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# RADIAL FUNCTIONS AND MAXIMAL ESTIMATES FOR RADIAL SOLUTIONS TO THE SCHRÖDINGER EQUATION

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## 1. Preliminaries and result

Let $f$ belong to the Schwartz space $\mathscr{S}(\boldsymbol{R}^n)$ and set

$$S_t f(x) = u(x,t) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{ix\cdot\xi} e^{it|\xi|^a} \hat{f}(\xi)\, d\xi, \quad x \in \boldsymbol{R}^n,\ t \in \boldsymbol{R},$$

where $a > 1$. Here $\hat{f}$ denotes the Fourier transform of $f$, defined by

$$\hat{f} = \int_{\boldsymbol{R}^n} e^{-i\xi\cdot x} f(x)\, dx.$$

In the particular case $a = 2$, it is well known that $u$ is the solution of the Schrödinger equation with initial data $f$, $i\partial u/\partial t = \Delta u$ and $\lim_{t\to 0} u(x,t) = f$ in the $L^2$ sense. We also introduce Sobolev spaces $H_s$ by setting

$$H_s = \{f \in \mathscr{S}' : \|f\|_{H_s} < \infty\}, \quad s \in \boldsymbol{R},$$

where

$$\|f\|_{H_s} = \left(\int_{\boldsymbol{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi\right)^{1/2}.$$

We shall here consider the maximal functions

$$S^* f(\xi) = \sup_{0<t<1} |S_t f(x)|, \quad x \in \boldsymbol{R}.$$

In [3], L. Carleson proposed the question under what condition does $u(x,t) \to f$ as $t \to 0$ pointwise a.e? To answer the question it is sufficient to get an a-priori estimate of the $S^* f$.

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Local estimate for $S^*f$

$$(1)\qquad \left(\int_{B(0;R)} |S^*f(x)|^q\,dx\right)^{1/q} \le C_R\|f\|_{H_s}$$

where $B(0;R) = \{x \in \mathbf{R}^n; |x| \le R\}$, have been studied in several papers. (see e.g. J. Bourgain [1], L. Carleson [3], B. E. Dahlberg and C. E. Kenig [5], C. E. Kenig and A. Ruiz [8], E. Prestini [10], P. Sjölin [12, 13], L. Vega [17], C. E. Kenig, G. Ponce and L. Vega [6]).

For $q = 2$, in the case $n = 1$, the inequalities (1) holds with $s = 1/4$ and the result is sharp, and for $n = 2$, (1) holds with $s = 1/2$, for $n \ge 3$, holds for $s > 1/2$. (see [12]).

For radial function, $q = 1$ and $n \ge 2$, Prestini [10] has proved (1) with $s \ge 1/4$ and his result is sharp, and Sjölin [13] has established (1) with $s = 1/4$ and $q = 4n/(2n-1)$ and, (1) does not hold for $q > 4n/(2n-1)$.

For global estimate of the type $\|S^*f\|_2 \le C\|F\|_{H_s}$ where the left hand side is the norm in $L^2(\mathbf{R}^n)$, there are also the several papers (see A. Carbery [2] and M. Cowling [4] and C. E. Kenig, G. Ponce and L. Vega [6] and P. Sjölin [14]).

Sjölin [14] has proved that if $a > 1$ then $s > an/4$ is a sufficient condition for all $n$ and if $n = 1$ then $s \ge a/4$ is a necessary condition.

The purpose of this paper is the following global estimate

THEOREM 1. *If $q = 4n/(2n-1)$, then for $f$ radial,*

$$(2)\qquad \left(\int_{\mathbf{R}^n} |S^*f(x)|^q\,dx\right)^{1/q} \le C\|f\|_{H_{1/4}}.$$

Theorem 1 is a direct consequence of the following

THEOREM 2. *Assume $2 \le q \le 8/3$. If $\alpha = q(2n-1)/4 - n$ and $f$ is radial, then*

$$\left(\int_{\mathbf{R}^n} |S^*f(x)|^q |x|^\alpha\,dx\right)^{1/q} \le C\|f\|_{H_{1/4}}.$$

We shall need the following lemma and we choose a real valued function $\rho(\xi)$ in $C_0^\infty$ such that $\rho(\xi) = 1$, $|\xi| \le 1$ and $\rho(\xi) = 0$, $|\xi| \ge 2$.

Lemma 1 [12, pp. 709–712]. *Let $N \geq 1$ be a natural number and $a > 1$, then*

$$\left| \int_{\boldsymbol{R}} e^{i\{x\xi + y|\xi|^a\}} \rho_N^2(\xi) |\xi|^{-1/2}\, d\xi \right| \leq C|x|^{-1/2},$$

*for $0 < y < 1$, $x \in \boldsymbol{R}$, where $\rho_N(\xi) = \rho(\xi/N)$.*

Lemma 2 [9, pp. 38]. *Let $1 \leq p \leq q \leq \infty$, $1/p + 1/p' = 1 (1/\infty = 1/-\infty = 0)$ and let $K(x,y)$ be a measurable function on $\boldsymbol{R}^{2n}$. Assume that a measurable function $K_1 > 0$, $K_2 > 0$ such that*

$$|K(x,y)| \leq K_1(x,y) K_2(x,y),$$
$$\operatorname*{ess.sup}_y \|K_1(x,y)\|_{L^q(\boldsymbol{R}_x^n)} \leq C_1 < \infty,$$
$$\operatorname*{ess.sup}_x \|K_2(x,y)\|_{L^{p'}(\boldsymbol{R}_y^n)} \leq C_2 < \infty,$$

*then*

$$Tf(y) = \int K(x,y) f(x)\, dx$$

*is a bounded operator from $L^p(\boldsymbol{R}_x^n)$ to $L^q(\boldsymbol{R}_y^n)$.*

## 2. Proof of the theorem 2

We assume $2 \leq q \leq 8/3$ and $1/p + 1/q = 1$. Let $t(x)$ denote a measurable function, $0 < t(x) < 1$. Then it is sufficient to prove the theorem with $S^*$ replaced by $S_{t(x)}$. Let $s = |x|$ and $r = |\xi|$. We linearize the operator $S_{t(x)}$, we obtain

$$Sf(s) = c_n s^{1-n/2} \int_0^\infty J_{n/2-1}(rs) e^{it(s)r^a} \hat{f}(r) r^{n/2}\, dr,$$

where $J_k(x)$ denote Bessel function. (see [16, pp. 155]).

Now we set $g(r) = \hat{f}(r)(1+r^2)^{1/8} r^{(n-1)/2}$ and

$$\begin{aligned} Pg(s) &= \frac{1}{c_n} Sf(s) s^{(2n-1)/4-1/q} \\ &= s^{(2n-1)/4-1/q+1-n/2} \int_0^\infty J_{n/2-1}(rs) e^{it(s)r^a} \hat{f}(r) r^{n/2}\, dr \\ &= s^{3/4-1/q} \int_0^\infty J_{n/2-1}(rs) e^{it(s)r^a} g(r)(1+r^2)^{-1/8} r^{1/2}\, dr. \end{aligned}$$

Hence we have to prove that

$$\left(\int_0^\infty |Pg(s)|^q\,ds\right)^{1/q} \le C\left(\int_0^\infty |g(r)|^2\,dr\right)^{1/2}.$$

We can set the adjoint of $P$

$$P^*g(r) = (1+r^2)^{-1/8} r^{1/2} \int_0^\infty J_{n/2-1}(rs) e^{-it(s)r^a} s^{3/4-1/q} g(s)\,ds.$$

It is sufficient to prove that

$$\left(\int_0^\infty |P^*g(r)|^2\,dr\right)^{1/2} \le C\left(\int_0^\infty |g(s)|^p\,ds\right)^{1/p}, \quad g \in L^p(0,\infty).$$

Now, it follows from the estimate

$$J_k(r) = \sqrt{2/\pi r}\cos(r - \pi m/2 - \pi/4) + O(r^{-3/2}) \quad \text{as } r \to \infty$$

(see [16, pp. 158]) that

$$|t^{1/2} J_{n/2-1}(t) - (b_1 e^{it} + b_2 e^{-it})| \le C/t, \quad t > 1,$$

and

$$|t^{1/2} J_{n/2-1}(t) - (b_1 e^{it} + b_2 e^{-it})| \le C, \quad 0 < t \le 1.$$

By using above inequalities, we can write that

$$\begin{aligned} P^*g(r) &= b_1(1+r^2)^{-1/8} \int_0^\infty e^{irs} e^{-it(s)r^a} s^{-\gamma} g(s)\,ds \\ &\quad + b_2(1+r^2)^{-1/8} \int_0^\infty e^{-irs} e^{-it(s)r^a} s^{-\gamma} g(s)\,ds + Q(r) \\ &= b_1 A(r) + b_2 B(r) + Q(r), \end{aligned}$$

where $\gamma = 1/q - 1/4$ and

$$|Q(r)| \le C(1+r^2)^{-1/8} \int_0^\infty \min(1, 1/rs) s^{-\gamma} |g(s)|\,ds.$$

We extend $A$ to $\boldsymbol{R}$ by setting

$$A(\xi) = (1+\xi^2)^{-1/8} \int_0^\infty e^{i(\xi s - t(s)|\xi|^a)} s^{-\gamma} g(s)\,ds, \quad -\infty < \xi < 0.$$

Then $B(\xi) = A(-\xi)$, $0 < \xi < \infty$, and to estimate $A$ and $B$ we have only to prove on $A$,

$$(3) \qquad \left( \int_{-\infty}^{\infty} |A(\xi)|^2 \, d\xi \right)^{1/2} \le C\|g\|_p,$$

where

$$\|g\|_p = \left( \int_0^{\infty} |g(s)|^p \, ds \right)^{1/p}.$$

Then set

$$A_N(\xi) = \rho_N(\xi)|\xi|^{-1/4} \int_0^{\infty} e^{i(s\xi - t(s)|\xi|^a)} s^{-\gamma} g(s) \, ds.$$

We shall prove that

$$\left( \int_{\boldsymbol{R}} |A_N(\xi)|^2 \, d\xi \right)^{1/2} \le C\|g\|_p$$

with $C$ independent of $N$, and (3) follow from this inequality.

We have

$$\begin{aligned}
&\int_{\boldsymbol{R}} |A_N(\xi)|^2 \, d\xi \\
&= \int_{\boldsymbol{R}} A_N(\xi)\overline{A_N(\xi)} \, d\xi \\
&= \int_{\boldsymbol{R}} \rho_N(\xi)^2 |\xi|^{-1/2} \left( \int_0^{\infty} e^{i(s\xi - t(s)|\xi|^a)} s^{-\gamma} g(s) \, ds \right) \cdot \left( \int_0^{\infty} e^{-i(s'\xi - t(s')|\xi|^a)} s'^{-\gamma} \overline{g(s')} \, ds' \right) d\xi \\
&= \int_0^{\infty} \int_0^{\infty} \left( \int_{\boldsymbol{R}} e^{i[(s-s')\xi - (t(s)-t(s'))|\xi|^a]} \rho_N(\xi)^2 |\xi|^{-1/2} \, d\xi \right) s^{-\gamma} g(s) s'^{-\gamma} \overline{g(s')} \, ds \, ds'.
\end{aligned}$$

Applying Lemma 1 to the inner integral, we therefore obtain

$$(4) \qquad \|A_N\|_2^2 \le C \int_0^{\infty} \int_0^{\infty} |s - s'|^{-1/2} s^{-\gamma} |g(s)| s'^{-\gamma} |g(s')| \, ds \, ds'.$$

We set

$$K(s, s') = |s - s'|^{-1/2} s^{-\gamma} s'^{-\gamma},$$

then (4) follows that

$$\|A_N\|_2^2 \le C \iint_0^\infty K(s,s')|g(s)|\,|g(s')|\,ds\,ds'$$
$$\le C\int_0^\infty |g(s)|\left\{\int K(s,s')|g(s')|\,ds'\right\}ds$$
$$\le C\|g\|_p\|h\|_q$$

where

$$h(s) = Hg(s) = \int K(s,s')|g(s')|\,ds'.$$

We shall prove that $H$ is a bounded operator from $L^q(\boldsymbol{R})$ to $L^p(\boldsymbol{R})$. Here we set

$$Hg(s) = \int_0^\infty K(s,s')|g(s')|\,ds'$$
$$= \left\{\int_0^{s'/2} + \int_{s'/2}^{s'} + \int_{s'}^{2s'} + \int_{2s'}^\infty\right\} K(s,s')|g(s')|\,ds'$$
$$= H_1 + H_2 + H_3 + H_4.$$

Because of the symmetry in $K$, it is sufficient to prove the estimate of $H_1$ and $H_2$.

To study $H_1$ and $H_2$, we shall use Lemma 2. First we shall consider $H_1$. For $0 \le s \le s'/2$, we have $|s'-s| = s'-s \ge s'-s'/2 = s'/2$ and

$$K(s,s') \le \sqrt{2}s^{-\gamma}s'^{-\gamma-1/2}.$$

If we choose

$$K_1 = s^{-\lambda}s'^{\lambda-1/q}, \quad K_2 = s^{\lambda-\gamma}s'^{-\lambda-\gamma+1/q-1/2},$$

where $\lambda = 1/q - 1/8$, then we have

$$\int_0^{s'/2} K_1^q(s,s')\,ds = s'^{\lambda q-1}\int_0^{s'/2} s^{-\lambda q}\,ds$$
$$= Cs'^{\lambda q-1-\lambda q+1}$$
$$= C,$$

$$\int_{2s}^\infty K_2^q(s,s')\,ds' = s^{\lambda q-\gamma q}\int_{2s}^\infty s'^{-\lambda q-\gamma q+1-q/2}\,ds'$$
$$= Cs^{\lambda q-\gamma q-\lambda q-\gamma q+1-q/2+1}$$
$$= C.$$

Next we shall consider $H_2$. For $s'/2 \le s \le s'$, we set

$$K_1 = |s - s'|^{-1/4} s'^{-\gamma}, \quad K_2 = |s - s'|^{-1/4} s^{-\gamma}.$$

Then, we obtain

$$\begin{aligned}\int_{s'/2}^{s'} K_1^q(s, s')\, ds &= s'^{\gamma q} \int_{s'/2}^{s'} |s - s'|^{-q/4}\, ds \\ &= s'^{-\gamma q} \int_0^{s'/2} s^{-q/4}\, ds \\ &= Cs'^{-\gamma q - q/4 + 1} \\ &= C.\end{aligned}$$

Similarly, $\int_s^{2s} K_2^q(s, s')\, ds = 1$. Hence $\|h\|_q \le C\|g\|_p$, and therefore $\|A_N\|_2 \le C\|g\|_p$.

Finally, we estimate $Q(r)$. To do so we have to prove that the integral operator with kernel

$$K(r, s) = (1 + r^2)^{-1/8} \min(1, 1/rs) s^{-\gamma}$$

is bounded from $L^p(\boldsymbol{R})$ to $L^2(\boldsymbol{R})$. This can be done by decomposing $K(r, s)$ as

$$K(r, s) \le K_1(r, s) K_2(r, s)$$

with

(5) $$\sup_r \int_0^\infty K_1^2\, ds < \infty$$

and

(6) $$\sup_s \int_0^\infty K_2^q\, dr < \infty.$$

Here we choose $K_1$, $K_2$ as follows:

$$K_1 = \begin{cases} s^{-1/4} r^{1/4} & \text{if } s \le 1/r, \quad 0 < r \le 1, \\ s^{-1} r^{-1/2} & \text{if } s \ge 1/r, \quad 0 < r \le 1, \\ s^{-\gamma-1} r^{-1/8} & \text{if } s \le 1/r, \quad 1 < r < \infty, \\ s^{-\gamma-1/2} r^{-1/4} & \text{if } s \ge 1/r, \quad 1 < r < \infty, \end{cases}$$

and

$$K_2 = \begin{cases} s^{1/4-\gamma}r^{-1/4} & \text{if } s \le 1/r, \quad 0 < r \le 1, \\ s^{-\gamma}r^{-1/2} & \text{if } s \ge 1/r, \quad 0 < r \le 1, \\ sr^{-1/8} & \text{if } s \le 1/r, \quad 1 < r < \infty, \\ s^{-1/2}r^{-1} & \text{if } s \ge 1/r, \quad 1 < r < \infty. \end{cases}$$

In the proof of (5) we first consider the case $0 < r \le 1$,

$$\begin{aligned} \int_0^\infty K_1^2(r,s)\,ds &= \int_0^{1/r} K_1^2\,ds + \int_{1/r}^\infty K_1^2\,ds \\ &= \int_0^{1/r} s^{-1/2}r^{1/2}\,ds + \int_{1/r}^\infty s^{-2}r^{-1}\,ds \\ &= 2 + 1 < \infty. \end{aligned}$$

For the case $1 < r < \infty$,

$$\begin{aligned} \int_0^\infty K_1^2(r,s)\,ds &= \int_0^{1/r} K_1^2\,ds + \int_{1/r}^\infty K_1^2\,ds \\ &= \int_0^{1/r} s^{-2\gamma-2}r^{-1/4}\,ds + \int_{1/r}^\infty s^{-2\gamma-1}r^{-1/2}\,ds \\ &= Cr^{2\gamma-5/4} + Cr^{2\gamma-1/2} \le 2C, \end{aligned}$$

since $\gamma = 1/4 - 1/q$, $q \ge 2$.

Similarly, in the proof of (6) we first consider the case $0 < s \le 1$,

$$\begin{aligned} \int_0^\infty K_2^q(r,s)\,dr &= \int_0^1 K_2^q\,dr + \int_1^{1/s} K_2^q\,dr + \int_{1/s}^\infty K_2^q\,dr \\ &= \int_0^1 s^{(1/4-\gamma)q}r^{-q/4}\,dr + \int_1^{1/s} s^q r^{-q/8}\,dr + \int_{1/s}^\infty s^{-q/2}r^{-q}\,dr \\ &= Cs^{q/2-1} + C(s^{9q/8-1} - s^q) + Cs^{q/2-1} \le 3C. \end{aligned}$$

For the case $1 < s < \infty$,

$$\int_0^\infty K_2^q\,dr = \int_0^{1/s} K_2^q\,dr + \int_{1/s}^1 K_2^q\,dr + \int_1^\infty K_2^q\,dr$$

$$= \int_0^{1/s} s^{(1/4-\gamma)q} r^{-q/4}\,dr + \int_{1/s}^1 s^{-\gamma q} r^{-q/2}\,dr + \int_1^\infty s^{-q/2} r^{-q}\,dr$$

$$= \begin{cases} Cs^{3q/4-2} + C\log s/\sqrt{s} + Cs^{-q/2}, & \text{for } q = 2, \\ Cs^{3q/4-2} + C(s^{q/4-1} - s^{3q/4-2}) + Cs^{-q/2}, & \text{for } 2 < q \le 8/3, \end{cases}$$

is bounded.

Arguing as above, the integral operator with kernel $K(r,s) = (1+r^2)^{-1/8}\min(1,1/rs)s^{-\gamma}$ is bounded operator from $L^p(\boldsymbol{R})$ to $L^2(\boldsymbol{R})$ by Lemma 2 and thus estimate for $Q(r)$ is established. Therefore, the proof of Theorem 2 is complete.

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