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ON UNIFORM WELL-POSEDNESS OF THE ABSTRACT CAUCHY PROBLEM

By

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§1. Introduction

We are interested in Duhamel's principle:

$$U(t,x) = U_H(t,x;s,\Phi) + \int_s^t U_H(t,x;\tau,F(\tau,x))\,d\tau. \tag{D}$$

Here $U(t,x)$ denotes a solution of the inhomogeneous Cauchy problem $[CP]_s$ $(0 \le s < T)$ for the system of linear differential equations:

$$\begin{cases} L(t,x;\partial_t,\partial_x)U(t,x) = F(t,x), t \in [s,T], x \in \boldsymbol{R}^n, \\ U(s,x) = \Phi(x) \end{cases}$$

and $U_H(t,x;s,\Phi)$ stands for a solution of the homogeneous Cauchy problem $[HCP]_s$ $(0 \le s < T)$ for the system of linear differential equations:

$$\begin{cases} L(t,x;\partial_t,\partial_x)U(t,x) = 0, t \in [s,T], x \in \boldsymbol{R}^n, \\ U(s,x) = \Phi(x), \end{cases}$$

where $L(t,x;\partial_t,\partial_x) = \partial_t I - A(t,x;\partial_x)$, I is the identity matrix of order n and $A(t,x;\partial_x)$ is a square matrix of order n.

When L is an ordinary differential operator, that is, $A(t,x;\partial_x)$ depends only on t, write $L = \partial_t I - A(t)$, the formula (D) becomes the constant variation formula of Lagrange

$$U(t) = \varphi(t)\varphi(s)^{-1}\Phi + \int_s^t \varphi(t)\varphi(\tau)^{-1}F(\tau)\,d\tau,$$

where $\varphi(t)$ is the fundamental matrix to $[HCP]_s$. In particular, if $A(t)$ is in-

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dependent of t, say $A(t) \equiv A$, then the formula above turns into

$$U(t) = \exp((t-s)A)\Phi + \int_s^t \exp((t-\tau)A)F(\tau)\,d\tau.$$

This fact remains available for a bounded linear operator A in any Banach space but, in general, does not for an unbounded linear operator A.

While all coefficients of the differential operator L depend only on t, I. G. Petrowsky [8] and L. Schwartz [9] examined the Cauchy problems $[CP]_s$, $[HCP]_s$ in $\mathscr{B}$ and $\mathscr{S}'$, respectively, where $\mathscr{B}$ is the set of infinitely differentiable functions on $\boldsymbol{R}^n$ whose derivatives of every order are all bounded and $\mathscr{S}'$ the dual space of the Schwartz space of rapidly decreasing functions on $\boldsymbol{R}^n$. They showed that the formula (D) holds if $[HCP]_s$ $(0 \le s < T)$ is uniformly well-posed. Then, especially, $[CP]_s$ is well-posed for any $s \in [0, T)$. As concerns this assertion, L. Schwartz [9] presumed that $[HCP]_s$ $(0 \le s < T)$ is generally not uniformly well-posed even though it is well-posed, and K. Kitagawa [5] gave such a concrete example.

In this paper we consider the relation between well-posedness of the inhomogeneous Cauchy problem $(CP)_s$ $(0 \le s < T)$ in a locally convex space X:

$$\begin{cases} (\partial_t - A(t))U(t) = F(t), t \in [s, T], \\ \qquad\qquad U(s) = \Phi \in X \end{cases}$$

and (uniform) well-posedness of the homogeneous Cauchy problem $(HCP)_s$ $(0 \le s < T)$ in X:

$$\begin{cases} (\partial_t - A(t))U(t) = 0, t \in [s, T], \\ \qquad\qquad U(s) = \Phi \in X. \end{cases}$$

Also we mention how it is concerned with Duhamel's principle.

DEFINITION 1. We call that a strict inductive limit F, say $F = \varinjlim F_n$, admits a partition of unity if there exist continuous linear maps $f_n : F \to F_n$ for $n = 1, 2, \ldots$ which satisfy the two conditions:

(i) For each l, $f_n \circ i_l = 0$ except for a finite number of $n = 1, 2, \ldots$.
(ii) $\sum_{n=1}^{\infty} i_n \circ f_n$ is the identity map in F, where $i_n : F_n \to F$ is the inclusion map.

ASSUMPTIONS. We suppose that X is a $\mathscr{LF}$-space, *i.e.*, X is the strict inductive limit of a family $\{X_n\}_{n=1}^{\infty}$ of increasing Fréchet spaces such that X_n is a closed proper vector subspace of X_{n+1} and that X admits a partition of unity (*e.g.*, $\mathscr{D}$, the space of infinitely differentiable functions on $\boldsymbol{R}^n$ with compact supports, see Example 8.4.2 (b) in P. P. Carreras-J. Bonet [1]). Here $X = \bigcup_{n=1}^{\infty} X_n$ is equipped with the finest locally convex topology such that the inclusion map $i_n : X_n \to X$ is continuous for all $n = 1, 2, \ldots$. The facts below are well-known (for instance, see pp. 288–292 in L. Narici-E. Beckenstein [7]):

(1) G is an open subset of X if and only if $G \cap X_n$ is open in X_n for all $n = 1, 2, \ldots$.

(2) X_n is a close vector subspace of X for $n = 1, 2, \ldots$.

(3) X is a Hausdorff space.

(4) X is non-metrizable.

(5) X is complete as uniform topological space.

$\mathscr{L}(X)$ denotes the set of continuous linear operators from X into itself and let $C^n([s, T], X)$ $(0 \leq n \leq \infty)$ be the set of n-times continuously differentiable functions defined on $[s, T]$ with values in X.

For a fixed integer m (≥ 1), we impose the following conditions:

(a) $A \in C^m([0, T], \mathscr{L}(X))$.

(b) $\{\partial_t^n A(t); t \in [0, T]\}$ $(n = 0, 1, \ldots, m)$ is equi-continuous, *i.e.*, for any continuous seminorm p on X there exists a continuous seminorm q on X such that

$$\sup_{t \in [0,T]} p(\partial_t^n A(t)x) \leq q(x)$$

holds for all $x \in X$.

Definition 2. (i) We say that $(CP)_s$ is m-wellposed if for any $F \in C^{m-1}([s, T], X)$ and $\Phi \in X$ there exists a unique solution $U(t; s, F, \Phi) \in C^m([s, T], X)$ of $(CP)_s$ with data (F, Φ).

(ii) We say that $(HCP)_s$ is m-wellposed if for any $\Phi \in X$ there exists a unique solution $U_H(t; s, \Phi) \in C^m([s, T], X)$ of $(HCP)_s$ with initial data Φ.

(iii) We say that $(HCP)_s$ $(0 \leq s < T)$ is uniformly m-wellposed if for each $s \in [0, T) (HCP)_s$ is m-wellposed and if the subset $\{X \ni \Phi \mapsto U_H(t; s, \Phi) \in X; (t, s) \in \Omega\}$ is equi-continuous in $\mathscr{L}(X)$, *i.e.*, for any continuous seminorm p on X there exists a continuous seminorm q on X such that

$$\sup_{(t,s) \in \Omega} p(U_H(t; s, \Phi)) \leq q(\Phi)$$

holds for all $\Phi \in X$, where $\Omega = \{(t, s) \in \boldsymbol{R}^2; 0 \leq s \leq t \leq T, s < T\}$.

Then we have our main result.

THEOREM 1. *The following statements are mutually equivalent.*
(i) $(CP)_s$ *is m-wellposed for any* $s \in [0, T)$.
(ii) $(HCP)_s$ $(0 \leq s < T)$ *is uniformly m-wellposed.*
(iii) $(HCP)_s$ *is m-wellposed for any* $s \in [0, T)$ *and Duhamel's principle*:

$$U(t; s, F, \Phi) = U_H(t; s, \Phi) + \int_s^t U_H(t; \tau, F(\tau))\, d\tau$$

holds, i.e., $U_H(t; \tau, F(\tau))$ *is continuous in* $\tau \in [s, t]$ *and*

$$U_H(t; s, \Phi) + \int_s^t U_H(t; \tau, F(\tau))\, d\tau$$

is a solution of $(CP)_s$ *with data* (F, Φ).

In K. Kitagawa [5], he showed Theorem 1 for any Fréchet space. Evidently (iii) implies (i), and (iii) easily follows from (ii). Thus it is substantial to prove that (ii) is deduced from (i). In supposing (i), we make use of the closed graph theorem of Dieudonné-Schwartz in order to verify the equi-continuity of the family $\{P(t, s); (t, s) \in \Omega\}$ of the linear maps $P(t, s)$ which send Φ to $U_H(t, s; \Phi)$. Then we have to check that $C^m([0, T], X)$ is a $\mathscr{LF}$-space. So as to do it, we will take advantage of the ε-tensor product of Grothendieck.

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§2. Preliminaries

We will prepare some notions to observe that $C^m([0, T], X)$ is a $\mathscr{LF}$-space. According to F. Treves [10], we first explain a tensor product of (infinite-dimensional) vector spaces. Let E, F be two vector spaces and φ a bilinear map of $E \times F$ into a third vector space G.

DEFINITION 3. We say that E and F are φ-linearly disjoint if the following property holds: Suppose that $\{x_1, x_2, \ldots, x_r\}$ and $\{y_1, y_2, \ldots, y_r\}$ are finite subsets of E and F, respectively, satisfying the relation

$$\sum_{j=1}^{r} \varphi(x_j, y_j) = 0.$$

Then, if $x_1, x_2, \dots, x_r$ are linearly independent, then $y_1 = y_2 = \cdots = y_r = 0$, and if $y_1, y_2, \dots, y_r$ are linearly independent, then $x_1 = x_2 = \cdots = x_r = 0$.

Definition 4. A tensor product of E and F is a pair (G, φ) consisting of a vector space G and of a bilinear map φ of $E \times F$ into G such that the following conditions are satisfied:

(i) The image $\varphi(E \times F)$ of $E \times F$ spans the whole space G.

(ii) E and F are φ-linearly disjoint.

The theorem below states the unique existence of a tensor product up to algebraic isomorphism.

Theorem 2 (Theorem 39.1 in F. Treves [10]). *Let E, F be two vector spaces.*

(i) *There is a tensor product of E and F.*

(ii) *If (G, φ) and (H, ψ) are two tensor products of E and F, there exists a one-to-one linear map u of G onto H with $u \circ \varphi = \psi$.*

Let $\mathbf{B}(E, F; G)$ stand for the space of continuous bilinear maps of $E \times F$ into G. When G is the scalar field ($\boldsymbol{R}$ or $\boldsymbol{C}$), we briefly write $\mathbf{B}(E, F)$ instead of $\mathbf{B}(E, F; G)$.

Now we note that there is a canonical bilinear map ϕ of $E \times F$ into $\mathbf{B}(E'_\sigma, F'_\sigma)$:

$$(x, y) \mapsto \phi(x, y) : (x', y') \mapsto \langle x, x' \rangle \langle y, y' \rangle.$$

Moreover, it follows that E and F are ϕ-linearly disjoint and that $\{\phi(x, y); (x, y) \in E \times F\}$ spans $\mathbf{B}(E'_\sigma, F'_\sigma)$ (see pp. 431–432 in F. Treves [10]). Hence we arrived at the next claim.

Proposition 1 (Proposition 42.4 in F. Treves [10]). *$\mathbf{B}(E'_\sigma, F'_\sigma)$ is a tensor product of E and F.*

Hereafter $\mathbf{B}(E'_\sigma, F'_\sigma)$ is denoted by $E \otimes F$.

By means of the proposition above, we may secondly introduce a topological tensor product, namely the ε-tensor product of Grothendieck.

Definition 5. We call ε-topology on $E \otimes F$ the topology brought over from $\mathbf{B}(E'_\sigma, F'_\sigma)$ in regarding the latter as a vector subspace of $B_\varepsilon(E'_\sigma, F'_\sigma)$, the space of separately continuous bilinear forms on $E'_\sigma \times F'_\sigma$ provided with the topology of uniform convergence on the product of an equi-continuous subset of E' and an equi-continuous subset of F'. Equipped with the ε-topology, the space $E \otimes F$ is written by $E \otimes_\varepsilon F$ and it is said the ε-tensor product of E and F. We also denote by $E \tilde{\otimes}_\varepsilon F$ the completion of $E \otimes_\varepsilon F$.

Next we will center on $C^m(Y, Z)$, the space of m-times continuously differentiable functions of Y into Z, where Y is an open subset of $\boldsymbol{R}^n$ and Z a locally convex space. The space $C^m(Y, Z)$ is endowed with the topology of uniform convergence of the functions together with their derivatives of order not greater than m on every compact subset of Y. Then we obtain the following representation theorem.

Theorem 3 (Theorem 44.1 in F. Treves [10]). *If Z is complete, then the topological isomorphism*

$$C^m(Y, Z) \cong C^m(Y) \tilde{\otimes}_\varepsilon Z$$

holds.

In addition, since we want to adopt this theorem for a strict inductive limit Z, it is significant to ensure the interchangeability of ε-tensor product and (strict) inductive limit. This theme has been researched by R. Hollstein [3] at the end of the 1970's.

Definition 6. A locally convex space E is said to have the countable neighbourhood property if for every sequence $\{U_n\}_{n=1}^\infty$ of zero-neighbourhoods of E there are positive constants a_n such that the intersection $\bigcap_{n=1}^\infty a_n U_n$ is a zero-neighbourhood in E.

Any normable vector space obviously has the countable neighbourhood property.

The next theorem assures the interchangeability of ε-tensor product and strict inductive limit under additional hypotheses.

Theorem 4 (Proposition 4.5 in R. Hollstein [3]). *Let E be a locally convex space and $F = \varinjlim F_n$ a strict inductive limit of locally convex spaces F_n. Then the inductive limit $\varinjlim (E \tilde{\otimes}_\varepsilon F_n)$ is strict. Further, if E has the countable neighbourhood property and if F admits a partition of unity, then we get the topological* *isomorphism*

$$E \tilde{\otimes}_\varepsilon \left(\varinjlim F_n\right) \cong \varinjlim (E \tilde{\otimes}_\varepsilon F_n).$$

Thus we have concluded the following fruit.

THEOREM 5. *$C^m([0,T],X)$ is a $\mathscr{L}\mathscr{F}$-space.*

PROOF. This immediately follows from Theorems 3 and 4. □

Finally, so as to use in the next section, we quote Dini's theorem and the closed graph theorem of Dieudonné and Schwartz.

THEOREM 6 (U. Dini). *Let E be a complete locally convex space. If a continuous function $f : [a,b] \to E$ has the continuous right derivative $\partial_t^+ f(t)$ in (a,b), then $f(t)$ is continuously differentiable in (a,b).*

PROOF. Fixing $c \in (a,b)$, we pick

$$g(t) = \int_c^t \partial_s^+ f(s)\, ds + f(c)$$

for $t \in (a,b)$. Then it easily follows that $g(t)$ is continuously differentiable in (a,b) and that $\partial_t g(t) = \partial_t^+ f(t)$ for every $t \in (a,b)$. Thus, if we set $h(t) = f(t) - g(t)$ for $t \in (a,b)$, then $h(t)$ is continuous on (a,b) and $\partial_t^+ h(t) = 0$ for all $t \in (a,b)$. Now we will observe that

$$p(h(t)) \le \varepsilon(t-c)$$

for any $\varepsilon > 0$ and $t \in (c,b)$, where p is a continuous seminorm on E. If this assertion was established, then as $\varepsilon \downarrow 0$ $p(h(t)) = 0$ for all $t \in (c,b)$. Since p is arbitrary, we know that $h(t) = 0$, *i.e.*, $f(t) = g(t)$ for every $t \in (c,b)$. This completes the proof.

At first, from $\partial_t^+ h(c) = 0$ there exists a positive constant δ_1 such that

$$p(h(t)) \le \varepsilon(t-c)$$

as long as $c < t < c + \delta_1 < b$. So, put

$$b' = \sup\{t \in [c,b); p(h(t)) \le \varepsilon(t-c)\}.$$

Then we see that $b' = b$. Indeed, if $b' < b$, then because of the continuity of $h(t)$

on (a,b)

$$p(h(b')) \leq \varepsilon(b' - c)$$

holds. By virtue of $\partial_t^+ h(b') = 0$, there is some positive constant δ_2 such that

$$p(h(b' + t)) \leq p(h(b')) + \varepsilon t$$

provided $0 < t < \delta_2$. It follows from this that

$$p(h(b' + t)) < \varepsilon(b' - c) + \varepsilon\delta_2 = \varepsilon(b' + \delta_2 - c)$$

for $t \in (0, \delta_2)$. Again, recalling the continuity of $h(t)$ on (a, b), we notice that

$$p(h(b' + \delta_2)) \leq \varepsilon(b' + \delta_2 - c)$$

as $t \uparrow \delta_2$. This contradicts the definition of b'. Therefore we have $b' = b$. □

THEOREM 7 (J. Dieudonné-L. Schwartz [2]). *Let E, F be two $\mathscr{LF}$-spaces and P a closed linear map of E into F with domain E. Then the map P is continuous.*

§3. Proof of Theorem 1.

First of all, we will begin to show that (ii) is derived from (i). Let $U(t; s, \Phi) \in C^m([s, T], X)$ be the solution of $(HCP)_s$ with initial data Φ. We set $A_n(s)\Phi = \partial_t^n U(t; s, \Phi)|_{t=s} = \Phi_n(s)$ for $n = 0, 1, \dots, m$. Let us express $U(t; s, \Phi)$ in terms of the solution to $(CP)_0$. For this aim, we define the function $U^0(t; s, \Phi) \in C^\infty(\boldsymbol{R}, X)$ by

$$U^0(t; s, \Phi) = \sum_{n=0}^{m} G_n(t; s, \Phi) = \sum_{n=0}^{m} \frac{\Phi_n(s)}{n!}(t - s)^n.$$

Denoting

$$V^0(t; s, \Phi) = U^0(t; s, \Phi) - U(t; s, \Phi) \in C^m([s, T], X),$$
$$F^0(t; s, \Phi) = (\partial_t - A(t))U^0(t; s, \Phi) \in C^{m-1}([s, T], X),$$

we obtain

$$\partial_t^n V^0(t; s, \Phi)|_{t=s} = 0 \qquad (n = 0, 1, \dots, m),$$
$$\partial_t^n F^0(t; s, \Phi)|_{t=s} = 0 \qquad (n = 0, 1, \dots, m - 1).$$

So $V(t; s, \Phi)$ and $F(t; s, \Phi)$ to which are prolonged $V^0(t; s, \Phi)$ and $F^0(t; s, \Phi)$, respectively, by putting 0 in $t \in [0, s]$, belong to $C^m([0, T], X)$ and $C^{m-1}([0, T], X)$.

Then $V(t;s,\Phi)$ is a solution of $(CP)_0$ with data $(F,0)$. Now, recalling that

$$U(t;s,\Phi) = U^0(t;s,\Phi) - V^0(t;s,\Phi),$$

we will estimate each term in the right hand side. Note that the operator $A_n(s) \in \mathscr{L}(X)$ is uniformly continuous in $s \in [0,T]$. Since the map

$$C^{m-1}([0,T],X) \ni F(t) \mapsto V(t;s,\Phi) = U(t;s,F,0) \in C^m([0,T],X)$$

is closed and linear by the assumptions on $A(t)$, in view of Theorems 5 and 7, for any continuous seminorm p on X there exist continuous seminorms q, r on X and some positive constant C such that

$$\begin{aligned}
\sup_{t\in[s,T]} p(V^0(t;s,\Phi)) &\le \sup_{t\in[0,T]} p(V(t;s,\Phi)) \\
&\le C \sup_{t\in[0,T]} \sum_{n=0}^{m-1} q(\partial_t^n F(t;s,\Phi)) \\
&\le C \sup_{t\in[0,T]} \sum_{n=0}^{m-1} \sum_{l=0}^{m} (q(\partial_t^{n+1} G_l(t;s,\Phi)) + q(\partial_t^n (A(t) G_l(t;s,\Phi)))) \\
&\le r(\Phi)
\end{aligned}$$

for all $\Phi \in X$. Also $U^0(t;s,\Phi)$ has an estimate of the same type.

Next, supposing (ii), we will prove (iii). It is sufficient to observe that Duhamel's principle

$$U(t;s,F,\Phi) = U_H(t;s,\Phi) + \int_s^t U_H(t;\tau,F(\tau))\,d\tau$$

holds. We may assume that $s=0$ and $\Phi=0$ without loss of generality. Choose $F \in C^{m-1}([0,T],X)$ and let $U_H(t;s,F(s)) \in C^m([s,T],X)$ be the solution of $(HCP)_s$ with initial data $F(s)$. Then $U_H(t;s,F(s))$ is uniformly continuous with respect to $(t,s) \in \Omega$. In fact, if $s \le s'$, then

$$\begin{aligned}
&U_H(t;s,F(s)) - U_H(t';s',F(s')) \\
&= \int_{t'}^t \partial_\tau U_H(\tau;s,F(s))\,d\tau + U_H(t';s,F(s)) - U_H(t';s',F(s')) \\
&= \int_{t'}^t A(\tau) U_H(\tau;s,F(s))\,d\tau + U_H(t';s',U_H(s';s,F(s)) - F(s)) \\
&\quad + U_H(t';s',F(s) - F(s')).
\end{aligned}$$

Consequently, because of the assumptions on $A(t)$ and the uniform well-posedness of $(HCP)_s$ $(0 \le s < T)$, for any continuous seminorm p on X there are some continuous seminorms q_1, q_2, q_3 on X such that

$$\begin{aligned}
&p(U_H(t;s,F(s)) - U_H(t';s',F(s'))) \\
&\le |t-t'|q_1(F(s)) + q_3\left(\int_s^{s'} A(\tau)U_H(\tau;s,F(s))\,d\tau\right) + q_3(F(s)-F(s')) \\
&\le |t-t'|q_1(F(s)) + |s-s'|q_2(F(s)) + q_3(F(s)-F(s'))
\end{aligned}$$

provided (t,s), $(t',s') \in \Omega$. Thus

$$U(t) = \int_0^t U_H(t;s,F(s))\,ds \in C^0([0,T],X) \qquad (0 \le t \le T)$$

is well-defined, and since $A(t)$ is a continuous linear operator, we see that

$$A(t)U(t) = \int_0^t A(t)U_H(t;s,F(s))\,ds.$$

Moreover, let us verify that

$$(*) \qquad \begin{cases} \partial_t^+ U(t) = F(t) + \displaystyle\int_0^t \partial_t U_H(t;s,F(s))\,ds, \\ U(0) = 0. \end{cases}$$

For $h > 0; t, t+h \in [0,T)$, elementary calculations show

$$\begin{aligned}
&\frac{U(t+h)-U(t)}{h} - \left(F(t) + \int_0^t \partial_t U_H(t,s,F(s))\,ds\right) \\
&= \int_0^1 (U_H(t+h;t+hs,F(t+hs)) - U_H(t;t,F(t)))\,ds \\
&\quad + \int_0^t ds \int_0^1 A(t+hs)(U_H(t+h\tau;s,F(s)) - U_H(t;s,F(s)))\,d\tau \\
&\quad + \int_0^t ds \int_0^1 (A(t+h\tau) - A(t))U_H(t;s,F(s))\,d\tau.
\end{aligned}$$

Thereby, for any continuous seminorm p on X there exists a continuous seminorm q on X such that

$$\begin{aligned}
&p\left(\frac{U(t+h)-U(t)}{h} - \left(F(t) + \int_0^t \partial_t U_H(t,s,F(s))\,ds\right)\right) \\
&\le \int_0^1 p(U_H(t+h;t+hs,F(t+hs)) - U_H(t;t,F(t)))\,ds \\
&\quad + \int_0^t ds \int_0^1 q(U_H(t+h\tau;s,F(s)) - U_H(t;s,F(s)))\,d\tau \\
&\quad + \text{const.}\, h \int_0^t q(U_H(t;s,F(s)))\,ds.
\end{aligned}$$

Hence (*) follows from the uniform continuity of $U_H(t; s, F(s))$ in $(t, s) \in \Omega$. Now, because the right hand side in (*) is continuous, by taking account of Theorem 6, we have

$$\begin{cases} \partial_t U(t) = F(t) + \displaystyle\int_0^t \partial_t U_H(t; s, F(s))\, ds \in C^0([0, T], X), \\ U(0) = 0. \end{cases}$$

This exhibits that $U(t) \in C^1([0, T], X)$ is the solution of $(CP)_0$ with data $(F, 0)$. Further, repeating the argument above, we can catch that $U(t) \in C^m([0, T], X)$.

Finally, leading (iii) to (i) is plain and so we omit it.

§4. Some Remarks and Related Topics

As can be seen from the preceding proof, (ii) in Theorem 1 generally implies (i) in that for arbitrary complete locally convex space. It is natural to suspect whether we can replace the fixed integer m with ∞ or not. Unfortunately, we cannot adapt our method to do it. Indeed, owing to Observation 8.3.6 (b) in P. P. Carreras-J. Bonet [1], we find that $C^\infty([0, T])$ does not have the countable neighbourhood property. Therefore we cannot apply Theorem 4. However, Theorem 3 remains valid for $m = \infty$.

We employed ε-tensor product in order to prove that $C^m([0, T], X)$ is a $\mathscr{LF}$-space. In the case of $m \geq 1$, it does not yet seem known if we have any proof without using ε-tensor product. On the other hand, for $m = 0$ there is a simple topological proof of

$$C\left([0, T], \varinjlim X_n\right) \cong \varinjlim C([0, T], X_n)$$

by J. Mujica (see Theorem 3′ in J. Mujica [6]). This fact is originated from the problem of what property $C(E, F)$ inherits from a locally convex space F if we yield some conditions to a topological space E. Its problem has systematically been investigated by J. Schmets et al. (see R. Hollstein [4]). By the way, in aid of (12.1.8) in L. Narici-E. Beckenstein [7], we can deduce that $C^m([0, T], X)$ is a $\mathscr{LF}$-space if it is barrelled.

Afterward, Theorem 7 is extended by A. Grothendieck, W. Slowikowski, D. A. Raikov, M. De Wilde and so on (see M. Valdivia [11]). But, it is even nowadays the most general result as a closed graph theorem such that both the domain space and the range space lie in the same class of locally convex spaces.

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