

REPRESENTATIONS OF A LINK GROUP

By

Moto-o TAKAHASHI

In this paper we shall consider the link L illustrated in the Figure 1, and construct representations of the link group of L .

The link group $\pi(L)$ has the following presentations:

$$\begin{aligned} \pi(L) &\simeq \langle x_0, x_1, x_2, x_3, x_4, x_5 | [x_i, x_{i-1}x_{i+1}^{-1}] = 1, i = 0, 1, \dots, 5 \pmod{6} \rangle \\ &\simeq \langle x_0, x_1, x_2, x_3, x_4, x_5, y_0, y_1, y_2, y_3, y_4, y_5 | \\ &\quad [x_i, y_i] = 1, x_{i-1}x_{i+1}^{-1}y_i^{-1} = 1, i = 0, 1, \dots, 5 \pmod{6} \rangle, \end{aligned}$$

where $[x, y] = xyx^{-1}y^{-1}$. Note that, in the last presentation, it holds that $y_0y_2y_4 = 1$ and $y_1y_3y_5 = 1$.

Now, representations of $\pi(L)$ to $\text{PSL}(2, \mathbf{C})$ can be constructed using the following theorem. We set $c(z) = (z + 1/z)/2$ and $s(z) = (z - 1/z)/2$.

THEOREM. Let $\lambda_0, \lambda_2, \lambda_4, \mu_0, \mu_2, \mu_4$ be complex numbers not equal to $0, \pm 1$. Suppose that we can take complex numbers $\lambda_1, \lambda_3, \lambda_5, \mu_1, \mu_3, \mu_5$ not equal to $0, \pm 1$, satisfying the following conditions (i) and (ii): for $i = 1, 3, 5 \pmod{6}$

- (i) $c(\lambda_i) = c(\mu_{i-1})c(\mu_{i+1}) + \{c(\lambda_{i-1})c(\lambda_{i+1}) - c(\lambda_{i+3})\}s(\mu_{i-1})s(\mu_{i+1})/\{s(\lambda_{i-1})s(\lambda_{i+1})\}$,
- (ii) $s(\lambda_i)c(\mu_i)/s(\mu_i) = -c(\lambda_{i-1})c(\mu_{i+1})s(\mu_{i-1})/s(\lambda_{i-1}) - c(\mu_{i-1})c(\mu_{i+1})s(\mu_{i+1})/s(\lambda_{i+1}) - c(\mu_{i+3})s(\lambda_{i+3})s(\mu_{i-1})s(\mu_{i+1})/\{s(\mu_{i+3})s(\mu_{i-1})s(\mu_{i+1})\}$.

Then, we can take $A_i \in \text{SL}(2, \mathbf{C})$ ($i = 0, 1, \dots, 5 \pmod{6}$) to construct a non-abelian representation of $\pi(L)$ to $\text{PSL}(2, \mathbf{C})$ by corresponding

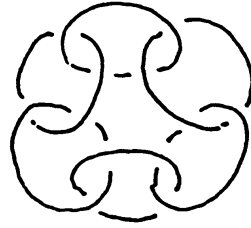


Figure 1

$$y_i \rightarrow A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1},$$

$$x_i \rightarrow A_i \begin{pmatrix} \mu_i & 0 \\ 0 & 1/\mu_i \end{pmatrix} A_i^{-1},$$

$i = 0, 1, \dots, 5 \pmod{6}$.

REMARK. The conditions (i) & (ii) are equivalent to the following (iii) & (iv), for (i)+(ii) makes (iii) and (i)-(ii) makes (iv):

$$(iii) \quad \frac{s(\mu_i/\lambda_i)}{s(\mu_i)} = \frac{s(\mu_{i+1}/\lambda_{i+1})s(\mu_{i-1}/\lambda_{i-1})}{s(\lambda_{i+1})s(\lambda_{i-1})} - \frac{s(\lambda_{i+3}\mu_{i+3})s(\mu_{i+1})s(\mu_{i-1})}{s(\mu_{i+3})s(\lambda_{i+1})s(\lambda_{i-1})},$$

$$(iv) \quad \frac{s(\lambda_i\mu_i)}{s(\mu_i)} = \frac{s(\lambda_{i+1}\mu_{i+1})s(\lambda_{i-1}\mu_{i-1})}{s(\lambda_{i+1})s(\lambda_{i-1})} - \frac{s(\mu_{i+3}/\lambda_{i+3})s(\mu_{i+1})s(\mu_{i-1})}{s(\mu_{i+3})s(\lambda_{i+1})s(\lambda_{i-1})}.$$

REMARK 2. For $i = 1, 3, 5 \pmod{6}$, $c(\lambda_i)$ is determined by (i). Let the value of the right-hand side of (i) be a , then λ_i is determined by the equation $\lambda_i + 1/\lambda_i = 2a$, or the quadratic equation $\lambda_i^2 - 2a\lambda_i + 1 = 0$. In some special cases the two roots of the equation $x^2 - 2ax + 1 = 0$ may be among $0, \pm 1$. Except these cases $\lambda_i (\neq 0, \pm 1)$ exists.

Suppose that λ_i exists. Then μ_i is determined by (ii). Let the value of the right-hand side of (ii) be b . Then μ_i is determined by the equation $(\mu_i + 1/\mu_i)/(\mu_i - 1/\mu_i) = b/s(\lambda_i)$, so

$$\mu_i = \pm \sqrt{-(s(\lambda_i) + b)/(s(\lambda_i) - b)}.$$

In some special cases these values are among $0, \pm 1$. Except these cases $\mu_i (\neq 0, \pm 1)$ exists.

REMARK 3. In the theorem we have assumed that $\lambda_i \neq \pm 1$. But in the case $\lambda_i = \pm 1$ (parabolic case), where $0 \leq i \leq 5$, if we modify (i), (ii) by re-

placing $s(\mu_i)/s(\lambda_i)$ by a_i and by replacing $y_i \rightarrow A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1}$ by $y_i \rightarrow A_i \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} A_i^{-1}$ and by replacing $x_i \rightarrow A_i \begin{pmatrix} \mu_i & 0 \\ 0 & 1/\mu_i \end{pmatrix} A_i^{-1}$ by $x_i \rightarrow A_i \begin{pmatrix} \pm 1 & a_i \\ 0 & \pm 1 \end{pmatrix} A_i^{-1}$. Then we also have a representation.

REMARK 4. By using this theorem, I would like to construct a bi-rational representation (i.e. a homomorphism into the group of bi-rational transformations of a certain algebraic variety) of the mapping class group of the closed orientable surface of genus 2. This will be carried out in the subsequent paper.

Roughly speaking, it follows from this theorem that the space of the non-equivalent representations of $\pi(L)$ to $\text{PSL}(2, \mathbb{C})$ has the complex dimension at least 6.

PROOF OF THE THEOREM. For $i = 0, 2, 4 \pmod{6}$, let

$$U_i = \begin{pmatrix} u_{i1} & u_{i2} \\ u_{i3} & u_{i4} \end{pmatrix},$$

where

$$u_{i1} = \frac{\lambda_{i+2} - \lambda_i \lambda_{i-2}}{\lambda_{i+2}(\lambda_{i-2}^2 - 1)}, \quad u_{i2} = \lambda_0 \lambda_2 \lambda_4 - 1,$$

$$u_{i3} = \frac{\lambda_i - \lambda_{i+2} \lambda_{i-2}}{\lambda_i(\lambda_{i+2}^2 - 1)(\lambda_{i-2}^2 - 1)}, \quad u_{i4} = \frac{\lambda_{i+2}(\lambda_{i-2} - \lambda_i \lambda_{i+2})}{\lambda_i(\lambda_{i+2}^2 - 1)}.$$

Then, it is easy to check that $U_0 U_2 U_4 = E$ and

$$\begin{pmatrix} \lambda_0 & 0 \\ 0 & 1/\lambda_0 \end{pmatrix} U_4 \begin{pmatrix} \lambda_2 & 0 \\ 0 & 1/\lambda_2 \end{pmatrix} U_0 \begin{pmatrix} \lambda_4 & 0 \\ 0 & 1/\lambda_4 \end{pmatrix} U_2 = E,$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover it holds that, for $i = 0, 2, 4 \pmod{6}$,

$$u_{i1} u_{i4} = \frac{c(\lambda_i) - c(\lambda_{i+2})c(\lambda_{i-2}) + s(\lambda_{i+2})s(\lambda_{i-2})}{2s(\lambda_{i+2})s(\lambda_{i-2})} \tag{1}$$

and

$$u_{i2}u_{i3} = \frac{c(\lambda_i) - c(\lambda_{i+2})c(\lambda_{i-2}) - s(\lambda_{i+2})s(\lambda_{i-2})}{2s(\lambda_{i+2})s(\lambda_{i-2})}.$$

So, by putting $A_0 = E$, $A_2 = U_4$, $A_4 = U_2^{-1}$, we have, for $i = 0, 2, 4 \pmod{6}$, $U_i = A_{i+2}^{-1}A_{i-2}$, and

$$\left\{ A_0 \begin{pmatrix} \lambda_0 & 0 \\ 0 & 1/\lambda_0 \end{pmatrix} A_0^{-1} \right\} \left\{ A_2 \begin{pmatrix} \lambda_2 & 0 \\ 0 & 1/\lambda_2 \end{pmatrix} A_2^{-1} \right\} \left\{ A_4 \begin{pmatrix} \lambda_4 & 0 \\ 0 & 1/\lambda_4 \end{pmatrix} A_4^{-1} \right\} = E.$$

For $i = 0, 2, 4 \pmod{6}$, let

$$Y_i = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1}, \quad X_i = A_i \begin{pmatrix} \mu_i & 0 \\ 0 & 1/\mu_i \end{pmatrix} A_i^{-1}.$$

Then, we have $Y_0 Y_2 Y_4 = E$, which corresponds the relator $y_0 y_2 y_4 = 1$.

Next we compute the trace of $X_{i-1} X_{i+1}^{-1}$, for $i = 1, 3, 5 \pmod{6}$.

$$\begin{aligned} & \text{Tr}(X_{i-1} X_{i+1}^{-1}) \\ &= \text{Tr} \left(A_{i-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} A_{i-1}^{-1} A_{i+1} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1} \right) \\ &= \text{Tr} \left(\begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} A_{i-1}^{-1} A_{i+1} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1} A_{i-1} \right) \\ &= \text{Tr} \left(\begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} U_{i+3} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & 1/\mu_{i+1} \end{pmatrix} U_{i+3}^{-1} \right) \\ &= \mu_{i-1} (u_{i+3,1} u_{i+3,4} / \mu_{i+1} - \mu_{i+1} u_{i+3,2} u_{i+3,3}) \\ &\quad + (\mu_{i+1} u_{i+3,1} u_{i+3,4} - u_{i+3,2} u_{i+3,3} / \mu_{i+1}) / \mu_{i-1} \\ &= \left(\frac{\mu_{i-1}}{\mu_{i+1}} + \frac{\mu_{i+1}}{\mu_{i-1}} \right) u_{i+3,1} u_{i+3,4} - \left(\mu_{i-1} \mu_{i+1} + \frac{1}{\mu_{i-1} \mu_{i+1}} \right) u_{i+3,2} u_{i+3,3} \\ &= \left(\frac{\mu_{i-1}}{\mu_{i+1}} + \frac{\mu_{i+1}}{\mu_{i-1}} \right) \frac{c(\lambda_{i+3}) - c(\lambda_{i+1})c(\lambda_{i-1}) + s(\lambda_{i+1})s(\lambda_{i-1})}{2s(\lambda_{i+1})s(\lambda_{i-1})} \\ &\quad - \left(\mu_{i-1} \mu_{i+1} + \frac{1}{\mu_{i-1} \mu_{i+1}} \right) \frac{c(\lambda_{i+3}) - c(\lambda_{i+1})c(\lambda_{i-1}) - s(\lambda_{i+1})s(\lambda_{i-1})}{2s(\lambda_{i+1})s(\lambda_{i-1})}. \end{aligned}$$

by (1). Now,

$$\begin{aligned} \frac{\mu_{i-1}}{\mu_{i+1}} + \frac{\mu_{i+1}}{\mu_{i-1}} &= 2(c(\mu_{i-1})c(\mu_{i+1}) - s(\mu_{i-1})s(\mu_{i+1})), \\ \mu_{i-1}\mu_{i+1} + \frac{1}{\mu_{i-1}\mu_{i+1}} &= 2(c(\mu_{i-1})c(\mu_{i+1}) + s(\mu_{i-1})s(\mu_{i+1})). \end{aligned}$$

So,

$$\begin{aligned} &\text{Tr}(X_{i-1}X_{i+1}^{-1}) \\ &= 2(c(\mu_{i-1})c(\mu_{i+1}) - s(\mu_{i-1})s(\mu_{i+1})) \\ &\quad \cdot (c(\lambda_{i+3}) - c(\lambda_{i+1})c(\lambda_{i-1}) + s(\lambda_{i+1})s(\lambda_{i-1}))/ (2s(\lambda_{i+1})s(\lambda_{i-1})) \\ &\quad - 2(c(\mu_{i-1})c(\mu_{i+1}) + s(\mu_{i-1})s(\mu_{i+1})) \\ &\quad \cdot (c(\lambda_{i+3}) - c(\lambda_{i+1})c(\lambda_{i-1}) - s(\lambda_{i+1})s(\lambda_{i-1}))/ (2s(\lambda_{i+1})s(\lambda_{i-1})) \\ &= 2(c(\mu_{i-1})c(\mu_{i+1}) + (c(\lambda_{i-1})c(\lambda_{i+1}) - c(\lambda_{i+3}))s(\mu_{i-1})s(\mu_{i+1}))/ (s(\lambda_{i-1})s(\lambda_{i+1})) \\ &= 2c(\lambda_i). \end{aligned}$$

Here we have used the hypothesis of the theorem. Thus,

$$\text{Tr}(X_{i-1}X_{i+1}^{-1}) = \lambda_i + \frac{1}{\lambda_i}.$$

$X_{i-1}X_{i+1}^{-1} \in \text{SL}(2, \mathbf{C})$ and $\lambda_i \neq \pm 1$. So, there exists an $A_i \in \text{SL}(2, \mathbf{C})$ such that

$$X_{i-1}X_{i+1}^{-1} = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1}.$$

Now we put, for $i = 1, 3, 5 \pmod{6}$,

$$Y_i = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1}, \quad X_i = A_i \begin{pmatrix} \mu_i & 0 \\ 0 & 1/\mu_i \end{pmatrix} A_i^{-1}.$$

Then, we have $Y_i = X_{i-1}X_{i+1}^{-1}$, which corresponds to the relator $y_i = x_{i-1}x_{i+1}^{-1}$, for $i = 1, 3, 5 \pmod{6}$.

Obviously, $[X_i, Y_i] = E$, for $i = 0, 1, \dots, 5 \pmod{6}$. It remains to prove that $Y_i \sim X_{i-1}X_{i+1}^{-1}$, for $i = 0, 2, 4 \pmod{6}$, where $A \sim B$ means that $A = \xi B$, for some scalar $\xi \neq 0$.

Now $Y_i \sim X_{i-1}X_{i+1}^{-1}$ is equivalent to

$$A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1} \sim \left\{ A_{i-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} A_{i-1}^{-1} \right\} \left\{ A_{i+1} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} A_{i+1}^{-1} \right\}$$

or

$$\begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} V_{i+1} \begin{pmatrix} \mu_{i+1} & 0 \\ 0 & 1/\mu_{i+1} \end{pmatrix} V_{i+1}^{-1} \sim V_i^{-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} V_i, \quad (2)$$

where $V_i = A_{i-1}^{-1}A_i$.

Now, for $i = 1, 3, 5 \pmod{6}$,

$$X_{i-1}X_{i+1}^{-1} = A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1}. \quad (3)$$

So,

$$A_{i-1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i-1} = A_{i-1}^{-1}A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1}A_{i-1} = V_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1}.$$

Also,

$$A_{i-1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i-1} = \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} U_{i+3} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1}.$$

Hence we have

$$V_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} V_i^{-1} = \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} U_{i+3} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1}.$$

Let

$$V_i = \begin{pmatrix} v_{i1} & v_{i2} \\ v_{i3} & v_{i4} \end{pmatrix}.$$

Then,

$$\begin{aligned} V_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} V_i^{-1} &= \begin{pmatrix} \lambda_i v_{i1} v_{i4} - v_{i2} v_{i3} / \lambda_i & (1/\lambda_i - \lambda_i) v_{i1} v_{i2} \\ (\lambda_i - 1/\lambda_i) v_{i3} v_{i4} & v_{i1} v_{i4} / \lambda_i - \lambda_i v_{i2} v_{i3} \end{pmatrix}. \\ \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} U_{i+3} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} U_{i+3}^{-1} &= \begin{pmatrix} \mu_{i-1} (u_{i+3,1} u_{i+3,4} / \mu_{i+1} - u_{i+3,2} u_{i+3,3} \mu_{i+1}) & \mu_{i-1} (\mu_{i+1} - 1/\mu_{i+1}) u_{i+3,1} u_{i+3,2} \\ (1/\mu_{i+1} - \mu_{i+1}) u_{i+3,3} u_{i+3,4} / \mu_{i-1} & (\mu_{i+1} u_{i+3,1} u_{i+3,4} - u_{i+3,2} u_{i+3,3} / \mu_{i+1}) / \mu_{i-1} \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned}
 v_{i1}v_{i2} &= -\frac{s(\mu_{i+1})\mu_{i-1}(\lambda_{i-1} - \lambda_{i+3}\lambda_{i+1})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_i)\lambda_{i-1}(\lambda_{i+1}^2 - 1)}, \\
 v_{i3}v_{i4} &= -\frac{s(\mu_{i+1})\lambda_{i-1}(\lambda_{i+3} - \lambda_{i-1}\lambda_{i+1})(\lambda_{i+1} - \lambda_{i-1}\lambda_{i+3})}{\mu_{i-1}s(\lambda_i)\lambda_{i+3}^2(\lambda_{i-1}^2 - 1)^2(\lambda_{i+1}^2 - 1)}, \\
 v_{i1}v_{i4} + v_{i2}v_{i3} &= \frac{c(\mu_{i-1})s(\mu_{i+1})\{c(\lambda_{i+1})c(\lambda_{i-1}) - c(\lambda_{i+3})\}}{s(\lambda_i)s(\lambda_{i+1})s(\lambda_{i-1})} + \frac{s(\mu_{i-1})c(\mu_{i+1})}{s(\lambda_i)},
 \end{aligned}$$

for $i = 1, 3, 5 \pmod{6}$. Hence, for $i = 0, 2, 4 \pmod{6}$,

$$\begin{aligned}
 v_{i+1,1}v_{i+1,2} &= -\frac{s(\mu_{i+2})\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_{i+1})\lambda_i(\lambda_{i+2}^2 - 1)}, \\
 v_{i+1,3}v_{i+1,4} &= -\frac{s(\mu_{i+2})\lambda_i(\lambda_{i-2} - \lambda_i\lambda_{i+2})(\lambda_{i+2} - \lambda_i\lambda_{i-2})}{\mu_i s(\lambda_{i+1})\lambda_{i-2}^2(\lambda_i^2 - 1)(\lambda_{i+2}^2 - 1)}, \\
 v_{i+1,1}v_{i+1,4} + v_{i+1,2}v_{i+1,3} &= \frac{c(\mu_i)s(\mu_{i+2})\{c(\lambda_{i+2})c(\lambda_i) - c(\lambda_{i-2})\}}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)} + \frac{s(\mu_i)c(\mu_{i+2})}{s(\lambda_{i+1})},
 \end{aligned} \tag{4}$$

Next, for $i = 1, 3, 5 \pmod{6}$, by (3),

$$\begin{aligned}
 A_{i+1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i+1} &= A_{i+1}^{-1}A_i \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} A_i^{-1}A_{i+1} \\
 &= V_{i+1}^{-1} \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} V_{i+1}.
 \end{aligned}$$

Also,

$$\begin{aligned}
 A_{i+1}^{-1}X_{i-1}X_{i+1}^{-1}A_{i+1} &= A_{i+1}^{-1}A_{i-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} A_{i-1}^{-1}A_{i+1} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix} \\
 &= U_{i+3}^{-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} U_{i+3} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix}
 \end{aligned}$$

Hence we have

$$V_{i+1}^{-1} \begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} V_{i+1} = U_{i+3}^{-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} U_{i+3} \begin{pmatrix} 1/\mu_{i+1} & 0 \\ 0 & \mu_{i+1} \end{pmatrix}.$$

Hence by (1),

$$v_{i+1,2}v_{i+1,4} = \frac{s(\mu_{i-1})\mu_{i+1}\lambda_{i-1}(\lambda_{i+1} - \lambda_{i-1}\lambda_{i+3})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_i)\lambda_{i+3}(\lambda_{i-1}^2 - 1)},$$

$$v_{i+1,1}v_{i+1,3} = \frac{s(\mu_{i-1})(\lambda_{i-1} - \lambda_{i+3}\lambda_{i+1})(\lambda_{i+3} - \lambda_{i-1}\lambda_{i+1})}{s(\lambda_i)\mu_{i+1}\lambda_{i-1}\lambda_{i+3}(\lambda_{i-1}^2 - 1)(\lambda_{i+1}^2 - 1)^2},$$

$$v_{i+1,1}v_{i+1,4} + v_{i+1,2}v_{i+1,3} = \frac{c(\mu_{i+1})s(\mu_{i-1})\{c(\lambda_{i+3}) - c(\lambda_{i+1})c(\lambda_{i-1})\}}{s(\lambda_i)s(\lambda_{i+1})s(\lambda_{i-1})} - \frac{s(\mu_{i+1})c(\mu_{i-1})}{s(\lambda_i)},$$

for $i = 1, 3, 5 \pmod{6}$. Hence, for $i = 0, 2, 4 \pmod{6}$, we have

$$v_{i2}v_{i4} = \frac{s(\mu_{i-2})\mu_i\lambda_{i-2}(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_{i-1})\lambda_{i+2}(\lambda_{i-2}^2 - 1)},$$

$$v_{i1}v_{i3} = \frac{s(\mu_{i-2})(\lambda_{i-2} - \lambda_{i+2}\lambda_i)(\lambda_{i+2} - \lambda_{i-2} - \lambda_i)}{s(\lambda_{i-1})\mu_i\lambda_{i-2}\lambda_{i+2}(\lambda_{i-2}^2 - 1)(\lambda_i^2 - 1)^2}, \quad (5)$$

$$v_{i1}v_{i4} + v_{i2}v_{i3} = \frac{c(\mu_i)s(\mu_{i-2})\{c(\lambda_{i+2}) - c(\lambda_i)c(\lambda_{i-2})\}}{s(\lambda_{i-1})s(\lambda_i)s(\lambda_{i-2})} - \frac{s(\mu_i)c(\mu_{i-2})}{s(\lambda_{i-1})}.$$

Now we prove (2). Let

$$\begin{pmatrix} \lambda_i & 0 \\ 0 & 1/\lambda_i \end{pmatrix} V_{i+1} \begin{pmatrix} \mu_{i+1} & 0 \\ 0 & 1/\mu_{i+1} \end{pmatrix} v_{i+1}^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix},$$

and

$$V_i^{-1} \begin{pmatrix} \mu_{i-1} & 0 \\ 0 & 1/\mu_{i-1} \end{pmatrix} v_i = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

($i = 0, 2, 4 \pmod{6}$) and we prove that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \sim \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Now,

$$a_1 = \lambda_i(v_{i+1,1}v_{i+1,4}\mu_{i+1} - v_{i+1,2}v_{i+1,3}/\mu_{i+1}),$$

$$b_1 = \lambda_i(1/\mu_{i+1} - \mu_{i+1})v_{i+1,1}v_{i+1,2},$$

$$c_1 = (\mu_{i+1}^{-1}/\mu_{i+1})v_{i+1,3}v_{i+1,4}/\lambda_i,$$

$$d_1 = (v_{i+1,1}v_{i+1,4}/\mu_{i+1} - v_{i+1,2}v_{i+1,3}\mu_{i+1})/\lambda_i,$$

$$a_2 = v_{i1}v_{i4}\mu_{i-1} - v_{i2}v_{i3}/\mu_{i-1},$$

$$b_2 = \mu_{i-1} - 1/\mu_{i-1})v_{i2}v_{i4},$$

$$c_2 = (1/\mu_{i-1} - \mu_{i-1})v_{i1}v_{i3},$$

$$d_2 = v_{i1}v_{i4}/\mu_{i-1} - v_{i2}v_{i3}\mu_{i-1}.$$

By (4) and (5),

$$b_1 = \lambda_1(-2s(\mu_{i+1})) \frac{-s(\mu_{i+2})\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_{i+1})\lambda_i(\lambda_{i+2}^2 - 1)}$$

$$= \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})} \cdot \frac{\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{\lambda_{i+2}},$$

$$b_2 = 2s(\mu_{i-1}) \frac{s(\mu_{i-2})\mu_i\lambda_{i-2}(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{s(\lambda_{i-1})\lambda_{i+2}(\lambda_{i-2}^2 - 1)}$$

$$= \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})} \cdot \frac{\mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)}{\lambda_{i+2}}.$$

Let

$$b_3 = \mu_i(\lambda_i - \lambda_{i-2}\lambda_{i+2})(\lambda_0\lambda_2\lambda_4 - 1)/\lambda_{i+2}.$$

Then,

$$b_1 = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})} b_3, \quad b_2 = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})} b_3. \quad (6)$$

Similarly, if we put

$$c_3 = -\frac{(\lambda_{i-2} - \lambda_i\lambda_{i+2})(\lambda_{i+2} - \lambda_i\lambda_{i-2})}{\mu_i\lambda_{i-2}^2\lambda_{i+2}(\lambda_i^2 - 1)^2},$$

then, we have

$$c_1 = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})} c_3, \quad c_2 = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})} c_3. \quad (7)$$

Next we compute a_1 and a_2 .

$$\begin{aligned}
a_1 &= \lambda_i(v_{i+1,1}v_{i+1,4}\mu_{i+1} - v_{i+1,2}v_{i+1,3}/\mu_{i+1}) \\
&= \lambda_i\{(v_{i+1,1}v_{i+1,4} + v_{i+1,2}v_{i+1,3})s(\mu_{i+1}) + (v_{i+1,1}v_{i+1,4} - v_{i+1,2}v_{i+1,3})c(\mu_{i+1})\} \\
&= \lambda_i\{(v_{i+1,1}v_{i+1,4} + v_{i+1,2}v_{i+1,3})s(\mu_{i+1}) + c(\mu_{i+1})\} \\
&= \lambda_i s(\mu_{i+1})\{(v_{i+1,1}v_{i+1,4} + v_{i+1,2}v_{i+1,3}) + c(\mu_{i+1})/s(\mu_{i+1})\} \\
&= \lambda_i s(\mu_{i+1}) \left\{ \frac{c(\mu_i)s(\mu_{i+2})\{c(\lambda_{i+2})c(\lambda_i) - c(\lambda_{i-2})\}}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)} + \frac{s(\mu_i)c(\mu_{i+2})}{s(\lambda_{i+1})} + \frac{c(\mu_{i+1})}{s(\mu_{i+1})} \right\} \\
&= \lambda_i \frac{s(\mu_{i+1})}{s(\lambda_{i+1})} \left\{ \frac{c(\mu_{i+1})s(\mu_{i+2})\{c(\lambda_{i+2})c(\lambda_i) - c(\lambda_{i-2})\}}{s(\lambda_{i+2})s(\lambda_i)} \right. \\
&\quad \left. + s(\mu_i)c(\mu_{i+2}) + \frac{s(\lambda_{i+1})c(\mu_{i+1})}{s(\mu_{i+1})} \right\}.
\end{aligned}$$

Here we have used (4).

Now by the hypothesis of the theorem,

$$\begin{aligned}
& s(\lambda_{i+1})c(\mu_{i+1})/s(\mu_{i+1}) \\
&= -s(\mu_i)c(\lambda_i)c(\mu_{i+2}) - s(\mu_{i+2})c(\mu_i)c(\lambda_{i+2})/s(\lambda_{i+2}) \\
&\quad - s(\lambda_{i-2})s(\mu_i)s(\mu_{i+2})c(\mu_{i-2})/\{s(\mu_{i-2})s(\lambda_i)s(\lambda_{i+2})\}.
\end{aligned}$$

So,

$$\begin{aligned}
a_1 &= \lambda_i \frac{s(\mu_{i+1})}{s(\lambda_{i+1})} \left\{ \frac{c(\mu_i)s(\mu_{i+2})\{c(\lambda_{i+2})c(\lambda_i) - c(\lambda_{i-2})\}}{s(\lambda_{i+2})s(\lambda_i)} \right. \\
&\quad + s(\mu_i)c(\mu_{i+2}) - \frac{s(\mu_i)}{s(\lambda_i)} c(\lambda_i)c(\mu_{i+2}) - \frac{s(\mu_{i+2})}{s(\lambda_{i+2})} c(\mu_i)c(\lambda_{i+2}) \\
&\quad \left. - \frac{s(\lambda_{i-2})s(\mu_i)s(\mu_{i+2})}{s(\mu_{i-2})s(\lambda_i)s(\lambda_{i+2})} c(\mu_{i-2}) \right\} \\
&= \frac{s(\mu_{i+1})s(\mu_{i+2})\lambda_i}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)} \left\{ c(\mu_i)c(\lambda_{i+2})c(\lambda_i) - c(\mu_i)c(\lambda_{i-2}) \right. \\
&\quad + s(\mu_i) \frac{c(\mu_{i+2})s(\lambda_{i+2})s(\lambda_i)}{s(\mu_{i+2})} - \frac{s(\mu_i)c(\lambda_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})} \\
&\quad \left. - c(\mu_i)c(\lambda_{i+2})s(\lambda_i) - \frac{s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})} \right\}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{s(\mu_{i+1})s(\mu_{i+2})\lambda_i}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)} \left\{ c(\mu_i)c(\lambda_{i+2})\{c(\lambda_i) - s(\lambda_i)\} \right. \\
 &\quad + \frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})\{s(\lambda_i) - c(\lambda_i)\}}{s(\mu_{i+2})} - c(\mu_i)c(\lambda_{i-2}) \\
 &\quad \left. - \frac{s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})} \right\} \\
 &= \frac{s(\mu_{i+1})s(\mu_{i+2})\lambda_i}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)} \left\{ \frac{c(\mu_i)c(\lambda_{i+2})}{\lambda_i} - \frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})\lambda_i} \right. \\
 &\quad \left. - c(\mu_i)c(\lambda_{i-2}) - \frac{s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})} \right\} \\
 &= \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})s(\lambda_i)} \left\{ c(\mu_i)c(\lambda_{i+2}) - \frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})} \right. \\
 &\quad \left. - \lambda_i c(\mu_i)c(\lambda_{i-2}) - \frac{\lambda_i s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})} \right\}.
 \end{aligned}$$

So, if we put

$$\begin{aligned}
 a_3 = \frac{1}{s(\lambda_i)} \left\{ c(\mu_i)c(\lambda_{i+2}) - \frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})} \right. \\
 \left. - \lambda_i c(\mu_i)c(\lambda_{i-2}) - \frac{\lambda_i s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})} \right\},
 \end{aligned}$$

then, we have

$$a_1 = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})} a_3. \tag{8}$$

Also,

$$\begin{aligned}
 a_2 &= v_{i1}v_{i4}\mu_{i-1} - v_{i2}v_{i3}/\mu_{i-1} \\
 &= (v_{i1}v_{i4} + v_{i2}v_{i3})s(\mu_{i-1}) + (v_{i1}v_{i4} - v_{i2}v_{i3})c(\mu_{i-1}) \\
 &= (v_{i1}v_{i4} + v_{i2}v_{i3})s(\mu_{i-1}) + c(\mu_{i-1}) \\
 &= s(\mu_{i-1})\{(v_{i1}v_{i4} + v_{i2}v_{i3}) + c(\mu_{i-1})/s(\mu_{i-1})\}
 \end{aligned}$$

$$\begin{aligned}
&= s(\mu_{i-1}) \left\{ \frac{c(\mu_i)s(\mu_{i-2})\{c(\lambda_{i+2}) - c(\lambda_i)c(\lambda_{i-2})\}}{s(\lambda_{i-1})s(\lambda_i)s(\lambda_{i-2})} - \frac{s(\mu_i)c(\mu_{i-1})}{s(\lambda_{i-1})} + \frac{c(\mu_{i-1})}{s(\mu_{i-1})} \right\} \\
&= \frac{s(\mu_{i-1})}{s(\lambda_{i-1})} \left\{ \frac{c(\mu_i)s(\mu_{i-2})\{c(\lambda_{i+2}) - c(\lambda_i)c(\lambda_{i-2})\}}{s(\lambda_i)s(\lambda_{i-2})} - s(\mu_i)c(\mu_{i-2}) + \frac{c(\mu_{i-1})s(\lambda_{i-1})}{s(\mu_{i-1})} \right\}.
\end{aligned}$$

Here we have used (5). Now, by the hypothesis of the theorem,

$$\begin{aligned}
\frac{c(\mu_{i-1})s(\lambda_{i-1})}{s(\mu_{i-1})} &= -\frac{s(\mu_{i-2})}{s(\lambda_{i-2})} c(\lambda_{i-2})c(\mu_i) - \frac{s(\mu_i)}{s(\lambda_i)} c(\mu_{i-2})c(\lambda_i) \\
&\quad - \frac{s(\lambda_{i+2})s(\mu_{i-2})c(\mu_i)}{s(\mu_{i+2})s(\lambda_{i-2})s(\lambda_i)} c(\mu_{i+2}).
\end{aligned}$$

Hence,

$$\begin{aligned}
a_2 &= \frac{s(\mu_{i-1})}{s(\lambda_{i-1})} \left\{ \frac{c(\mu_i)s(\mu_{i-2})\{c(\lambda_{i+2}) - c(\lambda_i)c(\lambda_{i-2})\}}{s(\lambda_i)s(\lambda_{i-2})} \right. \\
&\quad \left. - s(\mu_i)c(\mu_{i-2}) - \frac{s(\mu_{i-2})}{s(\lambda_{i-2})} c(\lambda_{i-2})c(\mu_i) - \frac{s(\mu_i)}{s(\lambda_i)} c(\mu_{i-2})c(\lambda_i) \right. \\
&\quad \left. - \frac{s(\lambda_{i+2})s(\mu_{i-2})s(\mu_i)}{s(\mu_{i+2})s(\lambda_{i-2})s(\lambda_i)} c(\mu_{i+2}) \right\} \\
&= \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})s(\lambda_i)} \left\{ c(\mu_i)c(\lambda_{i+2}) - c(\mu_i)c(\lambda_i)c(\lambda_{i-2}) \right. \\
&\quad \left. - \frac{s(\mu_i)c(\mu_{i-2})s(\lambda_{i-2})s(\lambda_i)}{s(\mu_{i-2})} - s(\lambda_i)c(\lambda_{i-2})c(\mu_i) \right. \\
&\quad \left. - \frac{s(\mu_i)s(\lambda_{i-2})}{s(\mu_{i-2})} c(\mu_{i-2})c(\lambda_i) - \frac{s(\lambda_{i+2})s(\mu_i)c(\mu_{i+2})}{s(\mu_{i+2})} \right\} \\
&= \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})s(\lambda_i)} \left\{ c(\mu_i)c(\lambda_{i+2}) - c(\mu_i)c(\lambda_{i-2})\{s(\lambda_i) + c(\lambda_i)\} \right. \\
&\quad \left. - \frac{c(\mu_{i-2})s(\lambda_{i-2})s(\mu_i)\{s(\lambda_i) + c(\lambda_i)\}}{s(\mu_{i-2})} - s(\lambda_i)c(\lambda_{i-2})c(\mu_i) - \frac{s(\lambda_{i+2})s(\mu_i)c(\mu_{i+2})}{s(\mu_{i+2})} \right\} \\
&= \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})s(\lambda_i)} \left\{ c(\mu_i)c(\lambda_{i+2}) - \frac{s(\mu_i)c(\mu_{i+2})s(\lambda_{i+2})}{s(\mu_{i+2})} \right. \\
&\quad \left. - \lambda_i c(\mu_i)c(\lambda_{i-2}) - \frac{\lambda_i s(\lambda_{i-2})s(\mu_i)c(\mu_{i-2})}{s(\mu_{i-2})} \right\}.
\end{aligned}$$

Hence

$$a_2 = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})} a_3. \tag{9}$$

Similarly, if we put

$$d_3 = -\frac{c(\mu_i)c(\lambda_{i+2})s(\mu_{i+2}) + s(\mu_i)s(\lambda_{i+2})c(\mu_{i+2})}{s(\lambda_i)s(\mu_{i+2})} + \frac{c(\mu_i)c(\lambda_{i-2})s(\mu_{i-2}) - s(\mu_i)s(\lambda_{i-2})c(\mu_{i-2})}{\lambda_i s(\lambda_i)s(\mu_{i-2})},$$

we have

$$d_1 = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})} d_3, \quad d_2 = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})} d_3. \tag{10}$$

By (6), (7), (8), (9), (10),

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \frac{s(\mu_{i+1})s(\mu_{i+2})}{s(\lambda_{i+1})s(\lambda_{i+2})} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$$

and

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \frac{s(\mu_{i-1})s(\mu_{i-2})}{s(\lambda_{i-1})s(\lambda_{i-2})} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \sim \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Thus, (2) is proved.

(Q.E.D)