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SPINOR-GENERATORS OF COMPACT EXCEPTIONAL LIE GROUPS F_4, E_6 AND E_7

By

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1. Introduction

We know that any element A of the group $SO(3)$ can be represented as

$$A = A_1A_2A_1', \quad A_1, A_1' \in SO_1(2), A_2 \in SO_2(2)$$

where $SO_k(2) = \{A \in SO(3) \mid Ae_k = e_k\}$ $(k = 1,2)$ ([1]). In the present paper, we shall show firstly that the similar results hold for the groups $SU(3)$ and $Sp(3)$ (Theorem 1). Secondly, we shall show that any element α of the simply connected compact Lie group F_4 (resp. E_6) can be represented as

$$\alpha = \alpha_1\alpha_2\alpha_1', \qquad \begin{array}{c} \alpha_1, \alpha_1' \in Spin_1(9),\ \alpha_2 \in Spin_2(9) \\ (\text{resp. } \alpha_1, \alpha_1' \in Spin_1(10),\ \alpha_2 \in Spin_2(10)) \end{array}$$

where $Spin_k(9) = \{\alpha \in F_4, \mid \alpha E_k = E_k\}$ (resp. $Spin_k(10) = \{\alpha \in E_6 \mid \alpha E_k = E_k\}$ (Theorem 5 (resp. Theorem 7))). Lastly, we shall show that any element α of the simply connected compact Lie group E_7 can be represented as

$$\alpha = \alpha_1\alpha_2\alpha_1'\alpha_2'\alpha_1'', \quad \alpha_1, \alpha_1', \alpha_1'' \in Spin_1(12), \alpha_2, \alpha_2' \in Spin_2(12)$$

where $Spin_k(12) = \{\alpha \in E_7 \mid \alpha\kappa_k = \kappa_k\alpha, \alpha\mu_k = \mu_k\alpha\}$ (Theorem 10).

In this paper we follow the notation of [2].

2. Spinor-generators of the groups $SO(3)$, $SU(3)$ and $Sp(3)$

Let $\boldsymbol{H}$ be the quaternion field with basis 1, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ over $\boldsymbol{R}$. Then we can express each element $a = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \in \boldsymbol{H}$ in the following polar form

$$a = r(\cos\theta + u\sin\theta), \quad u^2 = -1 (u \in \boldsymbol{H}), r = |a| = \sqrt{\sum_{k=0}^{3} a_k^2}, \theta \in \boldsymbol{R}.$$

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Hereafter, we briefly denote by $re^{u\theta}$ an element $r(\cos\theta + u\sin\theta)$ after the model of complex numbers.

The classical groups $SO(n)$, $SU(n)$ and $Sp(n)$ are respectively defined by

$$SO(n) = \{A \in M(n, \boldsymbol{R}) \mid {}^tAA = E, \det A = 1\},$$
$$SU(n) = \{A \in M(n, \boldsymbol{C}) \mid A^*A = E, \det A = 1\},$$
$$Sp(n) = \{A \in M(n, \boldsymbol{H}) \mid A^*A = E\}$$

where we follow the usual convention for matrices: $M(n, K)$ (= the set of square matrices of order n with coefficients in $K = \boldsymbol{R}, \boldsymbol{C}$ or $\boldsymbol{H}$), ${}^tA$, $A^*(= \overline{{}^tA})$, E (= the unit matrix) and det (= the determinant).

THEOREM 1. (1) *Any element* $A \in SO(3)$ *can be represented as*

$$A = A_1A_2A_1', \quad A_1, A_1' \in SO_1(2),\ A_2 \in SO_2(2)$$

where $SO_k(2) = \{A \in SO(3) \mid Ae_k = e_k\} \cong Spin(2)$ $(k = 1, 2)$, $e_1 = {}^t(1, 0, 0)$, $e_2 = {}^t(0, 1, 0)$.

(2) *Any element* $A \in SU(3)$ *can be represented as*

$$A = A_1A_2A_1', \quad A_1, A_1' \in SU_1(2),\ A_2 \in SU_2(2)$$

where $SU_k(2) = \{A \in SU(3) \mid Ae_k = e_k\} \cong Spin(3)$ $(k = 1, 2)$.

(3) *Any element* $A \in Sp(3)$ *can be represented as*

$$A = A_1A_2A_1', \quad A_1, A_1' \in Sp_1(2),\ A_2 \in Sp_2(2)$$

where $Sp_k(2) = \{A \in Sp(3) \mid Ae_k = e_k\} \cong Spin(5)$ $(k = 1, 2)$.

PROOF. It suffices to prove (3), because we can reduce (1) and (2) to the particular case of (3) in the proof below. First, for a given element $A \in Sp(3)$, suppose $Ae_1 = {}^t(a_1, a_2, a_3)$, $a_2 \neq 0$ $(a_k \in \boldsymbol{H}$ $(k = 1, 2, 3))$. Then there exist an element $u \in \boldsymbol{H}$ satisfying $u^2 = -1$ and a real number $\alpha \in \boldsymbol{R}$ such that $a_3a_2^{-1} = (|a_3|/|a_2|)e^{u\alpha}$. Choose $\theta \in \boldsymbol{R}$ such that $\cot\theta = |a_3|/|a_2|$ and set

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{u\alpha/2}\cos\theta & -e^{-u\alpha/2}\sin\theta \\ 0 & e^{u\alpha/2}\sin\theta & e^{-u\alpha/2}\cos\theta \end{pmatrix} \in Sp_1(2).$$

Then we get

$$B_1 A e_1 = {}^t(b_1, 0, b_3), \quad b_1, b_3 \in \boldsymbol{H}.$$

Next suppose $b_3 \neq 0$. Then there exist an element $v \in \boldsymbol{H}$ satisfying $v^2 = -1$ and a real number $\beta \in \boldsymbol{R}$ such that $b_1 b_3^{-1} = (|b_1|/|b_3|)e^{v\beta}$. Choose $\varphi \in \boldsymbol{R}$ such that $\cot \varphi = -|b_1|/|b_3|$ and set

$$B_2 = \begin{pmatrix} e^{-v\beta/2}\cos\varphi & 0 & -e^{v\beta/2}\sin\varphi \\ 0 & 1 & 0 \\ e^{-v\beta/2}\sin\varphi & 0 & e^{v\beta/2}\cos\varphi \end{pmatrix} \in Sp_2(2).$$

Then we get

$$B_2 B_1 A e_1 = {}^t(c_1, 0, 0), \quad c_1 \in \boldsymbol{H}.$$

Since $|c_1| = 1$, we can say $c_1 = e^{w\gamma}$ ($w^2 = -1$, $w \in \boldsymbol{H}$, $\gamma \in \boldsymbol{R}$). Set

$$B_2' = \begin{pmatrix} e^{-w\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{w\gamma} \end{pmatrix} \in Sp_2(2).$$

Then, since it follows $B_2' B_2 B_1 A e_1 = e_1$, *i.e.*, $B_2' B_2 B_1 A \in Sp_1(2)$, we can set $B_2' B_2 B_1 A = B_1' \in Sp_1(2)$. This implies

$$A = A_1 A_2 A_1', \quad A_1, A_1' \in Sp_1(2),\ A_2 \in Sp_2(2).$$

3. Some elements of $Spin_k(9)$, $Spin_k(10)$ and $Spin_k(12)$.

As for the definitions of $Spin_k(9)$, $Spin_k(10)$ and $Spin_k(12)$ ($k = 1, 2$), see Section 4, 5 and 6.

Lemma 2 (Section 4 and [2]). (1) *Let* $\alpha_1(a)$ *be the mapping* $\alpha(a)$ *defined in* [2] *Lemma* 2.(1). *Then* $\alpha_1(a)$ *belongs to* $Spin_1(9) \subset Spin_1(10) \subset Spin_1(12)$.

(2) *For* $a \in \mathfrak{C}$, $a \neq 0$, *let* $\alpha_2(a) : \mathfrak{J} \to \mathfrak{J}$ *be the mapping defined by changing all of the indices from* k *to* $k+1$ (*index modulo* 3) *in the definition of* $\alpha(a)$ *of* [2] *Lemma* 2.(1), *that is,*

$$\begin{cases} \xi_1' = \dfrac{\xi_3+\xi_1}{2} - \dfrac{\xi_3-\xi_1}{2}\cos 2|a| - \dfrac{(a,x_2)}{|a|}\sin 2|a| \\ \xi_2' = \xi_2 \\ \xi_3' = \dfrac{\xi_3+\xi_1}{2} + \dfrac{\xi_3-\xi_1}{2}\cos 2|a| + \dfrac{(a,x_2)}{|a|}\sin 2|a| \end{cases}$$

$$\begin{cases} x_1' = x_1\cos|a| + \dfrac{\overline{ax_3}}{|a|}\sin|a| \\ x_2' = x_2 - \dfrac{(\xi_3-\xi_1)a}{2|a|}\sin 2|a| - \dfrac{2(a,x_2)a}{|a|^2}\sin^2|a| \\ x_3' = x_3\cos|a| - \dfrac{\overline{x_1 a}}{|a|}\sin|a|, \end{cases}$$

where $\alpha_2(a)X = X'$. *Then* $\alpha_2(a)$ *belongs to* $Spin_2(9) \subset Spin_2(10) \subset Spin_2(12)$.

LEMMA 3 (Section 5 and [2]). (1) *Let* $\beta_1(a)$ *be the mapping* $\beta(a)$ *defined in* [2] *Lemma* 2.(2). *Then* $\beta_1(a)$ *belongs to* $Spin_1(10) \subset Spin_1(12)$.

(2) *For* $a \in \mathfrak{C}$, $a \neq 0$, *let* $\beta_2(a) : \mathfrak{J}^C \to \mathfrak{J}^C$ *be the mapping defined by changing all of the indices from* k *to* $k+1$ (*index modulo* 3) *in the definition of* $\beta(a)$ *of* [2] *Lemma* 2.(2), *that is,*

$$\begin{cases} \xi_1' = -\dfrac{\xi_3-\xi_1}{2} + \dfrac{\xi_3+\xi_1}{2}\cos 2|a| + i\dfrac{(a,x_2)}{|a|}\sin 2|a| \\ \xi_2' = \xi_2 \\ \xi_3' = \dfrac{\xi_3-\xi_1}{2} + \dfrac{\xi_3+\xi_1}{2}\cos 2|a| + i\dfrac{(a,x_2)}{|a|}\sin 2|a| \end{cases}$$

$$\begin{cases} x_1' = x_1\cos|a| + i\dfrac{\overline{ax_3}}{|a|}\sin|a| \\ x_2' = x_2 + i\dfrac{(\xi_3+\xi_1)a}{2|a|}\sin 2|a| - \dfrac{2(a,x_2)a}{|a|^2}\sin^2|a| \\ x_3' = x_3\cos|a| + i\dfrac{\overline{x_1 a}}{|a|}\sin|a|, \end{cases}$$

where $\beta_2(a)X = X'$. *Then* $\beta_2(a)$ *belongs to* $Spin_2(10) \subset Spin_2(12)$.

LEMMA 4 (Section 6 and [2]). (1) *Let* $\gamma_1(a)$ *be the mapping* $\gamma(a)$ *defined in* [2] *Lemma* 3.(1). *Then* $\gamma_1(a)$ *belongs to* $Spin_1(12)$.

(2) *For* $a \in \mathfrak{C}$, $a \neq 0$, *let* $\gamma_2(a) : \mathfrak{P}^C \to \mathfrak{P}^C$ *be the mapping defined by changing all of the indices from* k *to* $k+1$ (*index modulo* 3) *in the definition of* $\gamma(a)$ *of* [2] *Lemma* 3.(1), *that is,*

$$\begin{cases} \xi_1' = \xi_1 \\ \xi_2' = \dfrac{\xi_2 - \xi}{2} + \dfrac{\xi_2 + \xi}{2}\cos 2|a| + \dfrac{(a, y_2)}{|a|}\sin 2|a| \\ \xi_3' = \xi_3 \\ x_1' = x_1 \cos|a| - \dfrac{\overline{a y_3}}{|a|}\sin|a| \\ x_2' = x_2 + \dfrac{(\eta_2 + \eta)a}{2|a|}\sin 2|a| - \dfrac{2(a, x_2)a}{|a|^2}\sin^2|a| \\ x_3' = x_3 \cos|a| - \dfrac{\overline{y_1 a}}{|a|}\sin|a| \end{cases}$$

$$\begin{cases} \eta_1' = \eta_1 \\ \eta_2' = \dfrac{\eta_2 - \eta}{2} + \dfrac{\eta_2 + \eta}{2}\cos 2|a| - \dfrac{(a, x_2)}{|a|}\sin 2|a| \\ \eta_3' = \eta_3 \\ y_1' = y_1 \cos|a| + \dfrac{\overline{a x_3}}{|a|}\sin|a| \\ y_2' = y_2 - \dfrac{(\xi_2 + \xi)a}{2|a|}\sin 2|a| - \dfrac{2(a, y_2)a}{|a|^2}\sin^2|a| \\ y_3' = y_3 \cos|a| + \dfrac{\overline{x_1 a}}{|a|}\sin|a| \end{cases}$$

$$\begin{cases} \xi' = -\dfrac{\xi_2 - \xi}{2} + \dfrac{\xi_2 + \xi}{2}\cos 2|a| + \dfrac{(a, y_2)}{|a|}\sin 2|a| \\ \eta' = -\dfrac{\eta_2 - \eta}{2} + \dfrac{\eta_2 + \eta}{2}\cos 2|a| - \dfrac{(a, x_2)}{|a|}\sin 2|a|, \end{cases}$$

where $\gamma_2(a)(X, Y, \xi, \eta) = (X', Y', \xi', \eta')$. *Then* $\gamma_2(a)$ *belongs to* $Spin_2(12)$.

(3) *Let* $\delta_1(a)$ *be the mapping* $\delta(a)$ *defined in* [2] *Lemma* 3.(2). *Then* $\delta_1(a)$ *belongs to* $Spin_1(12)$.

(4) *For* $a \in \mathfrak{C}$, $a \neq 0$, *let* $\delta_2(a) : \mathfrak{P}^C \to \mathfrak{P}^C$ *be the mapping defined by changing all of the indices from* k *to* $k+1$ (*index modulo* 3) *in the definition of* $\delta(a)$ *of* [2] *Lemma* 3.(2), *that is,*

$$\begin{cases} \xi_1' = \xi_1 \\ \xi_2' = \dfrac{\xi_2 + \xi}{2} + \dfrac{\xi_2 - \xi}{2}\cos 2|a| - i\dfrac{(a, y_2)}{|a|}\sin 2|a| \\ \xi_3' = \xi_3 \\ x_1' = x_1 \cos|a| + i\dfrac{\overline{ay_3}}{|a|}\sin|a| \\ x_2' = x_2 - i\dfrac{(\eta_2 - \eta)a}{2|a|}\sin 2|a| - \dfrac{2(a, x_2)a}{|a|^2}\sin^2|a| \\ x_3' = x_3 \cos|a| + i\dfrac{\overline{y_1 a}}{|a|}\sin|a| \end{cases}$$

$$\begin{cases} \eta_1' = \eta_1 \\ \eta_2' = \dfrac{\eta_2 + \eta}{2} + \dfrac{\eta_2 - \eta}{2}\cos 2|a| - i\dfrac{(a, x_2)}{|a|}\sin 2|a| \\ \eta_3' = \eta_3 \\ y_1' = y_1 \cos|a| + i\dfrac{\overline{ax_3}}{|a|}\sin|a| \\ y_2' = y_2 - i\dfrac{(\xi_2 - \xi)a}{2|a|}\sin 2|a| - \dfrac{2(a, y_2)a}{|a|^2}\sin^2|a| \\ y_3' = y_3 \cos|a| + i\dfrac{\overline{x_1 a}}{|a|}\sin|a| \end{cases}$$

$$\begin{cases} \xi' = \dfrac{\xi_2 + \xi}{2} - \dfrac{\xi_2 - \xi}{2}\cos 2|a| + i\dfrac{(a, y_2)}{|a|}\sin 2|a| \\ \eta' = \dfrac{\eta_2 + \eta}{2} - \dfrac{\eta_2 - \eta}{2}\cos 2|a| + i\dfrac{(a, x_2)}{|a|}\sin 2|a| \end{cases}$$

where $\delta_2(a)(X, Y, \xi, \eta) = (X', Y', \xi', \eta')$. *Then* $\delta_2(a)$ *belongs to* $Spin_2(12)$.

4. $Spin(9)$-generators of the group F_4

The simply connected compact Lie group F_4 is given by

$$F_4 = \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}) \mid \alpha(X \times Y) = \alpha X \times \alpha Y\}.$$

The group F_4 has subgroups

$$Spin_k(9) = \{\alpha \in F_4 \mid \alpha E_k = E_k\} \ (k = 1, 2),$$

where $E_1 = (1,0,0;0,0,0)$, $E_2 = (0,1,0;0,0,0) \in \mathfrak{J}$, which is isomorphic to the usual spinor group $Spin(9)$ ([2], [3]).

THEOREM 5. *Any element* $\alpha \in F_4$ *can be represented as*

$$\alpha = \alpha_1 \alpha_2 \alpha_1', \quad \alpha_1, \alpha_1' \in Spin_1(9), \, \alpha_2 \in Spin_2(9).$$

PROOF. For a given element $\alpha \in F_4$, it suffices to show that there exist $\alpha_1 \in Spin_1(9)$ and $\alpha_2 \in Spin_2(9)$ such that $\alpha_2\alpha_1\alpha E_1 = E_1$. Now, for $\alpha E_1 = (\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = X_0$, choose $a \in \mathfrak{C}$ such that $(a, x_1) = 0$, $|a| = \pi/4$, and define $\alpha_1(a) \in Spin_1(9)$ of Lemma 2.(1). Then we get

$$\alpha_1(a)X_0 = (\xi_1', \xi_2', \xi_3'; x_1', x_2', x_3') = X_1. \quad \xi_1' = \xi_1, \xi_2' = \xi_3' \in \boldsymbol{R}, x_k' \in \mathfrak{C}.$$

If $x_1' \neq 0$, define $\alpha_1(\pi x_1'/4|x_1'|) \in Spin_1(9)$. Then we get

$$\alpha_1(\pi x_1'/4|x_1'|)X_1 = (\xi_1'', \xi_2'', \xi_3''; 0, x_2'', x_3'') = X_2, \quad \xi_1'' = \xi_1', \xi_k'' \in \boldsymbol{R}, x_k'' \in \mathfrak{C}.$$

The condition $X_2 \times X_2 = 0$ of the above form is equivalent to the following equations:

$$(*) \qquad \begin{aligned} &\xi_2''\xi_3'' = 0, \quad \xi_3''\xi_1'' = x_2''\overline{x_2''}, \quad \xi_1''\xi_2'' = x_3''\overline{x_3''}, \\ &\overline{x_2''x_3''} = 0, \quad \xi_2''x_2'' = 0, \quad \xi_3''x_3'' = 0. \end{aligned}$$

By the first equation $\xi_2''\xi_3'' = 0$ of $(*)$, it is enough to consider the two cases: (I) $\xi_2'' = 0$, (II) $\xi_2'' \neq 0$ and $\xi_3'' = 0$.

(I) Because of $(*)$ and $\xi_2'' = 0$, we have $x_3''\overline{x_3''} = 0$, hence $x_3'' = 0$. Therefore X_2 is of the form

$$X_2 = (\xi_1'', 0, \xi_3''; 0, x_2'', 0), \quad \xi_1'' = \xi_1', \, \xi_3'' \in \boldsymbol{R}, \, x_2'' \in \mathfrak{C}.$$

Choose $b \in \mathfrak{C}$ such that $(b, x_2'') = 0$, $|b| = \pi/4$, and define $\alpha_2(b) \in Spin_2(9)$ of Lemma 2.(2). Then

$$\alpha_2(b)X_2 = (\xi_1^{(3)}, 0, \xi_3^{(3)}; 0, x_2^{(3)}, 0) = X_3, \quad \xi_1^{(3)} = \xi_3^{(3)} \in \boldsymbol{R}, \, x_2^{(3)} \in \mathfrak{C}.$$

If $x_2^{(3)} = 0$, then by the condition $X_3 \times X_3 = 0$ we have that $(\xi_1^{(3)})^2 = (\xi_3^{(3)})^2 = x_2^{(3)}\overline{x_2^{(3)}} = 0$ so that $X_2 = 0$, which is a contradiction. Hence $x_2^{(3)} \neq 0$. Consider $\alpha_2(\pi x_2^{(3)}/4|x_2^{(3)}|) \in Spin_2(9)$. Then

$$\alpha_2(\pi x_2^{(3)}/4|x_2^{(3)}|)X_3 = (\xi_1^{(4)}, 0, \xi_3^{(4)}; 0, 0, 0) = X_4, \quad \xi_1^{(4)}, \xi_3^{(4)} \in \boldsymbol{R}.$$

From $X_4 \times X_4 = 0$, we have $\xi_3^{(4)}\xi_1^{(4)} = 0$. If $\xi_3^{(4)} = 0$, then $X_4 = E_1$ since $\xi_1^{(4)} = \mathrm{tr}(X_4) = \mathrm{tr}(E_1) = 1$. If $\xi_1^{(4)} = 0$, consider $\alpha_2(\pi/2) \in Spin_2(9)$. Then

$$\alpha_2(\pi/2)X_4 = (\xi_1^{(5)}, 0, 0; 0, 0, 0) = X_5, \quad \xi_1^{(5)} = \xi_3^{(4)} \in \boldsymbol{R}.$$

Thus we obtain $X_5 = E_1$.

(II) Because of the condition $\xi_2'' x_2'' = 0$ in $(*)$, we have $x_2'' = 0$. Therefore X_2 is of the form

$$X_2 = (\xi_1'', \xi_2'', 0; 0, 0, x_3''), \quad \xi_k'' \in \boldsymbol{R},\ x_3'' \in \mathfrak{C}.$$

Then $\alpha_1(\pi/2)X_2$ is nothing but X_2 in Case (I), so that Case (II) can be reduced to Case (I).

We have just completed the proof of Theorem 5.

5. *Spin*(10)-generators of the group E_6

The simply connected compact Lie group E_6 is given by

$$E_6 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \mid \alpha X \times \alpha Y = \tau\alpha\tau(X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}.$$

The group E_6 has subgroups

$$Spin_k(10) = \{\alpha \in E_6 \mid \alpha E_k = E_k\} \ (k = 1, 2),$$

which is isomorphic to the usual spinor group $Spin(10)$ ([2], [3]).

LEMMA 6. (1) *For any element*

$$X = (\xi_1, \xi_2, \xi_3; x_1, 0, 0), \quad \xi_k \in C,\ x_1 \in \mathfrak{C}^C$$

of $\mathfrak{J}^C$, *there exists some element* $\alpha_1 \in Spin_1(10)$ *such that*

$$\alpha_1 X = (\xi_1', \xi_2', \xi_3'; 0, 0, 0), \quad \xi_1' = \xi_1,\ \xi_k' \in C.$$

(2) *For any element*

$$X = (\xi_1, 0, 0; 0, x_2, x_3), \quad \xi_1 \in C,\ x_k \in \mathfrak{C}^C$$

of $\mathfrak{J}^C$, *there exists some element* $\alpha_1 \in Spin_1(9)$ *such that*

$$\alpha_1 X = (\xi_1, 0, 0; 0, x_2', x_3'), \quad \xi_1' = \xi_1 \in C,\ x_2' \in \mathfrak{C}^C,\ x_3' \in \mathfrak{C}.$$

PROOF. (1) For $x_1 = p + iq$ $(p, q \in \mathfrak{C})$, choose $a \in \mathfrak{C}$, $a \neq 0$, such that $(a, p) = (a, q) = 0$, and define $\alpha_1(\pi a/4|a|) \in Spin_1(9)$ of Lemma 2.(1). Then

$$\alpha_1(\pi a/4|a|)X = (\xi_1', \xi_2', \xi_3'; x_1', 0, 0) = X_1, \quad \xi_1' = \xi_1,\ \xi_2' = \xi_3' \in C,\ x_1' \in \mathfrak{C}^C.$$

Next, for $x_1' = p' + iq'$ $(p', q' \in \mathfrak{C})$, choose $b \in \mathfrak{C}$, $b \neq 0$, such that $(b, p') =$

$(b, q') = 0$, and define $\beta_1(\pi b/4|b|) \in Spin_1(10)$ of Lemma 3.(1). Then

$$\beta_1(\pi b/4|b|)X_1 = (\xi_1'', 0, 0; x_1'', 0, 0) = X_2, \quad \xi_1'' = \xi_1 \in C,\ x_1'' \in \mathfrak{C}^C.$$

Next, for $x_1'' = p'' + iq''$ $(p'', q'' \in \mathfrak{C})$, if $q'' \neq 0$, define $\alpha_1(\pi q''/4|q''|) \in Spin_1(9)$. Then

$$\alpha_1(\pi q''/4|q''|)X_2 = (\xi_1^{(3)}, \xi_2^{(3)}, \xi_3^{(3)}; p^{(3)}, 0, 0) = X_3,\ \xi_1^{(3)} = \xi_1, \xi_3^{(3)} = -\xi_2^{(3)} \in C, p^{(3)} \in \mathfrak{C}.$$

Finally, if $p^{(3)} \neq 0$, define $\beta_1(\pi p^{(3)}/4|p^{(3)}|) \in Spin_1(10)$. Then we get

$$\beta_1(\pi p^{(3)}/4|p^{(3)}|)X_3 = (\xi_1^{(4)}, \xi_2^{(4)}, \xi_3^{(4)}; 0, 0, 0), \quad \xi_1^{(4)} = \xi_1,\ \xi_k^{(4)} \in C$$

as desired.

(2) At first, we show that for any element

$$Z = (\zeta_1, 0, 0; 0, z_2, z_3), \quad \zeta_1 \in \boldsymbol{R},\ z_k \in \mathfrak{C},$$

there exists $\alpha_1 \in Spin_1(9)$ such that

$$\alpha_1 Z = (\zeta_1', 0, 0; 0, z_2', 0), \quad \zeta_1' \in \boldsymbol{R},\ z_2' \in \mathfrak{C}.$$

In fact, if $z_2 z_3 \neq 0$, choose $t > 0$ such that $\cot(t|z_2 z_3|) = -|z_2|/|z_3|$, and define $\alpha_1(t\overline{z_2 z_3}) \in Spin_1(9)$. Then we get ($z_3$-part of $\alpha_1(t\overline{z_2 z_3})Z) = 0$. If $z_2 = 0$, then $\alpha_1(\pi/2)Z$ is of the form as desired. Now for a given element $X = (\xi_1, 0, 0; 0, x_2, x_3) \in \mathfrak{J}^C$, express it as $X = Y + iZ$, $Y, Z \in \mathfrak{J}$ and apply the result above to Z, then we get the required form $\alpha_1 X = \alpha_1 Y + i\alpha_1 Z$.

Theorem 7. *Any element $\alpha \in E_6$ can be represented as*

$$\alpha = \alpha_1 \alpha_2 \alpha_1', \quad \alpha_1, \alpha_1' \in Spin_1(10), \alpha_2 \in Spin_2(10).$$

Proof. For a given element $\alpha \in E_6$, set $\alpha E_1 = (\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = X_0 \in \mathfrak{J}^C$. By Lemma 6.(1), we can take $\alpha_1 \in Spin_1(10)$ such that

$$\alpha_1 X_0 = (\xi_1', \xi_2', \xi_3'; 0, x_2', x_3') = X_1, \quad \xi_1' = \xi_1,$$

because the subspaces $\{(\xi_1, \xi_2, \xi_3; x_1, 0, 0) \in \mathfrak{J}^C\}$ and $\{(0, 0, 0; 0, x_2, x_3) \in \mathfrak{J}^C\}$ are invariant under the action of the elements of $Spin_1(10)$, respectively. From the condition $X_1 \times X_1 = 0$, we have $\xi_2' \xi_3' = 0$. As a result, the argument is divided into the following three cases:

(I) Case $\xi_2' = 0$, $\xi_3' \neq 0$. From $X_1 \times X_1 = 0$, we have $\xi_3' x_3' = 0$, hence $x_3' = 0$. Therefore X_1 is of the form

$$X_1 = (\xi_1', 0, \xi_3'; 0, x_2', 0), \quad \xi_1' = \xi_1,$$

Thus, for $X_1 \in \mathfrak{J}^C$, we can take $\alpha_2 \in Spin_2(10)$ such that

$$\alpha_2 X_1 = (\xi_1'', 0, \xi_3''; 0, 0, 0) = X_2,$$

in the same way as in Lemma 6.(1). Then, from $X_2 \times X_2 = 0$, we have $\xi_1''\xi_3'' = 0$. Combined with $\langle X_2, X_2 \rangle = 1$, we have also that

$$X_2 = (\xi_1'', 0, 0; 0, 0, 0), \ (\tau\xi_1'')\xi_1'' = 1 \quad \text{or} \quad X_2 = (0, 0, \xi_3''; 0, 0, 0), (\tau\xi_3'')\xi_3'' = 1.$$

Thus we obtain that there exist some elements $\varepsilon_2(t) \in Spin_2(10)$ and $\alpha_2(\pi/2) \in Spin_2(9)$ such that

$$\varepsilon_2(t) X_2 = E_1 \quad \text{or} \quad \varepsilon_2(t)\alpha_2(\pi/2) X_2 = E_1,$$

where $\varepsilon_2(t) \in Spin_2(10)$ is defined by

$$\varepsilon_2(t)(\xi_1, \xi_2, \xi_3; x_1, x_2, x_3) = (e^{it}\xi_1, \xi_2, e^{-it}\xi_3; e^{-it/2}x_1, x_2, e^{it/2}x_3), \quad t \in \boldsymbol{R}$$

(cf. [2] Lemma 10.(1)).

(II) Case $\xi_2' \neq 0$, $\xi_3' = 0$. From $X_1 \times X_1 = 0$, we have $\xi_2' x_2' = 0$, hence $x_2' = 0$. Therefore X_1 is of the form

$$X_1 = (\xi_1', \xi_2', 0; 0, 0, x_3'), \quad \xi_1' = \xi_1.$$

Thus, by considering $\alpha_1(\pi/2)X_1$, where $\alpha_1(\pi/2) \in Spin_1(9)$, this can be reduced to Case (I).

(III) Case $\xi_2' = \xi_3' = 0$. By Lemma 6.(2), we can take $\alpha_1' \in Spin_1(9)$ such that

$$\alpha_1' X_1 = (\xi_1'', 0, 0; 0, x_2'', x_3'') = X_2, \quad \xi_1'' = \xi_1, \ x_2'' \in \mathfrak{C}^C, \ x_3'' \in \mathfrak{C}.$$

Then, from $X_2 \times X_2 = 0$ we have $x_3''\overline{x_3''} = 0$, hence $x_3'' = 0$. Thus, for $X_2 = (\xi_1'', 0, 0; 0, x_2'', 0) \in \mathfrak{J}^C$, we can take $\alpha_2 \in Spin_2(10)$ such that

$$\alpha_2 X_2 = (\xi_1^{(3)}, 0, \xi_3^{(3)}; 0, 0, 0) = X_3,$$

because of the result for $Spin_2(10)$ similar to Lemma 6.(1) for $Spin_1(10)$. Hence this can be reduced to Case (I), because X_3 is nothing but X_2 in Case (I).

We have just completed the proof of Theorem 7.

6. *Spin*(12)-generators of the group E_7

The simply connected compact Lie group E_7 is given by

$$E_7 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \ \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}.$$

The group E_7 has subgroups

$$Spin_k(12) = \{\alpha \in E_7 \mid \alpha\kappa_k = \kappa_k\alpha, \alpha\mu_k = \mu_k\alpha\} \quad (k = 1, 2)$$

where κ_k and μ_k are defined by

$$\kappa_k(X, Y, \xi, \eta) = (-(E_k, X)E_k + 4E_k \times (E_k \times X), (E_k, Y)E_k - 4E_k \times (E_k \times Y), -\xi, \eta),$$

$$\mu_k(X, Y, \xi, \eta) = (2E_k \times Y + \eta E_k, 2E_k \times X + \xi E_k, (E_k, Y), (E_k, X)),$$

respectively, e.g., when $k = 1$, for $P = ((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta) \in \mathfrak{P}^C$,

$$\kappa_1 P = ((-\xi_1, \xi_2, \xi_3; x_1, 0, 0), (\eta_1, -\eta_2, -\eta_3; -y_1, 0, 0), -\xi, \eta),$$

$$\mu_1 P = ((\eta, \eta_3, \eta_2; -y_1, 0, 0), (\xi, \xi_3, \xi_2; -x_1, 0, 0), \eta_1, \xi_1).$$

Then $Spin_k(12)$ is isomorphic to the usual spinor group $Spin(12)$ ([2], [4]).

LEMMA 8. *For an element* $P = ((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta) \in \mathfrak{P}^C$ *satisfying* $P \times P = 0$, *it holds the following*

(1) $\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + 2(x_1, y_1) + 2(x_2, y_2) + 2(x_3, y_3) - 3\xi\eta = 0$,

(2) $\xi_2\xi_3 - \eta_1\eta - x_1\overline{x_1} = 0$, (3) $\xi_3\xi_1 - \eta_2\eta - x_2\overline{x_2} = 0$,

(4) $\xi_1\xi_2 - \eta_3\eta - x_3\overline{x_3} = 0$, (5) $\xi_1 x_1 + \eta y_1 - \overline{x_2 x_3} = 0$,

(6) $\xi_2 x_2 + \eta y_2 - \overline{x_3 x_1} = 0$, (7) $\xi_3 x_3 + \eta y_3 - \overline{x_1 x_2} = 0$,

(8) $\eta_2\eta_3 - \xi_1\xi - y_1\overline{y_1} = 0$, (9) $\eta_3\eta_1 - \xi_2\xi - y_2\overline{y_2} = 0$,

(10) $\eta_1\eta_2 - \xi_3\xi - y_3\overline{y_3} = 0$, (11) $\eta_1 y_1 + \xi x_1 - \overline{y_2 y_3} = 0$,

(12) $\eta_2 y_2 + \xi x_2 - \overline{y_3 y_1} = 0$, (13) $\eta_3 y_3 + \xi x_3 - \overline{y_1 y_2} = 0$,

(14) $\eta_3 x_1 + \xi_2 y_1 + \overline{y_2 x_3} = 0$, (15) $\eta_3 x_2 + \xi_1 y_2 + \overline{x_3 y_1} = 0$,

(16) $\eta_2 x_3 + \xi_1 y_3 + \overline{y_1 x_2} = 0$, (17) $\eta_1 x_3 + \xi_2 y_3 + \overline{x_1 y_2} = 0$.

PROOF. These are immediate from the straightforward computation of $P \times P = 0$. (Note that those are not all of the relations followed by $P \times P = 0$.)

LEMMA 9. (1) *For any element* $P \in \mathfrak{P}^C$, *there exists some element* $\alpha_1 \in Spin_1(12)$ *such that*

$$\alpha_1 P = ((\xi_1, 0, 0; 0, x_2, x_3), (\eta_1, \eta_2, \eta_3; 0, y_2, y_3), \xi, \eta).$$

In particular, if an element $P = ((0, \xi_2, \xi_3; x_1, 0, 0), (\eta_1, 0, 0; 0, 0, 0), 0, \eta) \in \mathfrak{P}^C$ *satisfies the conditions* $P \times P = 0$ *and* $\langle P, P\rangle = 1$, *then there exists some element*

$\alpha_1 \in Spin_1(12)$ *such that*

$$\alpha_1 P = \underset{\cdot}{1}, \quad \textit{where } \underset{\cdot}{1} = (0,0,0,1) \in \mathfrak{P}^C.$$

(2) *For any element* $P \in \mathfrak{P}^C$, *there exists some element* $\alpha_2 \in Spin_2(12)$ *such that*

$$\alpha_2 P = ((0,\xi_2,0;x_1,0,x_3),(\eta_1,\eta_2,\eta_3;y_1,0,y_3),\xi,\eta).$$

In particular, if an element $P = ((\xi_1,0,\xi_3;0,x_2,0),(0,\eta_2,0;0,0,0),0,\eta) \in \mathfrak{P}^C$ *satisfies the conditions* $P \times P = 0$ *and* $\langle P,P\rangle = 1$, *then there exists some element* $\alpha_2 \in Spin_2(12)$ *such that*

$$\alpha_2 P = \underset{\cdot}{1}.$$

PROOF. (1) The first half is the very [2] Proposition 4.(2). We shall now prove the latter half. For an element $P=((0,\xi_2,\xi_3;x_1,0,0),(\eta_1,0,0;0,0,0),0,\eta) \in \mathfrak{P}^C$, act $\alpha_1 \in Spin_1(12)$ that is given in the first half which is composed of the elements of $Spin_1(12)$ defined in Lemmas 2, 3 and 4, on P. Then we get

$$\alpha_1 P = ((0,0,0;0,0,0),(\eta_1',0,0;0,0,0),0,\eta') = P_1,$$

because the subspaces $\langle\mathfrak{P}^C\rangle_1$, $\langle\mathfrak{P}^C\rangle_1'$ and $\langle\mathfrak{P}^C\rangle_1''$ of $\mathfrak{P}^C$ are invariant under the action of the elements of $Spin_1(12)$ defined in Lemmas 2, 3 and 4, respectively, where

$$\langle\mathfrak{P}^C\rangle_1 = \{((\xi_1,0,0;0,0,0),(0,\eta_2,\eta_3;y_1,0,0),\xi,0) \in \mathfrak{P}^C\},$$

$$\langle\mathfrak{P}^C\rangle_1' = \{((0,\xi_2,\xi_3;x_1,0,0),(\eta_1,0,0;0,0,0),0,\eta) \in \mathfrak{P}^C\},$$

$$\langle\mathfrak{P}^C\rangle_1'' = \{((0,0,0;0,x_2,x_3),(0,0,0;0,y_2,y_3),0,0) \in \mathfrak{P}^C\}.$$

From $P \times P = 0$, we have $\eta_1'\eta' = 0$ by Lemma 8.(2). As a result, the argument is devided into the following three cases:

(I) Case $\eta_1' = 0, \eta' \neq 0$. P_1 is of the form $P_1 = ((0,0,0;0,0,0),(0,0,0;0,0,0),0,\eta')$. Now, for $\theta \in C$ satisfying $(\tau\theta)\theta = 1$, define the mapping $\epsilon_1(\theta) : \mathfrak{P}^C \to \mathfrak{P}^C$ as follows.

$$\begin{aligned}\epsilon_1(\theta)&((\xi_1,\xi_2,\xi_3;x_1,x_2,x_3),(\eta_1,\eta_2,\eta_3;y_1,y_2,y_3),\xi,\eta)\\ &= ((\theta^{-2}\xi_1,\xi_2,\xi_3;x_1,\theta^{-1}x_2,\theta^{-1}x_3),(\theta^2\eta_1,\eta_2,\eta_3;y_1,\theta y_2,\theta y_3),\theta^2\xi,\theta^{-2}\eta).\end{aligned}$$

Then $\epsilon_1(\theta) \in Spin_1(12)$. Therefore, noting that $(\tau\eta')\eta' = \langle P_1,P_1\rangle = 1$, choose $\theta \in C$ such that $\theta^2 = \eta'$ and set $\epsilon_1(\theta)$. Then we get $\epsilon_1(\theta)P_1 = \underset{\cdot}{1}$.

(II) Case $\eta'_1 \neq 0$, $\eta' = 0$. By considering $\gamma_1(\pi/2)P_1$, where $\gamma_1(\pi/2) \in Spin_1(12)$ of Lemma 4.(1), this can be reduced to Case (I).

(III) Case $\eta'_1 = \eta' = 0$. This does not occur, because $\langle P_1, P_1 \rangle = 1$.

(2) It is similarly verified by using $Spin_2(12)$ instead of $Spin_1(12)$ in the proof of (1).

THEOREM 10. *Any element $\alpha \in E_7$ can be represented as*

$$\alpha = \alpha_1\alpha_2\alpha'_1\alpha'_2\alpha''_1, \quad \alpha_1, \alpha'_1, \alpha''_1 \in Spin_1(12), \alpha_2, \alpha'_2 \in Spin_2(12).$$

PROOF. For a given element $\alpha \in E_7$, it suffices to show that there exist $\alpha_1, \alpha'_1 \in Spin_1(12)$ and $\alpha_2 \in Spin_2(12)$ such that $\alpha'_1\alpha_2\alpha_1\alpha\underset{\cdot}{1} = \underset{\cdot}{1}$. In fact, since an element $\alpha \in E_7$ belongs to $E_6(\subset E_7)$ if and only if α fixes an element $\underset{\cdot}{1}$, *i.e.*, $\alpha\underset{\cdot}{1} = \underset{\cdot}{1}$ ([4]), it follows $\alpha'_1\alpha_2\alpha_1\alpha \in E_6$, which implies that $\alpha \in E_7$ can be represented as a required form by Theorem 7. Now, set

$$\alpha\underset{\cdot}{1} = ((\xi_1, \xi_2, \xi_3; x_1, x_2, x_3), (\eta_1, \eta_2, \eta_3; y_1, y_2, y_3), \xi, \eta) = P_0 \in \mathfrak{P}^C.$$

Then, by Lemma 9.(1), we can take $\alpha_1 \in Spin_1(12)$ such that

$$\alpha_1 P_0 = ((\xi'_1, 0, 0; 0, x'_2, x'_3), (\eta'_1, \eta'_2, \eta'_3; 0, y'_2, y'_3), \xi', \eta') = P_1.$$

From $P_1 \times P_1 = 0$, we have $\eta'_1\eta' = 0$ by Lemma 8.(2). As a result, the argument is devided into the following three cases:

(I) Case $\eta'_1 = 0$, $\eta' \neq 0$. By Lemma 8.(6) and (7), we get $y'_2 = y'_3 = 0$. Furthermore we get $\xi' = 0$ by Lemma 8.(1). Therefore P_1 is of the form

$$P_1 = ((\xi'_1, 0, 0; 0, x'_2, x'_3), (0, \eta'_2, \eta'_3; 0, 0, 0), 0, \eta').$$

Then, by Lemma 8.(8), we have $\eta'_2\eta'_3 = 0$. Hence there are three cases to be considered.

(I.A) Case $\eta'_2 = 0$, $\eta'_3 \neq 0$. By Lemma 8.(15), we get $x'_2 = 0$, that is, P_1 is of the form

$$P_1 = ((\xi'_1, 0, 0; 0, 0, x'_3), (0, 0, \eta'_3; 0, 0, 0), 0, \eta').$$

Then, applying Lemma 9.(2) to $\alpha_1(\pi/2)P_1$, where $\alpha_1(\pi/2) \in Spin_1(9)$, we can obtain that there exists some element $\alpha_2 \in Spin_2(12)$ such that $\alpha_2\alpha_1(\pi/2)P_1 = \underset{\cdot}{1}$.

(I.B) Case $\eta'_2 \neq 0$, $\eta'_3 = 0$. By Lemma 8.(16), we get $x'_3 = 0$, that is, P_1 is of the form

$$P_1 = ((\xi'_1, 0, 0; 0, x'_2, 0), (0, \eta'_2, 0; 0, 0, 0), 0, \eta').$$

Thus we can easily obtain the required result by Lemma 9.(2).

(I.C) Case $\eta'_2 = \eta'_3 = 0$. P_1 is of the form

$$P_1 = ((\xi'_1, 0, 0; 0, x'_2, x'_3), (0, 0, 0; 0, 0, 0), 0, \eta').$$

Here we distinguish the following cases:

(I.C.1) When $x'_2 \neq 0$, $x'_3 \neq 0$. By Lemma 6.(2), we can take $\alpha'_1 \in Spin_1(9)$ such that

$$\alpha'_1 P_1 = ((\xi''_1, 0, 0; 0, x''_2, x''_3), (0, 0, 0; 0, 0, 0), 0, \eta''), \xi''_1 = \xi'_1, \eta'' = \eta' \in C,$$
$$x''_2 \in \mathfrak{C}^C, x''_3 \in \mathfrak{C}.$$

Then, by Lemma 8.(4) we have $x''_3 \overline{x''_3} = 0$, hence $x''_3 = 0$. Thus we easily obtain the required result by Lemma 9.(2).

(I.C.2) When $x'_2 = 0$, $x'_3 \neq 0$. Considering $\alpha_1(\pi/2)P_1$, where $\alpha_1(\pi/2) \in Spin_1(9)$, we can easily obtain the required result by Lemma 9.(2).

(I.C.3) When $x'_2 \neq 0$, $x'_3 = 0$. We can easily obtain the required result by Lemma 9.(2).

(I.C.4) When $x'_2 = x'_3 = 0$. We can easily obtain the required result by Lemma 9.(2).

(II) Case $\eta'_1 \neq 0$, $\eta' = 0$. By considering $\delta_1(\pi/2)P_1$, where $\delta_1(\pi/2) \in Spin_1(12)$ of Lemma 4.(3), this can be reduced to Case (I).

(III) Case $\eta'_1 = \eta' = 0$. P_1 is of the form

$$P_1 = ((\xi'_1, 0, 0; 0, x'_2, x'_3), (0, \eta'_2, \eta'_3; 0, y'_2, y'_3), \xi', 0).$$

Now, as is similar to Lemma 9.(1), we obtain that, for any element $P \in \mathfrak{P}^C$, there exists some element $\alpha_1 \in Spin_1(12)$ such that

$$\alpha_1 P = ((\xi_1, \xi_2, \xi_3; 0, x_2, x_3), (\eta_1, 0, 0; 0, y_2, y_3), \xi, \eta).$$

Note that the invariant subspaces $\langle \mathfrak{P}^C \rangle_1$, $\langle \mathfrak{P}^C \rangle'_1$ and $\langle \mathfrak{P}^C \rangle''_1$ of $\mathfrak{P}^C$ under the action of the elements of $Spin_1(12)$ defined in Lemmas 2, 3 and 4. Then, applying the result above to the present Case (III), we can take $\alpha'_1 \in Spin_1(12)$ such that

$$\alpha'_1 P_1 = ((\xi''_1, 0, 0; 0, x''_2, x''_3), (0, 0, 0; 0, y''_2, y''_3), \xi'', 0) = P_2.$$

Therefore we have $\xi''_1 \xi'' = 0$ by Lemma 8.(8). Hence there are three cases to be considered.

(III.A) Case $\xi''_1 = 0$, $\xi'' \neq 0$. By Lemma 8.(12) and (13), we get $x''_2 = x''_3 = 0$. Then P_2 is of the form

$$P_2 = ((0, 0, 0; 0, 0, 0), (0, 0, 0; 0, y''_2, y''_3), \xi'', 0).$$

Thus, by considering $\gamma_1(\pi/2)P_2$, where $\gamma_1(\pi/2) \in Spin_1(12)$ of Lemma 4.(1), this can be reduced to Case (I.C).

(III.B) Case $\xi_1'' \neq 0$, $\xi'' = 0$. By Lemma 8.(15) and (16), we get $y_2'' = y_3'' = 0$. Therefore this is reduced to Case (I.C).

(III.C) Case $\xi_1'' = \xi'' = 0$. P_2 is of the form

$$P_2 = ((0,0,0;0,x_2'',x_3''),(0,0,0;0,y_2'',y_3''),0,0).$$

Here we distinguish the following cases:

(III.C.1) When $x_2'' \neq 0$. By Lemma 9.(2), there exists some element $\alpha_2 \in Spin_2(12)$ such that

$$\alpha_2 P_2 = ((0,\xi_2^{(3)},0;x_1^{(3)},0,x_3^{(3)}),(\eta_1^{(3)},\eta_2^{(3)},\eta_3^{(3)};y_1^{(3)},0,y_3^{(3)}),\xi^{(3)},\eta^{(3)}) = P_3.$$

Here, by Lemma 8.(3), we have $\eta_2^{(3)}\eta^{(3)} = 0$. Hence there are three cases to be considered.

(III.C.1.1) Case $\eta_2^{(3)} = 0$, $\eta^{(3)} \neq 0$. By Lemma 8.(5) and (7), we get $y_1^{(3)} = y_3^{(3)} = 0$. Furthermore, we get $\xi^{(3)} = 0$ by Lemma 8.(1). Then P_3 is of the form

$$P_3 = ((0,\xi_2^{(3)},0;x_1^{(3)},0,x_3^{(3)}),(\eta_1^{(3)},0,\eta_3^{(3)};0,0,0),0,\eta^{(3)}).$$

Here, by Lemma 8.(9), we have $\eta_3^{(3)}\eta_1^{(3)} = 0$. Hence there are three cases to be considered.

(III.C.1.1.1) Case $\eta_1^{(3)} = 0$, $\eta_3^{(3)} \neq 0$. By Lemma 8.(14), we get $x_1^{(3)} = 0$. Then, considering $\alpha_2(\pi/2)P_3$, where $\alpha_2(\pi/2) \in Spin_2(9)$, we can easily obtain the required result by Lemma 9.(1).

(III.C.1.1.2) Case $\eta_1^{(3)} \neq 0$, $\eta_3^{(3)} = 0$. By Lemma 8.(17), we get $x_3^{(3)} = 0$. Then we can easily obtain the required result by Lemma 9.(1).

(III.C.1.1.3) Case $\eta_1^{(3)} = \eta_3^{(3)} = 0$. P_3 is of the form

$$P_3 = ((0,\xi_2^{(3)},0;x_1^{(3)},0,x_3^{(3)}),(0,0,0;0,0,0),0,\eta^{(3)}).$$

Here we distinguish the following cases:

(III.C.1.1.3.(i)) When $x_1^{(3)} \neq 0$, $x_3^{(3)} \neq 0$. As is similar to Lemma 6.(2), we obtain that there exists some element $\alpha_2' \in Spin_2(9)$ such that

$$\alpha_2' P_3 = ((0,\xi_2^{(4)},0;x_1^{(4)},0,x_3^{(4)}),(0,0,0;0,0,0),0,\eta^{(4)}) = P_4, \quad \begin{array}{l} \xi_2^{(4)} = \xi_2^{(3)}, \eta^{(4)} = \eta^{(3)} \in C, \\ x_1^{(4)} \in \mathfrak{C}, x_3^{(4)} \in \mathfrak{C}^C. \end{array}$$

Then, by Lemma 8.(2), we have $x_1^{(4)}\overline{x_1^{(4)}} = 0$, hence $x_1^{(4)} = 0$. Thus, considering $\alpha_2(\pi/2)P_4$, where $\alpha_2(\pi/2) \in Spin_2(9)$, we can easily obtain the required result by Lemma 9.(1).

(III.C.1.1.3.(ii)) When $x_1^{(3)} = 0$, $x_3^{(3)} \neq 0$. Considering $\alpha_2(\pi/2)P_3$, where $\alpha_2(\pi/2) \in Spin_2(9)$, we can easily obtain the required result by Lemma 9.(1).

(III.C.1.1.3.(iii)) When $x_1^{(3)} \neq 0$, $x_3^{(3)} = 0$. We easily obtain the required result by Lemma 9.(1).

(III.C.1.1.3.(iv)) When $x_1^{(3)} = x_3^{(3)} = 0$. We easily obtain the required result by Lemma 9.(1).

(III.C.1.2) Case $\eta_2^{(3)} \neq 0$, $\eta^{(3)} = 0$. By considering $\gamma_2(\pi/2)P_3$, where $\gamma_2(\pi/2) \in Spin_2(12)$ of Lemma 4.(2), this can be reduced to Case (III.C.1.1).

(III.C.1.3) Case $\eta_2^{(3)} = \eta^{(3)} = 0$. This does not occur. In fact, note that the subspace $\langle \mathfrak{P}^C \rangle_2$ of $\mathfrak{P}^C$ is invariant under the action of the elements of $Spin_2(12)$ defined in Lemmas 2, 3 and 4, where $\langle \mathfrak{P}^C \rangle_2 = \{((\xi_1, 0, \xi_2, ; 0, x_2, 0), (0, \eta_2, 0; 0, 0, 0), 0, \eta) \in \mathfrak{P}^C\}$. Then, for $P_3 = \alpha_2 P_2$, that is,

$$((0, \xi_2^{(3)}, 0; x_1^{(3)}, 0, x_3^{(3)}), (\eta_1^{(3)}, 0, \eta_3^{(3)}; y_1^{(3)}, 0, y_3^{(3)}), \xi^{(3)}, 0)$$

$$= \alpha_2((0, 0, 0; 0, x_2'', x_3''), (0, 0, 0; 0, y_2'', y_3''), 0, 0),$$

where $\alpha_2 \in Spin_2(12)$, the condition $\eta_2^{(3)} = \eta^{(3)} = 0$ contradicts $x_2'' \neq 0$.

(III.C.2) When $x_2'' = 0$, $x_3'' \neq 0$. By considering $\alpha_1(\pi/2)P_2$, where $\alpha_1(\pi/2) \in Spin_1(9)$, this can be reduced to Case (III.C.1).

(III.C.3) When $x_2'' = x_3'' = 0$, $y_3'' \neq 0$. By considering $\gamma_1(\pi/2)P_2$, where $\gamma_1(\pi/2) \in Spin_1(12)$, this can be reduced to Case (III.C.1).

(III.C.4) When $x_2'' = x_3'' = y_3'' = 0$, $y_2'' \neq 0$. By considering $\alpha_1(\pi/2)P_2$, where $\alpha_1(\pi/2) \in Spin_1(9)$, this can be reduced to Case (III.C.3).

(III.C.5) When $x_2'' = x_3'' = y_2'' = y_3'' = 0$. It is obvious that this does not occur.

We have just completed the proof of Theorem 10.

Conjecture. We know that the simply connected compact exceptional Lie group E_8 has subgroups $Ss_k(16) = (E_8)^{\sigma_k}$ (where $\sigma_k = \exp \pi\kappa_k$), $k = 1, 2, 3$ (which is isomorphic to $Spin(16)/Z_2$ not $SO(16)$). Now the authors do not know if $Ss_1(16)$ and $Ss_2(16)$ generate the group E_8?

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