partial_1 v(t))_{H^s}\\ &\le n^2C_5(t)(\gamma^4\varepsilon_0^{-1}\|(\log q_h)^{3/2}v(t)\|_{H^s}^2 + \|\Lambda_t^{1/2}\partial_1 v(t)\|_{H^s}^2)\\ &\le n^2C_6(t)(\gamma^4\varepsilon_0^{-2}\|(\log q_h)\Lambda_t^{1/2}v(t)\|_{H^s}^2 + \|\Lambda_t^{1/2}\partial_1 v(t)\|_{H^s}^2).\end{aligned}\tag{3.13}$$

$$\begin{aligned}&-\mathrm{Re}((\log q_h(D_x))\partial_1 v(t),(\log q_h(D_x))v(t))_{H^s}\\ &\le C_7(t)(\varepsilon_0^{-1}\|(\log q_h)^{3/2}v(t)\|_{H^s}^2 + \|\Lambda_t^{1/2}\partial_1 v(t)\|_{H^s}^2)\\ &\le C_8(t)(\varepsilon_0^{-2}\|(\log q_h)\Lambda_t^{1/2}v(t)\|_{H^s}^2 + \|\Lambda_t^{1/2}\partial_1 v(t)\|_{H^s}^2).\end{aligned}\tag{3.14}$$

Moreover, by means of (3.5), (v) and (vi) in Lemma 2.1

$$-\mathrm{Re}(r_3(a^\sharp)v(t),\partial_1 v(t))_{H^s} \le n^3\gamma^6(C_9(t)\|v(t)\|_{H^s}^2 + \|\partial_1 v(t)\|_{H^s}^2).\tag{3.15}$$

Since

$$\begin{aligned}|b(t,x,\xi)|^2 &\le A_0(a(t,x,\xi)+|\partial_t a(t,x,\xi)|)\\ &\le A_0(a(t,x,\xi)+(\log q_h(\xi))^2) + A_0|\partial_t a(t,x,\xi)|\\ &\le (A_0+1)(a(t,x,\xi)+(\log q_h(\xi))^2)\Lambda_t(t,x,\xi)\end{aligned}$$

from (1.4) and (1.8), we have

$$\|bv(t)\|_{H^s}^2 \le C_{10}(t)\{\mathrm{Re}((a^\sharp+(\log q_h)^2)\Lambda_t v(t),v(t))_{H^s} + C_s(t)\|(\log q_h)\Lambda_t^{1/2}v(t)\|_{H^s}^2\}.$$

Hence

$$\begin{aligned}&-\mathrm{Im}(bv(t),\partial_1 v(t))_{H^s} - \frac{1}{2}\|\partial_1 v(t)\|_{H^s}^2\\ &\le C_{10}(t)\{\mathrm{Re}((a^\sharp+(\log q_h)^2)\Lambda_t v(t),v(t))_{H^s}\\ &\quad + C_s(t)\|(\log q_h)\Lambda_t^{1/2}v(t)\|_{H^s}^2\}.\end{aligned}\tag{3.16}$$

Besides, from (3.6)–(3.8) and (vii) in Lemma 2.1 it is evident that

$$-\gamma \operatorname{Im}(\{b,\Lambda\}v(t),\partial_1 v(t))_{H^s} \leq n\gamma(C_{11}(t)\|(\log q_h))v(t)\|_{H^s}^2 + \|\partial_1 v(t)\|_{H^s}^2), \tag{3.17}$$

$$-\operatorname{Im}(r_2(b)v(t),\partial_1 v(t))_{H^s} \leq n^2\gamma^4(C_{12}(t)\|v(t)\|_{H^s}^2 + \|\partial_1 v(t)\|_{H^s}^2), \tag{3.18}$$

$$\operatorname{Re}(\tilde{c}v(t),\partial_1 v(t))_{H^s} \leq C_{13}(t)(\|v(t)\|_{H^s}^2 + \|\partial_1 v(t)\|_{H^s}^2), \tag{3.19}$$

$$\operatorname{Re}(\tilde{d}v(t),\partial_1 v(t))_{H^s} \leq C_{14}(t)(\|v(t)\|_{H^s}^2 + \|\partial_1 v(t)\|_{H^s}^2), \tag{3.20}$$

$$\operatorname{Re}(K^{-1}f(t),\partial_1 v(t))_{H^s} \leq \frac{1}{2}(\|K^{-1}f(t)\|_{H^s}^2 + \|\partial_1 v(t)\|_{H^s}^2). \tag{3.21}$$

By the way, due to (3.2), for any $(\gamma >)\ \rho > 0$

$$-\operatorname{Re}((\gamma\Lambda_t + R_1)\partial_1 v(t),\partial_1 v(t))_{H^s} \leq (-\gamma+\rho)\|\Lambda_t^{1/2}\partial_1 v(t)\|_{H^s}^2, \tag{3.22}$$

$$\begin{aligned} &-\operatorname{Re}((a^\sharp + (\log q_h(D_x))^2)(\gamma\Lambda_t + R_1)v(t),v(t))_{H^s} \\ &\leq (-\gamma+\rho)\{\operatorname{Re}((a^\sharp + (\log q_h)^2)\Lambda_t v(t),v(t))_{H^s} \\ &\quad + C_s(t)\|(\log q_h)\Lambda_t^{1/2}v(t)\|_{H^s}^2\}. \end{aligned} \tag{3.23}$$

While, owing to $|\partial_t a(t,x,\xi)| \leq (a + (\log q_h(\xi))^2)\Lambda_t(t,x,\xi)$

$$\begin{aligned} ((\partial_t a^\sharp)v(t),v(t))_{H^s} &\leq C_{15}(t)\{\operatorname{Re}((a^\sharp + (\log q_h)^2)\Lambda_t v(t),v(t))_{H^s} \\ &\quad + C_s(t)\|(\log q_h)\Lambda_t^{1/2}v(t)\|_{H^s}^2\}. \end{aligned} \tag{3.24}$$

Therefore, on account of (3.11), it follows from (3.12)–(3.14), (3.16), (3.23) and (3.24) that

$$\begin{aligned} &2(nC_4(t) + C_{10}(t))\{\operatorname{Re}((a^\sharp + (\log q_h)^2)\Lambda_t v(t),v(t))_{H^s} \\ &\quad + C_s(t)\|(\log q_h)\Lambda_t^{1/2}v(t)\|_{H^s}^2\} \\ &\quad + 2\varepsilon_0^{-2}(n^2\gamma^4 C_6(t) + C_8(t))\|(\log q_h(D_x))\Lambda_t^{1/2}v(t)\|_{H^s}^2 \\ &\quad - 2\operatorname{Re}((a^\sharp + (\log q_h(D_x))^2)(\gamma\Lambda_t + R_1)v(t),v(t))_{H^s} + ((\partial_t a^\sharp)v(t),v(t))_{H^s} \\ &\leq 2(-\gamma + nC_4(t) + n^2\gamma^4\varepsilon_0^{-2}C_6(t) + \varepsilon_0^{-2}C_8(t) + C_{10}(t) + C_{15}(t) + \rho) \\ &\quad \times \{\operatorname{Re}((a^\sharp + (\log q_h)^2)\Lambda_t v(t),v(t))_{H^s} + C_s(t)\|(\log q_h)\Lambda_t^{1/2}v(t)\|_{H^s}^2\}. \end{aligned} \tag{3.25}$$

On the other hand, in aid of (3.12)–(3.14) and (3.22)

$$2(n\gamma^2\varepsilon_0^{-1}C_3(t)+n^2C_6(t)+C_8(t))\|\Lambda_t^{1/2}\partial_1 v(t)\|_{H^s}^2$$
$$-2\operatorname{Re}((\gamma\Lambda_t+R_1)\partial_1 v(t),\partial_1 v(t))_{H^s}$$
$$\leq 2(-\gamma+n\gamma^2\varepsilon_0^{-1}C_3(t)+n^2C_6(t)+C_8(t)+\rho)\|\Lambda_t^{1/2}\partial_1 v(t)\|_{H^s}^2 \tag{3.26}$$

So we choose

$$\varepsilon_0\geq\gamma^2+1,$$
$$\gamma\geq\sup_{t\in[0,T]}(nC_3(t)+nC_4(t)+n^2C_6(t)+C_8(t)+C_{10}(t)+C_{15}(t))+\rho.$$

Consequently, picking

$$c(t)=2n^3\gamma^6(C_9(t)+1)+2n^2\gamma^4(C_{12}(t)+1)$$
$$+2n\gamma(C_{11}(t)+1)+2(C_{13}(t)+C_{14}(t))+2,$$

we get from (3.9)–(3.21) and (3.25)–(3.26)

$$\frac{d}{dt}E(t)\leq c(t)E(t)+\|K^{-1}f(t)\|_{H^s}^2.$$

Further, by Gronwall's inequality

$$E(t)\leq e^{\int_0^t c(\tau)\,d\tau}\left(E(0)+\int_0^t\|K^{-1}f(\tau)\|_{H^s}^2\,d\tau\right) \tag{3.27}$$

holds. Here, record that

$$E(t)\geq\|v(t)\|_{H^s}^2+\|\partial_1 v(t)\|_{H^s}^2=\|K^{-1}u(t)\|_{H^s}^2+\|K^{-1}u_t(t)\|_{H^s}^2 \tag{3.28}$$

from (3.1). Meanwhile, by $a^\sharp(0,x,\xi)+(\log q_h(\xi))^2\leq C\langle\xi\rangle_h^2$

$$E(0)\leq C_h\|K^{-1}u(0)\|_{H^{s+1}}^2+\|K^{-1}u_t(0)\|_{H^s}^2. \tag{3.29}$$

Thus, summing up (3.27), (3.28) and (3.29), we have arrived at the desired (1.7).

Next we shall proceed to a standard verification of the existence of C^∞ solution to the problem (1.5), (1.6). In advance, we may rewrite (1.7) like

$$\|u(t,\cdot)\|_{H^s}^2+\|u_t(t,\cdot)\|_{H^s}^2\leq C_{\gamma,h}\left(\|u_0\|_{H^{s+\ell+1}}^2+\|u_1\|_{H^{s+\ell}}^2+\int_0^t\|f(\tau,\cdot)\|_{H^{s+\ell}}^2\,d\tau\right), \tag{3.30}$$

where $\ell=\ell(\gamma)>0$ is the same in (2.11). Now let us consider the following strictly hyperbolic Cauchy problem on $[0,T]\times\boldsymbol{R}_x^n$

$$\{L(t,x,\partial_t,\partial_x)-\varepsilon\Delta_x\}w(t,x)=f(t,x), \tag{3.31}$$

$$w(0,x)=u_0(x), w_t(0,x)=u_1(x), \tag{3.32}$$

where $\varepsilon>0$ and $\Delta_x=\sum_{j=1}^n \partial_{x_j}^2$ (Laplacian). Thanks to the theory of strictly hyperbolic equations, we know that for any $f\in C^0([0,T];H^{s+\ell})$, $u_0\in H^{s+\ell+1}$ and $u_1\in H^{s+\ell}$ there exists a unique solution $u^{(\varepsilon)}\in C^0([0,T];H^{s+\ell+1})\cap C^1([0,T];H^{s+\ell})$ to the problem (3.31), (3.32) (for instance, see Theorem 23.2.2 in L. Hörmander [6]). Since the estimate (3.30) is still available for the problem (3.31), (3.32), we adapt (3.30) to the sequence $\{u^{(\varepsilon)}\}$ and so that

$$\|u^{(\varepsilon)}(t)\|_{H^s}^2+\|u_t^{(\varepsilon)}(t)\|_{H^s}^2\le C'_{\gamma,h}\left(\|u_0\|_{H^{s+\ell+1}}^2+\|u_1\|_{H^{s+\ell}}^2+\int_0^t\|f(\tau)\|_{H^{s+\ell}}^2\,d\tau\right)$$

with constant independent of ε. Hence, by virtue of the diagonal argument we can extract a subsequence of $\{u^{(\varepsilon)}\}$ such that

$$u^{(\varepsilon)}\to w_1 \text{ weakly in } L^2([0,T];H^s),$$

$$u_t^{(\varepsilon)}\to w_2 \text{ weakly in } L^2([0,T];H^s).$$

These imply $w_2(t,x)=\partial_t w_1(t,x)$ and $w_1(t,x)$ is just a unique solution of the problem (1.5), (1.6) which belongs to $C^1([0,T];H^s)$. Of course, the solution possesses the finite propagation property as described in the statement of Theorem 1.2 (*cf.*, in detail, Theorem 4.13 in J. Chazarain-A. Piriou [2]). The C^∞-regularity of the solution also follows by differentiating the equation (1.5) in t:

$$\partial_t^{m+2}u(t,x)=\sum_{j=0}^m\binom{m}{j}\partial_t^j\{a^\sharp(t,x,\partial_x)+b(t,x,\partial_x)+c(t,x)\}\cdot\partial_t^{m-j}u(t,x)$$
$$+\sum_{j=0}^m\binom{m}{j}\partial_t^j d(t,x)\cdot\partial_t^{m+1-j}u(t,x)+\partial_t^m f(t,x,u(t,x)).$$

§4. Proof of Theorem 1.1

From now we shall prove Theorem 1.1. Our method relies on a successive approximation (precisely, the contraction mapping principle).

To begin with, taking the estimate (1.7) into account, we introduce the function space X_m^T as the completion of $C^0([0,T];H^{m+\ell}(\boldsymbol{R}^n))\cap C^1([0,T];H^{m+\ell-1}(\boldsymbol{R}^n))$ by its norm

$$\|u\|_m^T = \sup_{t\in[0,T]} \{\|K(t,x,D_x)^{-1}u(t,\cdot)\|^2_{H^m(\boldsymbol{R}^n)} + \|K(t,x,D_x)^{-1}u_t(t,\cdot)\|^2_{H^{m-1}(\boldsymbol{R}^n)}\}^{1/2},$$

where $\ell = \ell(\gamma) > 0$ is the identical in (2.11). Under the guarantee of Theorem 1.2, the approximate sequence $\{u^{(\nu)}(t,x)\}$ to the solution $u(t,x)$ of the problem (1.1), (1.2) is yielded by the recursive procedure below:

$$\begin{cases} L(t,x,\partial_t,\partial_x)u^{(\nu)}(t,x) = f(t,x,u^{(\nu-1)}(t,x)), \\ u^{(\nu)}(0,x) = u_0(x), u_t^{(\nu)}(0,x) = u_1(x) \end{cases}$$

for $\nu = 1,2,\dots,$ where $u^{(0)}(t,x) \equiv 0$. So we define the mapping $\Phi : X_m^T \to X_m^T$ as follows: for $w \in X_m^T$, $u = \Phi(w)$ is the solution of the linear problem

$$\begin{cases} L(t,x,\partial_t,\partial_x)u = f(t,x,w), \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) \end{cases}$$

which exists uniquely by Theorem 1.2 and which belongs to X_m^T for m large enough, as can be seen later. Let $B_m^T(r) = \{u \in X_m^T; \|u\|_m^T \le r\}$ for $r \ge 1$. Then our chief task is to find some large $r > 1$ and small $T = T(m,r) > 0$ for sufficiently large $m \gg 1$ such that

$$\Phi : B_m^T(r) \to B_m^T(r), \tag{4.1}$$

$$\|\Phi(u) - \Phi(w)\|_m^T \le \frac{1}{2}\|u - w\|_m^T. \tag{4.2}$$

At first it holds that X_m^T forms a Banach algebra with norm $\|\cdot\|_m^T$ for m large enough. Indeed, note from (iv) in Lemma 2.2 that

$$C_\gamma^{-1}\|\tilde{K}_0(t,D_x)u\|_{H^m} \le \|\tilde{K}(t,x,D_x)u\|_{H^m} \le C_\gamma\|\tilde{K}_0(t,D_x)u\|_{H^m}, \tag{4.3}$$

where $\tilde{K}_0(t,D_x) = e^{-\gamma\Lambda}(t,0,D_x)$. Furthermore, when we put $p_{\gamma\Lambda} = \tilde{K}_0 p \tilde{K}_0^{-1}$ for $p \in S(q_h(\xi)^{s/2}, g_0)$, the next inequality is valid for every integer $k \ge 0$:

$$|p_{\gamma\Lambda}|_k^{(s)} \le C(k,\gamma)\|p\|^{(s,\gamma\Lambda)}_{[4\gamma(M+N+1)]+n+3+k}, \tag{4.4}$$

where $[s]$ means the maximal integer not greater than a real number s and

$$\|p\|_\ell^{(s,\gamma\Lambda)} = \max_{|\alpha|+|\beta|\le\ell} \sup_{\xi\in\boldsymbol{R}^n} \|\tilde{K}_0\partial_\xi^\alpha D_x^\beta p(x,\xi)\|_{L^2(\boldsymbol{R}_x^n)}/q_h(\xi)^{(s-|\alpha|)/2}.$$

In fact, it follows from Theorem 2.6 (1) of Chapter 2 in H. Kumano-go [9] that

$$p_{\gamma\Lambda}(t,x,\xi) = \text{Os-}\iint e^{-iy\cdot\eta-\gamma\Lambda(t,0,\xi+\eta)+\gamma\Lambda(t,0,\xi)} p(x+y,\xi)\,dy\tilde{d}\eta.$$

Also, because of Taylor's formula

$$p_{\gamma\Lambda}(t,x,\xi) = \sum_{|\alpha|<k} D_x^\alpha p(x,\xi)\lambda_\alpha(t,\xi) + r_k(t,x,\xi) = r(t,x,\xi) + r_k(t,x,\xi),$$

where

$$\lambda_\alpha(t,\xi) = D_\eta^\alpha(e^{\gamma\Lambda(t,0,\xi)-\gamma\Lambda(t,0,\xi+\eta)})|_{\eta=0},$$

$$r_k(t,x,\xi) = \sum_{|\alpha|=k} \frac{1}{\alpha!}\,\text{Os-}\iint e^{-iy\cdot\eta-\gamma\Lambda(t,0,\xi+\eta)+\gamma\Lambda(t,0,\xi)} y^\alpha \int_0^1 D_x^\alpha p(x+\theta y,\xi)\,d\theta dy\tilde{d}\eta.$$

Here, since $|\partial_\xi^\beta\lambda_\alpha(t,\xi)| \le C_{\alpha,\beta} q_h(\xi)^{-|\alpha|/2}$ by (i) of Lemma 2.2, we obtain, along with the Sobolev embedding theorem

$$\begin{aligned}|\partial_\xi^\alpha D_x^\beta r(t,x,\xi)| &\le \sum_{|\delta|<k}\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}|\partial_\xi^{\alpha'} D_x^{\beta+\delta} p(x,\xi)|\,|\partial_\xi^{\alpha''}\lambda_\delta(t,\xi)|\\ &\le C_\alpha \|p\|_{|\alpha|+|\beta|+k+n}^{(s,\gamma\Lambda)} q_h(\xi)^{(s-|\alpha|)/2}.\end{aligned}$$

Meanwhile, as to $r_k(t,x,\xi)$

$$\begin{aligned}r_k(t,x,\xi) = \sum_{|\delta|=k}\frac{1}{\delta!}\,\text{Os-}\iint & e^{-iy\cdot\eta-\gamma\Lambda(t,0,\xi+\eta)+\gamma\Lambda(t,0,\xi)}\mu_\delta(t,\xi+\eta)\\ &\times\int_0^1 D_x^\delta p(x+\theta y,\xi)\,d\theta dy\tilde{d}\eta,\end{aligned}$$

where $\mu_\delta(t,\xi+\eta) = e^{\gamma\Lambda(t,0,\xi+\eta)} D_\eta^\delta e^{-\gamma\Lambda(t,0,\xi+\eta)}$. Then

$$\begin{aligned}&\partial_\xi^\alpha D_x^\beta r_k(t,x,\xi)\\ &= \sum_{|\delta|=k}\frac{1}{\delta!}\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\iiint_0^1 e^{-iy\cdot\eta}(\partial_\xi^{\alpha'}\mu_\delta(t,\xi+\eta))(\partial_\xi^{\alpha''}D_x^{\beta+\delta}p(x+\theta y,\xi))\,d\theta dy\tilde{d}\eta\\ &= \sum_{|\delta|=k}\frac{1}{\delta!}\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\int e^{-\gamma\Lambda(t,0,\xi+\eta)+\gamma\Lambda(t,0,\xi)}(\partial_\xi^{\alpha'}\mu_\delta(t,\xi+\eta))F_{(\beta+\delta)}^{(\alpha'')}(x,\xi,\eta)\,\tilde{d}\eta,\end{aligned}$$

where

$$F^{(\alpha'')}_{(\beta+\delta)}(x,\xi,\eta) = \int_0^1 \int e^{-iy\cdot\eta}\partial_\xi^{\alpha''} D_x^{\beta+\delta} p(x+\theta y,\xi)\, dy d\theta$$
$$= \int_0^1 \int e^{-i(y-x)\cdot\eta/\theta}\partial_\xi^{\alpha''} D_x^{\beta+\delta} p(y,\xi)\frac{dy}{\theta^n} d\theta$$
$$= \int_0^1 e^{ix\cdot\eta/\theta}\theta^{-n}\partial_\xi^{\alpha''} D_x^{\beta+\delta}\hat{p}\left(\frac{\eta}{\theta},\xi\right) d\theta$$

and $\hat{p}(\eta,\xi)$ stands for the Fourier image of $p(x,\xi)$ with respect to x. Thereby, because $\gamma\Lambda \le 0$, (iii) in Lemma 2.2 and

$$|\partial_\xi^{\alpha'}\mu_\delta(t,\xi+\eta)| \le C_{\alpha',\delta} q_h(\xi+\eta)^{-(|\alpha'|+|\delta|)/2},$$

we can see that

$$|\partial_\xi^\alpha D_x^\beta r_k(t,x,\xi)|$$
$$\le C\sum_{|\delta|=k}\sum_{\alpha'+\alpha''=\alpha}\int q_h(\xi+\eta)^{-k/2} q_h(\eta)^{2\gamma(M+N+1)} e^{-\gamma\Lambda(t,0,\eta)}|F^{(\alpha'')}_{(\beta+\delta)}(x,\xi,\eta)|\, \tilde{d}\eta$$
$$\le Cq_h(\xi)^{-k/2}\sum\sum\int q_h(\eta)^{k/2} q_h(\eta)^{2\gamma(M+N+1)} e^{-\gamma\Lambda(t,0,\eta)}|F^{(\alpha'')}_{(\beta+\delta)}|\, \tilde{d}\eta$$
$$\le Cq_h(\xi)^{-k/2}\sum\sum\int\int_0^1 q_h(\eta)^{k/2+2\gamma(M+N+1)} e^{-\gamma\Lambda(t,0,\eta)}\theta^{-n}\left|\partial_\xi^{\alpha''} D_x^{\beta+\delta}\hat{p}\left(\frac{\eta}{\theta},\xi\right)\right| d\theta \tilde{d}\eta$$
$$\le Cq_h(\xi)^{-k/2}\sum\sum\int q_h(\eta)^{k/2+2\gamma(M+N+1)} e^{-\gamma\Lambda(t,0,\eta)}|\partial_\xi^{\alpha''} D_x^{\beta+\delta}\hat{p}(\eta,\xi)|\, \tilde{d}\eta$$
$$\le Cq_h(\xi)^{(s-k)/2}\|p\|^{(s,\gamma\Lambda)}_{[k+4\gamma(M+N+1)+(n+1)/2]+1+k+|\alpha|+|\beta|}.$$

Now, employing (4.3) and (4.4) with $s=0$ and $k=[m/2]$, we enjoy

$$\|K(t,x,D_x)^{-1}(u(t,\cdot)w(t,\cdot))\|_{H^m}$$
$$\le \|(I-R)^{-1}\|_{H^m\to H^m}\|\tilde{K}(t,x,D_x)(u(t,\cdot)w(t,\cdot))\|_{H^m}$$
$$\le C_{\gamma,m}(t)\sum_{|\alpha|\le m}\|\partial_x^\alpha\{\tilde{K}_0(t,D_x)(u(t)w(t))\}\|_{L^2}$$
$$\le C_{\gamma,m}(t)(\|\tilde{K}_0(t,D_x)u(t,\cdot)\|_{H^{m/2+[4\gamma(M+N+1)]+n+3}}\|\tilde{K}_0(t,D_x)w(t,\cdot)\|_{H^m}$$
$$+\|\tilde{K}_0(t,D_x)u(t,\cdot)\|_{H^m}\|\tilde{K}_0(t,D_x)w(t,\cdot)\|_{H^{m/2+[4\gamma(M+N+1)]+n+3}})$$

$$\leq C_{\gamma,m}(t)\|\tilde{K}_0(t,D_x)u(t,\,\cdot\,)\|_{H^m}\|\tilde{K}_0(t,D_x)w(t,\,\cdot\,)\|_{H^m}$$
$$\leq C_{\gamma,m}(t)\|K(t,x,D_x)^{-1}u(t,\,\cdot\,)\|_{H^m}\|K(t,x,D_x)^{-1}w(t,\,\cdot\,)\|_{H^m} \tag{4.5}$$

as long as $m \geq 2[4\gamma(M+N+1)] + 2n + 6$.

Next let us look for some nonnegative continuous function $\psi_{\gamma,m}(t,s)$ on $[0,T] \times \boldsymbol{R}_s^+$, increasing in each argument, such that

$$\|K(t,x,D_x)^{-1}f(t,\,\cdot\,,u(t,\,\cdot))\|_{H^m}$$
$$\leq \psi_{\gamma,m}(t,\|K(t,x,D_x)^{-1}u(t,\,\cdot\,)\|_{H^{m_0}})\|K(t,x,D_x)^{-1}u(t,\,\cdot\,)\|_{H^m} \tag{4.6}$$

for sufficiently large $m \geq m_0$ with $m_0 \geq [n/2]+1$. As can be known from the observation below, we may seek the above-like function $\tilde{\psi}_{\gamma,m}$ with second argument $\|K^{-1}u\|_{H^{m/2+[4\gamma(M+N+1)]+n+3}}$ instead of $\|K^{-1}u\|_{H^{m_0}}$ independent of m by use of (4.5). However, it is not enough to derive the positivity of the lifespan of the solution to (1.1), (1.2) in H^∞. For that aim we require an explicit representation of $\partial_x^\alpha f(t,x,u(t,x))$ ($|\alpha| \leq m$). In order to procure it, heeding that

$$\partial_{x_j} f(t,x,u(t,x)) = \left.\frac{\partial f(t,x,y)}{\partial x_j}\right|_{y=u(t,x)} + \left.\frac{\partial f(t,x,y)}{\partial y}\frac{\partial y}{\partial x_j}\right|_{y=u(t,x)}$$
$$= \left(\frac{\partial}{\partial x_j} + \frac{\partial u(t,x)}{\partial x_j}\frac{\partial}{\partial y}\right) f(t,x,y)|_{y=u(t,x)},$$

we express

$$\partial_x^\alpha f(t,x,u(t,x)) = \prod_{j=1}^n \left(\frac{\partial}{\partial x_j} + \frac{\partial u(t,x)}{\partial x_j}\frac{\partial}{\partial y}\right)^{\alpha_j} f(t,x,y)|_{y=u(t,x)}$$
$$= \sum_{|\beta|+j=|\alpha|} c_{\beta,j}^{(\alpha)}(t,x)\partial_x^\beta \partial_y^j f(t,x,y)|_{y=u(t,x)}$$
$$+ \sum_{\substack{|\beta|+j \leq |\alpha|-1 \\ j=1,\ldots,|\alpha|-1}} c_{\beta,j}^{(\alpha)}(t,x)\partial_x^\beta \partial_y^j f(t,x,y)|_{y=u(t,x)}.$$

Then the coefficients $c_{\beta,j}^{(\alpha)}(t,x)$ are determined by the following relations:

$$\begin{cases} c_{\beta,j}^{(\alpha+e_k)}(t,x) = c_{\beta-e_k,j}^{(\alpha)}(t,x) + \dfrac{\partial u(t,x)}{\partial x_k} c_{\beta,j-1}^{(\alpha)}(t,x), \quad \text{if } |\beta|+j = |\alpha|+1, \\[2ex] c_{\beta,j}^{(\alpha+e_k)}(t,x) = \dfrac{\partial c_{\beta,j}^{(\alpha)}(t,x)}{\partial x_k} + c_{\beta-e_k,j}^{(\alpha)}(t,x) + \dfrac{\partial u(t,x)}{\partial x_k} c_{\beta,j-1}^{(\alpha)}(t,x), \\[2ex] \text{if } |\beta|+j \leq |\alpha| \quad \text{and if } j = 1,\ldots,|\alpha|, \end{cases}$$

where e_k is the unit vector with k-th component 1 in $\boldsymbol{R}^n$, $c^{(\alpha)}_{\beta,0}(t,x) = 1$ for $|\beta| = |\alpha|$, $c^{(\alpha)}_{0,0}(t,x) = 0$ for $|\alpha| \geq 1$, $c^{(\alpha)}_{-\beta,j}(t,x) = 0$ for $|\beta| = 1$ and for $j = 1, \dots, |\alpha| + 1$, and $c^{(\alpha)}_{\beta,-1}(t,x) = 0$ for $|\beta| = |\alpha| + 1$. The above relations involve that we can represent

$$\partial_x^\alpha f(t,x,u(t,x)) = \sum_{|\delta|+j=|\alpha|} \frac{\alpha!}{\delta!} \sum_{k=0}^{j} \frac{\partial_x^\delta \partial_y^k f(t,x,y)}{k!}\Big|_{y=u(t,x)} \times \sum_{\substack{\beta_1+\cdots+\beta_k=\beta \\ |\beta|=j}} \frac{(\partial_x^{\beta_1} u(t,x)) \cdots (\partial_x^{\beta_k} u(t,x))}{\beta_1! \cdots \beta_k!}. \tag{4.7}$$

Hence, combining (4.3) with (4.7), we can deduce that

$$\begin{aligned}
\|K(t,x,D_x)^{-1} f(t,\cdot,u(t,\cdot))\|_{H^m} &\leq \|(I-R)^{-1}\|_{H^m \to H^m} \|\tilde{K}(t,x,D_x) f\|_{H^m} \\
&\leq C_m(t) \|(I-R)^{-1}\|_{H^m \to H^m} \sum_{|\alpha| \leq m+\ell} \|\partial_x^\alpha f(t,x,u(t,x))\|_{L_x^2} \\
&\leq C_{\gamma,m}(t) \sum_{|\alpha| \leq m+\ell} \sum_{|\delta|+j=|\alpha|} \sum_{k=0}^{j} \sum_{\substack{\beta_1+\cdots+\beta_k=\beta \\ |\beta|=j}} \left\| (\partial_x^\delta \partial_y^k f(t,x,y)|_{y=u(t,x)}) \prod_{i=1}^{k} \partial_x^{\beta_i} u \right\|_{L_x^2},
\end{aligned}$$

where, on account of (2.11) and the Gagliardo-Nirenberg inequality:

$$\left\| \prod_{i=1}^{k} \partial^{\delta_i} v \right\|_{L^2} \leq c(k) \|v\|_{L^\infty}^{k-1} \sum_{|\delta|=j} \|\partial^\delta v\|_{L^2}$$

provided $|\delta_1| + \cdots + |\delta_k| = j$ (see Lemma 3.10 of Chapter 13 in M. E. Taylor [13])

$$\begin{aligned}
&\left\| (\partial_x^\delta \partial_y^k f(t,x,y)|_{y=u(t,x)}) \prod_{i=1}^{k} \partial_x^{\beta_i} u \right\|_{L_x^2} \\
&\leq \|\partial_x^\delta \partial_y^k f(t,x,y)|_{y=u(t,x)}\|_{L_x^\infty} \left\| \prod_{i=1}^{k} \partial_x^{\beta_i} u \right\|_{L_x^2} \\
&\leq C_{\gamma,k}(t) \|u(t)\|_{L^\infty}^{k-1} \sum_{|\varepsilon|=j} \|\partial_x^\varepsilon u(t)\|_{L_x^2} \\
&\leq C_{\gamma,k}(t) \|\tilde{K}_0(t,D_x) u(t)\|_{H^{m_0}}^{k-1} \sum_{|\varepsilon|=j-\ell} \|\tilde{K}_0(t,D_x)^{-1}\|_{L^2 \to H^\ell} \|\partial_x^\varepsilon (\tilde{K}_0(t,D_x) u(t))\|_{L_x^2}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\|K(t,x,D_x)^{-1}f(t,\cdot,u(t,\cdot))\|_{H^m}\\
&\le C_{\gamma,m}(t)\sum_{|\alpha|\le m+\ell}(1+\|\tilde{K}_0u(t)\|_{H^{m_0}}+\cdots+\|\tilde{K}_0u(t)\|_{H^{m_0}}^{|\alpha|-1})\|\tilde{K}_0u(t)\|_{H^{|\alpha|-\ell}}\\
&\le C_{\gamma,m}(t)\sum_{j=0}^{m+\ell-1}\|\tilde{K}_0(t,D_x)u(t)\|_{H^{m_0}}^{j}\|\tilde{K}_0(t,D_x)u(t)\|_{H^m}\\
&\le C_{\gamma,m}(t)\sum_{j=0}^{m+\ell-1}\|K(t,x,D_x)^{-1}u(t,\cdot)\|_{H^{m_0}}^{j}\|K(t,x,D_x)^{-1}u(t,\cdot)\|_{H^m}.
\end{aligned}$$

Thus we gain (4.6). Here we used the inequality $\|v\|_{L^\infty}\le C_m\|v\|_{H^m}$ for $m>n/2$.

Now, in aid of (1.7) and (4.6)

$$\|\Phi(u)\|_m^T\le C_h e^{\int_0^T c(t)\,dt}\left\{C_{\gamma,m,u_0,u_1}+\int_0^T\psi_{\gamma,m}(t,\|K^{-1}u(t)\|_{H^{m_0}})\|K^{-1}u(t)\|_{H^m}\,dt\right\},$$

which leads to (4.1) if $T=T_{m,r}>0$ is small enough, *e.g.*, $r=2C_h e^{\max_{t\in[0,1]}c(t)}C_{\gamma,m,u_0,u_1}$ and $0<T\le 1$ with $T\psi_{\gamma,m}\ (1,\sup_{t\in[0,1]}\|K^{-1}u(t)\|_{H^{m_0}})r\le C_{\gamma,m,u_0,u_1}$. Analogously, we have by means of Taylor's formula

$$\begin{aligned}
\|\Phi(u)-\Phi(w)\|_m^T&\le e^{\int_0^T c(t)\,dt}\int_0^T\|K^{-1}\{f(t,\cdot,u)-f(t,\cdot,w)\}\|_{H^m}\,dt\\
&\le C_{\gamma,m}T\|u-w\|_m^T\le\frac{1}{2}\|u-w\|_m^T
\end{aligned}$$

if $T\le 1/2C_{\gamma,m}$. Therefore we conclude that Φ is a contraction on $B_m^T(r)$, whose unique fixed point is the local solution $u(t,x)$ to the problem (1.1), (1.2).

For now, the lifespan T of the solution, determined in the foregoing procedure, may depend on m, and so it may happen that T_m tends to 0 as m does to ∞. But we can select $T_m=T_{m_0}$ for all $m\ge m_0$ with $m_0\ge[n/2]+1$. To this end, we shall consider the following strictly hyperbolic Cauchy problem on $[0,T_{m_0}]\times\boldsymbol{R}_x^n(\varepsilon>0)$:

$$\{L(t,x,\partial_t,\partial_x)-\varepsilon\Delta_x\}w(t,x)=f(t,x,w(t,x)),\tag{4.8}$$

$$w(0,x)=u_0(x),w_t(0,x)=u_1(x).\tag{4.9}$$

It we apply the preceding inference to this problem, then there exists a local solution $u_\varepsilon\in X_{m_0}^{T_{m_0}}$ on some interval $[0,T_{m_0}]$. In the same manner as Section 3 we also know that for the solution $u_\varepsilon(t,x)$ to the problem (4.8), (4.9)

$$\varepsilon^2\|K^{-1}u_\varepsilon(t)\|^2_{H^{m+1}}+\|K^{-1}u'_\varepsilon(t)\|^2_{H^m}$$
$$\le e^{\int_0^t\tilde c(s)\,ds}\left(\tilde C_h\|K^{-1}u_0\|^2_{H^{m+1}}+\|K^{-1}u_1\|^2_{H^m}+\int_0^t\|K^{-1}f(s,\cdot,u_\varepsilon(s,\cdot))\|^2_{H^m}\,ds\right),$$

which derives $u_\varepsilon\in X^{T_{m_0}}_{m_0+1}$ since $u_0,u_1\in C_0^\infty(\boldsymbol{R}^n)\subset H^\infty(\boldsymbol{R}^n)$ and $f(t,x,u_\varepsilon(t,x))\in C^0([0,T_{m_0}];\boldsymbol{R}^n_x\times H^{m_0+\ell}(\boldsymbol{R}^n))$. The above argument can be repeated by induction on $m(\ge m_0)$.

Now we apply (1.7) to $u_\varepsilon(t,x)$, and then in view of (4.6)

$$\|K^{-1}u_\varepsilon(t)\|^2_{H^m}+\|K^{-1}u'_\varepsilon(t)\|^2_{H^m}$$
$$\le e^{\int_0^t c(s)\,ds}\left(C_h\|K^{-1}u_0\|^2_{H^{m+1}}+\|K^{-1}u_1\|^2_{H^m}+\int_0^t\|K^{-1}f(s,\cdot,u_\varepsilon(s,\cdot))\|^2_{H^m}\,ds\right)$$
$$\le e^{\int_0^t c(s)\,ds}\left(C_h\|K^{-1}u_0\|^2_{H^{m+1}}+\|K^{-1}u_1\|^2_{H^m}\right.$$
$$\left.+\psi_{\gamma,m}\left(T_{m_0},\sup_{s\in[0,T_{m_0}]}\|K^{-1}u_\varepsilon(s)\|^2_{H^{m_0}}\right)\int_0^t\|K^{-1}u_\varepsilon(s)\|^2_{H^m}\,ds\right). \tag{4.10}$$

Because $\sup_{s\in[0,T_{m_0}]}\|K^{-1}u_\varepsilon(s)\|^2_{H^{m_0}}\le B<\infty$ from $u_\varepsilon\in X^{T_{m_0}}_{m_0}$, (4.10) means that

$$\|K^{-1}u_\varepsilon(t)\|^2_{H^m}+\|K^{-1}u'_\varepsilon(t)\|^2_{H^m}$$
$$\le c(h,B)\left(\|K^{-1}u_0\|^2_{H^{m+1}}+\|K^{-1}u_1\|^2_{H^m}+\int_0^t\|K^{-1}u_\varepsilon(s)\|^2_{H^m}\,ds\right).$$

Therefore, defining

$$E^\varepsilon_m(t)=\int_0^t(\|K^{-1}u_\varepsilon(s)\|^2_{H^m}+\|K^{-1}u'_\varepsilon(s)\|^2_{H^m})\,ds,$$

we get

$$\frac{d}{dt}E^\varepsilon_m(t)\le c(h,B)(\|K^{-1}u_0\|^2_{H^{m+1}}+\|K^{-1}u_1\|^2_{H^m}+E^\varepsilon_m(t))$$

and by Gronwall's inequality

$$E^\varepsilon_m(t)\le c(h,B,T_{m_0})(E^\varepsilon_m(0)+\|K^{-1}u_0\|^2_{H^{m+1}}+\|K^{-1}u_1\|^2_{H^m}),$$

that is,

$$\|K^{-1}u_\varepsilon(t)\|^2_{H^m}+\|K^{-1}u'_\varepsilon(t)\|^2_{H^m}$$
$$\le\tilde c(h,B,T_{m_0})(\|K^{-1}u_0\|^2_{H^{m+1}}+\|K^{-1}u_1\|^2_{H^m}) \tag{4.11}$$

on $[0,T_{m_0}]$, where the constant in the right hand side does not depend on ε. As before, we may rewrite (4.11) like

$$\|u_\varepsilon(t)\|_{H^m}^2 + \|u_\varepsilon'(t)\|_{H^m}^2 \le c(h, B, T_{m_0}, \gamma)(\|u_0\|_{H^{m+\ell+1}}^2 + \|u_1\|_{H^{m+\ell}}^2)$$

on $[0, T_{m_0}]$. Hence, for each $m \ge m_0$ $\{u_\varepsilon\}$ is a bounded sequence in $C^1([0, T_{m_0}]; H^m)$, and by (1.1) a fortiori in $C^2([0, T_{m_0}]; H^{m-1})$. Thus we can extract a subsequence which converges in $C^1([0, T_{m_0}]; H^m)$ for all $m \ge m_0$.

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Institute of Mathematics
University of Tsukuba
1-1-1, Tennōdai, Tsukuba-shi,
Ibaraki 305-8571,
Japan