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NON-c_i-SELF-DUAL QUATERNIONIC YANG-MILLS CONNECTIONS AND L_2-GAP THEORY

By

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1. Introduction

In the context with the 4-dimensional Yang-Mills theory, it would be of interest to study the Yang-Mills theory on several cases which appear naturally. From this point of view, Nitta ([12]), Mamone Capria and Salamon ([8]) developed Yang-Mills theory on quaternion-Kähler manifold and gave the notion of c_1- and c_2-self-dual connections which reasonably corresponds to the self-dual or anti-self-dual connections on 4-dimensional manifold ([2]).

In this note, we will give two properties for c_1- and c_2-self-dual connections on quaternion-Kähler manifolds; (i) the existence of quaternionic Yang-Mills connections which are neither c_1- nor c_2-connections, and (ii) the gap phenomena for quaternionic Yang-Mills connections by L_2-norm. These results seem natural consequence as higher dimensional analogues to 4-dimensional Yang-Mills theory.

There are remarkable results on the construction c_1- and c_2-self-dual connections by Kametani, Nagatomo and Nitta ([6], [9], [10], [11]). As a counter part of this result, we can consider the question whether there exist non-c_1- and c_2-self-dual connections on the compact quaternionic Kähler symmetric spaces, so called *Wolf spaces*. On the other hand, in 4-dimensional Yang-Mills theory, Itoh [3] found the non-self-dual Yang-Mills connections on S^4 and CP^2. The non-self-duality of the canonical invariant G-connections on S^4 and CP^2 requires the injectivity of the isotropy homomorphisms. Namely, if the isotropy group of base space is embedded into the structure group G, then the canonical connection is not (anti-) self-dual. Employing the ideas in [3] crusiously, we will give the existence of non-c_i-self-dual Yang-Mills connections in higher dimensions. Namely, we show that the canonical invariant connections on a homogeneous G-bundle with some structure group G on a Wolf space give the non-c_i-self-dual Yang-Mills connections. It is also the non-c_i-self-dual quaternionic Yang-Mills connections.

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Secondly, we will discuss on the gap phenomena for quaternionic Yang-Mills fields. This problem has been studied in [14] by using the pointwise norm (cf. [14]). Replacing the pointwise norm to the L_2-norm, we will show the gap phenomena again for quaternionic Yang-Mills fields. It can be also viewed as a higher-dimensional context to the 4-dimensional gap phenomena via L_2-norm for Yang-Mills fields (cf. [13]).

2. Preliminaries

A quaternion-Kähler manifold (M, g) is a Riemannian $4n$-manifold whose holonomy group is contained in $Sp(n) \cdot Sp(1)$, $n > 1$. In the case of $n = 1$, we add the assumption that (M, g) is Einstein and half-conformally flat. It is known that the bundle $\wedge^2 T^* M$ of 2-forms on a quaternion-Kähler manifold (M, g) has the following irreducible decomposition as a representation of $Sp(n) \cdot Sp(1)$ (cf. [8], [12]):

$$\wedge^2 T^* M = S^2 \boldsymbol{H} \oplus S^2 \boldsymbol{E} \oplus (S^2 \boldsymbol{H} \oplus S^2 \boldsymbol{E})^{\perp}, \tag{2.1}$$

where $\boldsymbol{H}$ and $\boldsymbol{E}$ are the vector bundles associated with the standard representations of $Sp(1)$ and $Sp(n)$, respectively. Let P be a principal bundle with a compact Lie group G as the structure group over a quaternion-Kähler manifold (M, g). Let $Ad(P) = P \times_{Ad} \mathfrak{g}$ be the vector bundle associated to P via the adjoint representation of G on its Lie algebra $\mathfrak{g}$. The curvature form F^{∇} on P descends to a 2-form on M with values in $Ad(P)$. Corresponding to the decomposition (2.1), we write the curvature F^{∇} as $F^{\nabla} = F^1 + F^2 + F^3$. A connection ∇ is said to be c_i-self-dual ($i = 1, 2$ or 3) if $F^j = 0$ for all $j \neq i$. Each c_i-self-dual connection is a Yang-Mills connection (cf. [8], [12], [2]). Moreover, if M is a compact, a c_1- or c_2-self-dual connection is characterized as a connection minimizing the Yang-Mills functional $YM(\nabla) = 1/2 \int_M |F^{\nabla}|^2 \, dv_g$.

DEFINITION 2.1 ([14]). *A connection ∇ on a principal G-bundle P over a compact quaternion-Kähler manifold (M, g) is called a quaternionic Yang-Mills connection if $\Delta^{\nabla} (F^{\nabla} \wedge \Omega^{n-1}) = 0$ where Ω is the fundamental 4-form on (M, g) and Δ^{∇} is the Laplacian on $Ad(P)$.*

Note that in the case of $n = 1$, the quaternionic Yang-Mills connections are Yang-Mills connections, and vice versa. Each c_i-self-dual connection is a quaternionic Yang-Mills connection. Moreover, a quaternionic Yang-Mills connection is a Yang-Mills connection (Proposition 1.1 in [14]).

Let $M = K/H$ be a compact oriented Riemannian homogeneous space with a reductive decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ and $P = (P, \pi, M, G)$ be a principal bundle such that elements of K acts on P as automorphisms i.e. $\Phi_k \circ \pi = \pi \circ \overline{\Phi}_k$ for all $k \in K$ and $\overline{\Phi}_k \circ R_g = R_g \circ \overline{\Phi}_k$ for all $k \in K$ and all $g \in G$ where $\overline{\Phi} : K \times P \to P$ is a left action, Φ is the induced action of K on M and R is the action of G by right translations on the fibers of P. Fix u_0 in P over $o = eH$ in M. The K-action induces the isotropy homomorphisms $\lambda : H \to G$ by $\overline{\Phi}_h(u_0) = R_{\lambda(h)}(u_0)$. A connection ω on P is called *invariant* if and only if $\overline{\Phi}_k^* \omega = \omega$ for all $k \in K$. We then obtain a one-to-one correspondence between the set of K-invariant connections ω on P and the set of linear maps $\Lambda : \mathfrak{m} \to \mathfrak{g}$ such that $\Lambda_{\mathfrak{m}} \circ ad_h = ad_{\lambda(h)} \circ \Lambda_{\mathfrak{m}}$ for any $h \in H$. The correspondence is given by $\Lambda(X) = \lambda(X)$ if $X \in \mathfrak{h}$ or $\Lambda(X) = \Lambda_{\mathfrak{m}}(X)$ if $X \in \mathfrak{m}$ and the invariant connection ω and curvature F^ω on P are then given by

$$\omega_{u_0}(\tilde{X}) = \Lambda(X), \quad X \in \mathfrak{k}$$

$$F^\omega_{u_0}(\tilde{X}, \tilde{Y}) = [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) - \lambda([X, Y]_{\mathfrak{h}}), \quad X, Y \in \mathfrak{m}$$

where $\tilde{X}$, $\tilde{Y}$ are the vector fields in P induced by X, Y. The K-invariant connection in P defined by $\Lambda_{\mathfrak{m}} \equiv 0$ is called the *canonical connection* according to the decomposition $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$. Its curvature satisfies $F^\omega_{u_0}(\tilde{X}, \tilde{Y}) = -1/2\lambda([X, Y]_{\mathfrak{h}})$ for X, $Y \in \mathfrak{m}$ (cf. [5]).

Compact quaternionic Kähler symmetric spaces were classified by Wolf [16], called *Wolf spaces*. Wolf spaces are quotients $M = K/H$ of a compact simple centerless Lie group K by a closed subgroup H with the splitting $H = L \cdot A$ where A is isomorphic to $Sp(1)$.

THEOREM 2.1 ([15]). *Let P be a K-homogeneous principal G-bundle over a Wolf space and λ be the corresponding isotropy homomorphism of H into G. For a canonical K-invariant connection ω on P,*

(1) *ω is a c_1-self-dual if and only if $\lambda|L = 0$,*

(2) *ω is a c_2-self-dual if and only if $\lambda|Sp(1) = 0$,*

(3) *ω is a c_3-self-dual if and only if $\lambda = 0$, in this case, P is trivial and ω is flat.*

3. Non-c_i-self-dual quaternionic Yang-Mills connections

THEOREM 3.1. *Let G be a classical Lie group $Sp(r)$, $SU(r)$ or $SO(r)$. Let r satisfy in the table below the inequality corresponding to a Wolf space $M = K/H$.*

Then, there exists a K-homogeneous G-bundle over $M = K/H$ whose canonical invariant connections is not c_i-self-dual, $i = 1, 2, 3$.

manifold	$Sp(r)$	$SU(r)$	$SO(r)$
$\boldsymbol{H}P^n$	$r \geq n+1$	$r \geq 2n+2$	$r \geq 4n$
$G_2(\boldsymbol{C}^{n+2})$	$r \geq n+1$	$r \geq n+2$ $(n = 1, 2)$	$r \geq 6$ $(n = 2)$
		$r \geq n+4$ $(n \geq 3)$	$r \geq 2n+3$ $(n \neq 2)$
$G_4(\boldsymbol{R}^{n+4})$	$r \geq 2$ $(n = 1)$	$r \geq 4$ $(n = 1, 2)$	$r \geq n+4$
	$r \geq 3$ $(n = 2)$	$r \geq n+4$ $(n \geq 3)$	
	$r \geq n+2$ $(n \geq 3)$		
$G_2/(SU(2) \cdot Sp(1))$	$r \geq 2$	$r \geq 4$	$r \geq 4$
$F_4/(Sp(3) \cdot Sp(1))$	$r \geq 4$	$r \geq 8$	$r \geq 15$
$(E_6/Z_3)/(SU(6) \cdot Sp(1))$	$r \geq 7$	$r \geq 8$	$r \geq 15$
$E_7/(Spin(12) \cdot Sp(1))$	$r \geq 13$	$r \geq 14$	$r \geq 15$
$E_8/(E_7 \cdot Sp(1))$	$r \geq 57$	$r \geq 59$	$r \geq 115$

PROOF. In general, the canonical invariant connections on a homogeneous G-bundle on a compact symmetric space has parallel curvature i.e. $\nabla_i F^{\nabla}_{jk} = 0$ for any i, j, k ([3], [5]) and hence it gives a quaternionic Yang-Mills connection i.e. $\nabla_i F^{\nabla}_{ij} = 0$ for any i, j (Proposition 1.1 in [14]). It is also a Yang-Mills connection i.e. $\sum_i \nabla_i F^{\nabla}_{ij} = 0$ for any j. From Theorem 2.1 ([15]), if $\mathfrak{h}$ is embedded into $\mathfrak{g}$ by a homomorphism λ, then the λ induces as the isotropy representation a K-homogeneous G-bundle over $M = K/H$ whose canonical invariant connection is not c_i-self-dual. Hence, with respect to given $\mathfrak{h}$, we may find such the Lie algebra $\mathfrak{g}$. Elementary embeddings between Lie algebras are known as the following.

$$(3.1)\qquad \begin{cases} \mathfrak{sp}(r) \hookrightarrow \mathfrak{su}(2r) \hookrightarrow \mathfrak{u}(2r) \hookrightarrow \mathfrak{so}(4r), \\ \mathfrak{so}(r) \hookrightarrow \mathfrak{su}(r) \hookrightarrow \mathfrak{u}(r) \hookrightarrow \mathfrak{sp}(r), \\ \mathfrak{sp}(1) \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3), \quad \mathfrak{sp}(2) \simeq \mathfrak{so}(5), \quad \mathfrak{su}(4) \simeq \mathfrak{so}(6), \\ \mathfrak{u}(1) \simeq \mathfrak{so}(2) \simeq \boldsymbol{R}, \quad spin(n) \simeq \mathfrak{so}(n), \quad \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3). \end{cases}$$

Note that

$$\boldsymbol{H}P^1 = G_4(\boldsymbol{R}^5) = S^4, \quad G_2(\boldsymbol{C}^3) = \boldsymbol{C}P^2, \quad G_2(\boldsymbol{C}^4) = G_4(\boldsymbol{R}^6).$$

$\boldsymbol{HP}^n = (Sp(n+1)/\boldsymbol{Z}_2)/(Sp(n)\cdot Sp(1))$:
$\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)\ni(x,y)\mapsto\lambda(x,y)\in\mathfrak{sp}(n+1)$ defined by $\lambda(x,y):=\mathrm{diag}(x,y)$. For $N>n+1$, we defined by $\lambda(x,y):=\mathrm{diag}(x,y,0)$. Using (3.1), we see that

$$\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)\hookrightarrow\mathfrak{su}(2r)\oplus\mathfrak{su}(2)\hookrightarrow\mathfrak{su}(2n+2).$$

Hence we get $r\geq 2n+2$ for $SU(r)$. Since $\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)\ni(x,y)\mapsto\lambda(x,y)\in\mathfrak{so}(4n)$ defined by $\lambda(x,y)v:=xv-vy$, $v\in\boldsymbol{R}^{4n}$, we have $r\geq 4n$ for $SO(r)$.

$G_2(\boldsymbol{C}^{n+2})=(SU(n+2)/\boldsymbol{Z}_{n+2})/U(n)\cdot Sp(1)$:
Using (3.1), we have $\mathfrak{u}(n)\oplus\mathfrak{sp}(1)\hookrightarrow\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)\hookrightarrow\mathfrak{sp}(n+1)$ for any n. Using (3.1), we also have $\mathfrak{u}(n)\oplus\mathfrak{sp}(1)\hookrightarrow\mathfrak{so}(2n)\oplus\mathfrak{so}(3)\hookrightarrow\mathfrak{so}(2n+3)$ for any $n\neq 2$. In the case of $n=2$, $\mathfrak{u}(2)\oplus\mathfrak{sp}(1)\simeq\boldsymbol{R}\oplus\mathfrak{su}(2)\oplus\mathfrak{sp}(1)\simeq\mathfrak{so}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{sp}(1)\simeq\mathfrak{so}(2)\oplus\mathfrak{so}(3)\oplus\mathfrak{so}(3)\simeq\mathfrak{so}(2)\oplus\mathfrak{so}(4)\hookrightarrow\mathfrak{so}(6)$. When $n=1$, it has shown by Itoh [3]. Using (3.1), we get $\mathfrak{u}(n)\oplus\mathfrak{sp}(1)\simeq\boldsymbol{R}\oplus\mathfrak{su}(n)\oplus\mathfrak{su}(2)\hookrightarrow\mathfrak{su}(2)\oplus\mathfrak{su}(n)\oplus\mathfrak{su}(2)\hookrightarrow\mathfrak{su}(n+4)$ for any $n\geq 3$. In the case of $n=2$, $\mathfrak{u}(2)\oplus\mathfrak{sp}(1)\simeq\boldsymbol{R}\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)\simeq\mathfrak{so}(2)\oplus\mathfrak{so}(4)\hookrightarrow\mathfrak{so}(6)\simeq\mathfrak{su}(4)$. When $n=1$, it has shown by Itoh [3].

$E_8/(E_7\cdot SP(1))$:
For the wolf space $E_8/(E_7\cdot Sp(1))$ we use the fact that E_7 is closed subgroup of $U(56)$ (cf. [17]) and $\mathfrak{u}(n)\hookrightarrow\mathfrak{su}(n+1)$.

The same argument can be applied to the others. □

By generalizing the argument in Itoh [3, Theorem 3], we have the following.

Lemma 3.1. *Let P be a $Sp(n+1)$-homogeneous G-bundle over $\boldsymbol{HP}^n$ induced by an injective isotropy homomorphism λ of H into G. Then the canonical $Sp(n+1)$-invariant connection ω is not weakly stable.*

Proof. The curvature tensor of $\boldsymbol{HP}^n$ with quaternionic sectional curvature 4 is defined by

$$R(X,Y)=X\wedge Y+\sum_{\alpha=1}^{3}J_\alpha X\wedge J_\alpha Y-2\sum_{\alpha=1}^{3}\langle J_\alpha X,Y\rangle J_\alpha. \tag{3.2}$$

We fix a Λ in $\mathrm{Hom}_H(\mathfrak{m},\mathfrak{g})$. Since $\Lambda\circ\mathrm{ad}_h=\mathrm{ad}_{\lambda(h)}\circ\Lambda$ for any $h\in H$, the $\mathrm{Ad}(P)$-valued 1-form A induced by Λ is parallel, $\delta^\omega A=d^\omega A=0$. Then $\omega_t=\omega+tA$ gives a deformation of ω. Since F^{ω_t} is invariant under K, $|F^{\omega_t}|^2$ is constant. Thus, we have the following:

$$\frac{1}{2}\frac{d^2}{dt^2}\int_{\boldsymbol{HP}^n}|F^{\omega_t}|^2\,dv|_{t=0}=\mathrm{vol}(\boldsymbol{HP}^n)\langle F^\omega,[\Lambda,\Lambda]\rangle$$

for a deformation ω_t with $(d/dt)\omega_t|_{t=0} = A$. Using (3.2) and the same argument in Theorem 3 in [3], we have

$$\langle F^{\omega}, [\Lambda, \Lambda]\rangle = -n \sum_j |\Lambda(e_j)|^2,$$

where $\{e_j\}_{j=1,2,\dots,4n}$ is the orthonormal basis of $\mathfrak{m}$. Thus, if $\Lambda \neq 0$, then $(1/2)(d^2/dt^2)\int_{HP^n} |F^{\omega_t}|^2\, dv|_{t=0} < 0$. Therefore ω is not weakly stable. □

4. Gap phenomena for quaternionic Yang-Mills fields

Let (M, g) be a compact quaternion-Kähler manifold. The Riemannian curvature operator R acting on $\wedge^2 TM$ has a splitting $R = R_1 + R_2 + R_3$ with respect to the decomposition (2.1). By using the result in [7] we can write the curvature operator R_i as $R_i = \mu_i I_{\wedge^2 TM}$ where μ_i $(i = 1$ or $2)$ is a positive constant. Since R_3 is negative semi-definite, we put $\mu_3 = 0$. We set $\lambda_i = s/2n - 2\mu_i$ $(i = 1, 2$ or $3)$ where s is the scalar curvature of (M, g).

THEOREM 4.1. *Let ∇ be a quaternionic Yang-Mills connection over a compact quaternion-Kähler manifold (M, g). Assume $F^3 = 0$.*

(1) *There exists a constant*

$$\varepsilon_1 = \frac{n+2}{3} \min\left\{\frac{(2n-1)^2 s^2 V}{8(4n-1)^2}, \frac{1}{2}\left(\frac{s}{2n} - 2\mu_1\right)^2 V\right\}$$

such that

$$k < 0, \quad YM(\nabla) \leq 4\pi^2 c_2 k + \varepsilon_1 \Rightarrow F^1 \equiv 0.$$

(2) *There exists a constant*

$$\varepsilon_2 = \frac{n+2}{2n+1} \min\left\{\frac{(2n-1)^2 s^2 V}{8(4n-1)^2}, \frac{1}{2}\left(\frac{s}{2n} - 2\mu_2\right)^2 V\right\}$$

such that

$$k > 0, \quad YM(\nabla) \leq 4\pi^2 c_1 k + \varepsilon_2 \Rightarrow F^2 \equiv 0.$$

Where $k = -1/(8\pi^2)\int_M tr(F^{\nabla} \wedge F^{\nabla}) \wedge \Omega^{n-1}$, $c_1 = 6n/(2n+1)!$, $c_2 = -1/(2n-1)!$.

PROOF. We will write the Bochner-Weitzenböck formula for any $\mathfrak{g}$-valued 2-forms ϕ (cf. [14, [1]).

$$(4.1) \qquad \langle \Delta^{\nabla}\phi, \phi\rangle - \langle \nabla^*\nabla\phi, \phi\rangle = \left\langle \phi \circ \left(\frac{s}{2n} I - 2R\right), \phi\right\rangle - \langle [F^{\nabla}, \phi], \phi\rangle.$$

For convenience we put $A = (c_1 - c_2)/c_1$ and $\phi = AF^1$. Substituting $\phi = AF^1$ into (4.1) and using $F^3 = 0$, $[F^2, F^1] = 0$ (cf. Proposition 3.3 in [14]), we have

$$\langle \Delta^\nabla F^1, F^1 \rangle - \langle \nabla^* \nabla F^1, F^1 \rangle = \lambda_1 |F^1|^2 - \langle [F^1, F^1], F^1 \rangle, \tag{4.2}$$

where $(s/2nI - 2R_1)_{X,Y} = (s/2n) X \wedge Y - 2R_1(X \wedge Y) = (s/2n - 2\mu_1) X \wedge Y$, X, $Y \in T_xM$. Hence we put $\lambda_1 = s/2n - 2\mu_1$. Note that $\Delta^\nabla(F^\nabla \wedge \Omega^{n+1}) = 0$ and $F^3 = 0$ hold if and only if $\Delta^\nabla F^1 = 0$ (see Proposition 3.1 in [14]). Using the Kato's inequality $\int |\nabla F^1| \geq \int |d|F^1||$, $|[F^1, F^1]| \leq \sqrt{2}|F^1| \cdot |F^1|$ (cf. [14], [1], [13]) and integrating over the compact quaternion-Kähler manifold M, we obtain the inequality

$$\int \langle \Delta^\nabla F^1, F^1 \rangle \geq \int |d|F^1||^2 + \lambda_1 \int |F^1|^2 - \sqrt{2} \int |F^1| \cdot |F^1|. \tag{4.3}$$

To get the L_{2n}-estimates we use the following Sobolev inequality due to [4] for the case $\dim M = 4n$:

$$\|\varphi\|^2_{4n/2n-1} \leq \frac{2(4n-1)}{(2n-1)sV^{1/2n}} \|d\,|\varphi|\,\|_2^2 + V^{-1/(2n)} \|\varphi\|_2^2 \tag{4.4}$$

holding for all functions $\varphi \in C^\infty(M)$ where V is the volume of M, s is the scalar curvature and $\|\cdot\|_p$ denotes the L_p-norm. We now apply the Hölder's inequality to the integrand of the last term on the right hand side of (4.3) to get:

$$\int \langle \Delta^\nabla F^1, F^1 \rangle \geq \int |d\,|F^1||^2 + \lambda_1 \int |F^1|^2 - \sqrt{2} \|F^1\|_{2n} \cdot \|F^1\|^2_{4n/2n-1}. \tag{4.5}$$

Applying the Sobolev inequality (4.4) to the first term on the right hand side of (4.3), we have

$$\begin{aligned} \int \langle \Delta^\nabla F^1, F^1 \rangle \geq & \left(\lambda_1 - \frac{(2n-1)s}{2(4n-1)} \right) \|F^1\|_2^2 \\ & + \left(\frac{(2n-1)s}{2(4n-1)} V^{1/(2n)} - \sqrt{2} \|F^1\|_{2n} \right) \|F^1\|^2_{4n/2n-1}. \end{aligned} \tag{4.6}$$

In the case of $\lambda_1 - (2n-1)s/2(4n-1) > 0$, if we take $\|F^1\|_{2n} < (2n-1)s/(2\sqrt{2}(4n-1))V^{1/(2n)}$ from (4.6), then we conclude that $F^1 \equiv 0$. In the case of $\lambda_1 - (2n-1)s/2(4n-1) \leq 0$, we use (4.6) together with the following inequality which is obtained immediately from (4.5):

$$\int \langle \Delta^\nabla F^1, F^1 \rangle \geq \lambda_1 \|F^1\|_2^2 - \sqrt{2} \|F^1\|_{2n} \cdot \|F^1\|^2_{4n/2n-1}. \tag{4.7}$$

In fact, if $\|F^1\|_{2n} \le 1/(\sqrt{2})\lambda_1 V^{1/(2n)}$, then (4.7) implies

$$\int \langle \Delta^\nabla F^1, F^1 \rangle \ge \lambda_1 \|F^1\|_2^2 - \lambda_1 V^{1/(2n)} \|F^1\|_{4n/2n-1}^2 \tag{4.8}$$

which is positive if $\|F^1\|_2^2 - V^{1/(2n)}\|F^1\|_{4n/2n-1}^2 \ge 0$. On the other hand, if $\|F^1\|_{2n} \le 1/(\sqrt{2})\lambda_1 V^{1/(2n)}$, then we get by (4.6)

$$\int \langle \Delta^\nabla F^1, F^1 \rangle \ge \left(\lambda_1 - \frac{(2n-1)s}{2(4n-1)}\right)(\|F^1\|_2^2 - V^{1/(2n)}\|F^1\|_{4n/2n-1}^2) \tag{4.9}$$

which is positive if $\|F^1\|_2^2 - V^{1/(2n)}\|F^1\|_{4n/2n-1}^2 \le 0$, since we are in the case where $\lambda_1 - (2n-1)s/2(4n-1) \le 0$. If we take

$$\delta = \min\left\{\frac{(2n-1)s}{2\sqrt{2}(4n-1)} V^{1/(2n)}, \frac{1}{\sqrt{2}}\lambda_1 V^{1/(2n)}\right\},$$

we have $F^1 \equiv 0$. Namely, if $\|F^1\|_{2n} \le \delta$, then, from (4.8) and (4.9), we conclude that $F^1 \equiv 0$.

Applying the Hölder inequality, we have

$$\|F^1\|_2 \le \|F^1\|_{2n} \cdot V^{(n-1)/(2n)}.$$

Therefore, by using $\|F^1\|_{2n}^2 \le \delta^2$, we get

$$\|F^1\|_2^2 \le \delta^2 \cdot V^{(n-1)/n}. \tag{4.10}$$

On the other hand, from [2]

$$2YM(\nabla) = 8\pi^2 c_2 k + \frac{c_1 - c_2}{c_1}\|F^1\|_2^2 + \frac{c_3 - c_2}{c_3}\|F^3\|_2^2.$$

Using (4.10) and $F^3 \equiv 0$, we obtain

$$YM(\nabla) \le 4\pi^2 c_2 k + \frac{c_1 - c_2}{2c_1}\delta^2 V^{(n-1)/n}$$

Hence, according to take ε_1 as follows:

$$\varepsilon_1 = \frac{n+2}{3}\min\left\{\frac{(2n-1)^2 s^2}{8(4n-1)^2} V, \frac{1}{2}\left(\frac{s}{2n} - 2\mu_1\right)^2 V\right\},$$

if it satisfies $YM(\nabla) = 4\pi^2 c_2 k + \varepsilon_1$, then $F^1 \equiv 0$. We complete the proof of (1) of Theorem 4.1. The same argument can be applied to (2) of Theorem 4.1. □

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