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REDUCTION PROPERTY AND DIMENSIONAL ORDER PROPERTY

By

Kentaro WAKAI

1. Introduction

Let T be a theory with a relational language L including a unary predicate P. Let M be a model of T and N the L^--structure $P^M = \{a \in |M| : M \models P(a)\}$ where $L^- \subset L - \{P\}$. The following question seems to be natural:

QUESTION. Which properties of $T^- = Th(N)$ are also possessed by T (under certain conditions)?

There are a few papers treating the question. In [HP] Hodges and Pillay have shown that if T is minimal over P (definition 2.3) and every automorphism of N can be extended to an automorphism of M (they call M is a symmetric extension of N), then N is $\aleph_0$-categorical iff M is $\aleph_0$-categorical. In [KT] Kikyo and Tsuboi defined the $\emptyset$-reduction property, the reduction property, the strong reduction property, and the uniform reduction property. These reduction properties are model theoretical rephrasing of symmetry. They have shown that if T is minimal over P and has the uniform reduction property (i.e., for each L-formula $\varphi(\bar{x}\bar{y})$, there is an L^--formula $\psi(\bar{x}\bar{z})$ such that $(\forall\bar{y})(\exists\bar{z} \in P)(\forall\bar{x} \in P)[\varphi(\bar{x}\bar{y}) \leftrightarrow \psi^P(\bar{x}\bar{z})]$ holds), then T^- is λ-stable iff T is λ-stable and T^- is unidimensional iff T is unidimensional.

In this paper, we mainly deal with the $\emptyset$-reduction property (definition 2.1). The $\emptyset$-reduction property together with the minimality condition ensures that T is not far from T^- if T is stable. But the $\emptyset$-reduction property is not so strong for an unstable theory. In fact there is a theory T such that T has the $\emptyset$-reduction property over P, is minimal over P, and the number of models of T is more than that of T^-.

EXAMPLE. Let A be a model whose theory has uncountably many countable models. Let

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(i) $L = \{P, R\} \cup$ the language of A,
(ii) $L^- = L - \{P\}$,
(iii) $M = \langle M; P^M, R^M, \text{the structure of } A\rangle$,
(iv) $M = A \cup B \cup C$,
(v) $P^M = B \cup C$,
(vi) C is the set of all bijections from A to B,
(vii) $R^M(x, y, f)$ iff $x \in A \wedge y \in B \wedge f \in C \wedge f(x) = y$.

Since $(\forall xyz)\neg R(x, y, z)$ holds in $N, T = Th(M)$ has the $\varnothing$-reduction property over P and $T^- = Th(N)$ has a unique countable model. T is minimal over P because each element of C is a bijection from A to B. But T has uncountably many countable models since $Th(A)$ does.

When T is superstable, we can easily prove that if T has the $\varnothing$-reduction property over P and is minimal over P, then the number of a-models of T is equal to that of T^- (corollary 3.2). Furthermore, we prove the following:

THEOREM (4.1, 5.1). Let T be a superstable theory with the $\varnothing$-reduction property over P. If T is minimal over P, then

(1) T has the DOP (dimensional order property) iff T^- has the DOP.
(2) T is deep iff T^- is deep.

For a stable theory, the $\varnothing$-reduction property implies a reduction property for formulas with parameters (in [KT], it was called the reduction property). The key point of the proofs of the above theorems is that under this reduction property, a type of an element in N is determined by its "L^--reduction" of the type (see lemma 2.4).

2. Preliminaries

Let T be a theory with a relational language L including a unary predicate P. Let M be a model of T and N the L^--structure $P^M = \{a \in |M| : M \models P(a)\}$ where $L^- \subset L - \{P\}$. As usual, we work in the big model $\mathcal{M}$ of T. We may assume that a model of $T^- = Th(N)$ is an L^--elementary substructure of $P^{\mathcal{M}}$. The character M will denote an elementary submodel of $\mathcal{M}$ in T and the character N will denote the set P^M which is a model of T^-. M and N may have subscript. For notational convenience, we usually assume that if M and N have the same subscript then N is the restriction of M to P. For example, N_i for P^{M_i}.

We write $\bar{a}, \bar{b}, \ldots$ for finite tuples of elements of $\mathcal{M}$ and $\bar{x}, \bar{y}, \ldots$ for finite tuples of variables. When φ is an L^--formula, we write φ^P for the restriction of φ, that is, the formula obtained from φ by restricting all the variables to P. For example, if $\varphi(x) = (\forall y)(\exists z)\psi(xyz)$ and ϕ is open, then $\varphi^P(x) \equiv (\forall y)(P(y) \rightarrow (\exists z)(\psi(xyz) \wedge P(x) \wedge P(z)))$. We write $(\forall \bar{x} \in P)\varphi(\bar{x}\bar{y})$ to express the formula $(\forall x_1 \cdots \forall x_n)[P(x_1) \wedge \cdots \wedge P(x_n) \rightarrow \varphi(\bar{x}\bar{y})]$ where $\bar{x} = x_1 \cdots x_n$.

DEFINITION 2.1 ([KT, definition 1]). (1) *We say that M has the $\emptyset$-reduction property over N if every $L(\emptyset)$-definable relation on N is $L^-(\emptyset)$-definable in N, i.e., for any $L(\emptyset)$-formula $\varphi(\bar{x})$, there is an $L^-(\emptyset)$-formula $\psi(\bar{x})$ such that $M \models (\forall \bar{x} \in P)[\varphi(\bar{x}) \leftrightarrow \psi^P(\bar{x})]$.*

(2) *We say that M has the reduction property over N if every $L(M)$-definable relation on N is $L^-(N)$-definable in N.*

If some model of T has the $\emptyset$-reduction property over N, then every model of T has the property. So we say that T has the $\emptyset$-reduction property over P if some model of T has this property.

The following lemma was used in [KT] without proof. For the sake of completeness, we prove it here.

LEMMA 2.2 ([KT, pp. 902]). *If T is stable and has the $\emptyset$-reduction property over P, then every model M of T has the reduction property over N.*

PROOF. Let M be a model of T, $\varphi(\bar{x}\bar{y})$ an L-formula, and $\bar{a} \in M$. We want to find an L^--formula $\psi(\bar{x}\bar{z})$ and a tuple $\bar{b} \in N$ such that $M \models (\forall \bar{x} \in P)[\varphi(\bar{x}\bar{a}) \leftrightarrow \psi^P(\bar{x}\bar{b})]$. For this, it is sufficient to show that $M \models [\varphi(\bar{c}\bar{a}) \leftrightarrow \psi^P(\bar{c}\bar{b})]$ for every $\bar{c} \in N$. By the stability, there are an L-formula $\varphi'(\bar{x}\bar{z})$ and a tuple $\bar{b} \in N$ such that $M \models \varphi(\bar{c}\bar{a}) \leftrightarrow \varphi'(\bar{c}\bar{b})$ for every $\bar{c} \in N$. By the $\emptyset$-reduction property, there is an L^--formula $\psi(\bar{x}\bar{z})$ such that $M \models (\forall \bar{x}\bar{z} \in P)[\varphi'(\bar{x}\bar{z}) \leftrightarrow \psi^P(\bar{x}\bar{z})]$. □

DEFINITION 2.3 ([KT, definition 1]). *We say that T is minimal over P if every model M of T is a minimal model over N.*

We write $tp^-(\bar{a}/B)$ for the L^--type of $\bar{a} \in N$ over $B \subset N$.

LEMMA 2.4. *Let $\bar{a}$ and $\bar{b}$ be tuples from $P^{\mathcal{M}}$.*

(1) *$tp(\bar{a}/M)$ does not fork over N.*

(2) *If M has the reduction property over N, then $tp(\bar{a}/M) = tp(\bar{b}/M)$ iff $tp^-(\bar{a}/N) = tp^-(\bar{b}/N)$.*

(3) *If M has the reduction property over N, then $tp(\bar{a}/M\bar{b})$ does not fork over M iff $tp^-(\bar{a}/N\bar{b})$ does not fork over N in the sense of T^-.*

(4) *If T has the $\varnothing$-reduction property over P, then $tp(\bar{a}/N)$ is stationary in the sense of T.*

(*For only if part of* (2) *and* (3), *we don't need the reduction property.*)

PROOF. (1) Let $\varphi(\bar{x})$ be a formula in $tp(\bar{a}/M)$. We show that $\varphi(\bar{x})$ has a realization in N. Since $M \models (\exists\bar{x})(\varphi(\bar{x}) \wedge \bar{x} \in P)$, we can find a tuple $\bar{b} \in N$ realizing $\varphi(\bar{x})$.

(2) ($\Rightarrow$) Clear.

($\Leftarrow$) Let $\varphi(\bar{x})$ be a formula in $tp(\bar{a}/M)$. By the reduction property, there is an $L^-(N)$ formula $\psi(\bar{x})$ such that $M \models (\forall\bar{x} \in P)[\varphi(\bar{x}) \leftrightarrow \psi^P(\bar{x})]$. Hence $\psi(\bar{x}) \in tp^-(\bar{a}/N)$. And $\psi(\bar{x}) \in tp^-(\bar{b}/N)$ since $tp^-(\bar{a}/N) = tp^-(\bar{b}/N)$. Hence $\varphi(\bar{x}) \in tp(\bar{b}/M)$. So $tp(\bar{a}/M) = tp(\bar{b}/M)$.

(3) ($\Rightarrow$) We show that $tp^-(\bar{a}/N\bar{b})$ is an heir over N in the sense of T^-. If $\varphi(\bar{x}\bar{b}) \in tp^-(\bar{a}/N\bar{b})$ and $\varphi(\bar{x}\bar{y})$ is an $L^-(N)$-formula, then $\varphi^P(\bar{x}\bar{b}) \in tp(\bar{a}/M\bar{b})$. Since $\varphi^P(\bar{x}\bar{y})$ is an $L(M)$-formula, we can find a tuple $\bar{b}' \in N$ such that $\varphi^P(\bar{x}\bar{b}') \in tp(\bar{a}/M\bar{b})$. Hence $\varphi(\bar{x}\bar{b}') \in tp^-(\bar{a}/N\bar{b})$.

($\Leftarrow$) We show that $tp(\bar{a}/M\bar{b})$ is an heir over M. If $\varphi(\bar{x}\bar{b}) \in tp(\bar{a}/M\bar{b})$ and $\varphi(\bar{x}\bar{y})$ is an $L(M)$-formula, then, by the reduction property, there is an $L^-(N)$-formula $\psi(\bar{x}\bar{y})$ such that $M \models (\forall\bar{x}\bar{y} \in P)[\psi^P(\bar{x}\bar{y}) \leftrightarrow \varphi(\bar{x}\bar{y})]$. Since $\psi(\bar{x}\bar{b}) \in tp^-(\bar{a}/N\bar{b})$ and $\psi(\bar{x}\bar{y})$ is an $L^-(N)$-formula, we can find a tuple $\bar{b}' \in N$ such that $\psi(\bar{x}\bar{b}') \in tp^-(\bar{a}/N\bar{b})$. Hence $\varphi(\bar{x}\bar{b}') \in tp(\bar{a}/M\bar{b})$.

(4) If $tp(\bar{a}/N)$ is not stationary in the sense of T, then we can find tuples $\bar{b}, \bar{c} \in P^{\mathcal{M}}$ and a model $M' \supset N$ of T such that $tp(\bar{b}/M')$ and $tp(\bar{c}/M')$ are different non-forking extensions of the type $tp(\bar{a}/M)$. By (2) and (3), $tp^-(\bar{b}/N')$ and $tp^-(\bar{c}/N')$ are different non-forking extensions of the type $tp^-(\bar{a}/N)$ in the sense of T^-. This contradicts the stationarity of $tp^-(\bar{a}/N)$ in the sense of T^-. $\square$

Note. We can see $\kappa_r(T) \geq \kappa_r(T^-)$ from (3).

3. a-models of T and T^-

If M is an a-model of T, then N is an a-model of T^- because $\kappa_r(T) \geq \kappa_r(T^-)$. The following lemma shows that for any a-model N of T^-, there is an a-model M such that $P^M = N$.

LEMMA 3.1. *Assume that T is stable, has the $\emptyset$-reduction property over P, and $\kappa_r(T) = \kappa_r(T^-)$. Let N be an a-model of T^-. If M' is an a-prime model over N in the sense of T, then $N' = N$.*

PROOF. If $N' \neq N$, we can choose an element $a \in N' - N$. Since M' is a-prime over N, there is a subset $A \subset N$ such that $|A| < \kappa_r(T)$ and $stp(a/A)$ isolates $tp(a/N)$ in the sense of T. Let $\{E_i(xa; \bar{b}_i) : \bar{b}_i \in A, i < \lambda\}$ be an enumeration of $stp(a/A)$ where each $E_i(xy; \bar{b}_i)$ is a finite equivalence relation over A in the sense of T. By the $\emptyset$-reduction property, for each $i < \lambda$, we can find an L^--formula $E_i'(xy; \bar{z}_i)$ such that $(\forall xy\bar{z}_i \in P)[E_i(xy; \bar{z}_i) \leftrightarrow E_i'^P(xy; \bar{z}_i)]$ holds in M. $E_i'(xy; \bar{b}_i)$ is a finite equivalence relation over A in the sense of T^- because so is $E_i(xy; \bar{b}_i)$ in the sense of T. Hence $\{E_i'(xy; \bar{b}_i) : i < \lambda\} \subset stp^-(a/A)$. If we choose an element $c \in N$ as a realization of $stp^-(a/A)$, c also realizes $stp(a/A)$, so c realizes $tp(a/N)$. Hence $tp(a/N) = tp(c/N)$. But this is a contradiction since $a \notin N$. □

COROLLARY 3.2. *Let T be stable, minimal over P, has the $\emptyset$-reduction property over P, and $\kappa_r(T) = \kappa_r(T^-)$.*

(1) *For every a-model N of T^-, there is a unique a-model M of T such that $P^M = N$. Moreover, $|M| = |N|$.*

(2) *Every a-model M of T is a-prime over $N = P^M$.*

(3) *The map which takes M to $N = P^M$ is a bijection between a-models of T and a-models of T^- of the same cardinality.*

When T is superstable, the number of a-models is classified by the dimensional order property (definition 4.2) and the deepness (definition 5.4) ([Sh]). We consider these properties in following two sections.

4. Dimensional order property

In this section, we show:

THEOREM 4.1. *If T is superstable, has the $\emptyset$-reduction property over P and is minimal over P, then T has the DOP (dimensional order property) iff T^- has the DOP. The minimality condition is not necessary for if part.*

First, we recall the definition of the dimensional order property.

DEFINITION 4.2 ([Sh, definition X.2.1]). *let T be superstable. We say that T has the dimensional order property (DOP) if there are a-models $M_i (i = 0,1,2,3)$ and a regular type $p \in S(M_3)$ such that:*

- $M_0 \prec M_1, M_2$,
- *M_1 and M_2 are independent over M_0,*
- *M_3 is a-prime over $M_1 M_2$,*
- *p is orthogonal to M_1 and M_2.*

To prove the theorem 4.1, we need the following three lemmas.

LEMMA 4.3. *Assume that T is stable, has the $\varnothing$-reduction property over P and $\kappa_r(T) = \kappa_r(T^-)$. Let N_0, N_1 be a-models, $N_0 \prec N_1$, M_0 a-prime over N_0, and M_2 an a-prime model over $M_0 N_1$. Then M_2 is a-prime over N_1 and $P^{M_2} = N_1$.*

PROOF. Let M_1 be an a-prime model over N_1. By lemma 3.1, $P^{M_0} = N_0$ and $P^{M_1} = N_1$. We can find an elementary submodel $M_0' \prec M_1$ which is isomorphic to M_0 over N_0 since M_0 is a-prime over N_0. By lemma 2.4(1)(4), $tp(N_1/N_0)$ is stationary and $tp(N_1/M_0')$ does not fork over N_0. Hence $tp(M_0'/N_1) = tp(M_0/N_1)$. So we can embed M_2 into M_1 over N_1. Hence M_2 is a-prime over N_1 and $P^{M_2} = N_1$. □

LEMMA 4.4. *Assume that T is stable, has the $\varnothing$-reduction property over P and $\kappa_r(T) = \kappa_r(T^-)$. Let $\bar{a}$ be a tuple from $P^{\mathcal{M}}$. Then $tp(\bar{a}/M)$ is regular in the sense of T iff $tp^-(\bar{a}/N)$ is regular in the sense of T^-.*

PROOF. ($\Rightarrow$) If $tp^-(\bar{a}/N)$ is not regular in the sense of T^-, then there is a forking extension $tp^-(\bar{b}/C)$ which is not orthogonal to $tp^-(\bar{a}/N)$ in the sense of T^-. We may assume that

- $tp^-(\bar{a}/C)$ does not fork over N in the sense of T^-,
- $\bar{a}$ and $\bar{b}$ are dependent over C in the sense of T^-.

Let N' be an a-model of T^- such that $C \subset N'$ and $tp^-(\bar{a}\bar{b}/N')$ does not fork over C. By lemma 3.1, there is an a-model M' of T such that $P^{M'} = N'$. By lemma 2.4,

- $tp(\bar{a}/M')$ does not fork over M,
- $tp(\bar{b}/M')$ is a forking extension of $tp(\bar{a}/M)$,
- $\bar{a}$ and $\bar{b}$ are dependent over M'.

Hence $tp(\bar{a}/M)$ is not regular.

($\Leftarrow$) If $tp(\bar{a}/M)$ is not regular, there is a forking extension $tp(\bar{b}/C)$ which is not orthogonal to $tp(\bar{a}/M)$. We may assume that:

- $tp(\bar{a}/C)$ does not fork over M,
- $\bar{a}$ and $\bar{b}$ are dependent over C in the sense of T.

Let M' be a model of T such that $C \subset M'$ and $tp(\bar{a}\bar{b}/M')$ does not fork over C. By lemma 2.4,

- $tp^-(\bar{a}/N')$ does not fork over N in the sense of T^-,
- $tp^-(\bar{b}/N')$ is a forking extension of $tp^-(\bar{a}/N)$ in the sense of T^-,
- $\bar{a}$ and $\bar{b}$ are dependent over N' in the sense of T^-.

Hence $tp^-(\bar{a}/N)$ is not regular in the sense of T^-. □

LEMMA 4.5. *Assume that T is stable, has the $\emptyset$-reduction property over P. Let $\bar{a}$ be a tuple from $P^{\mathcal{M}}$ and $M_0 \prec M_1$.*

(1) *If T is minimal over P, M_0 and M_1 a-models, and $tp(\bar{a}/M_1)$ orthogonal to M_0, then $tp^-(\bar{a}/N_1)$ is orthogonal to N_0 in the sense of T^-.*

(2) *If $tp^-(\bar{a}/N_1)$ is orthogonal to N_0 in the sense of T^-, then $tp(\bar{a}/M_1)$ is orthogonal to M_0.*

PROOF. (1) If $tp^-(\bar{a}/N_1)$ is non-orthogonal to N_0 in the sense of T^-, we can choose a tuple $\bar{b} \in P^{\mathcal{M}}$ such that:

- $tp^-(\bar{b}/N_1)$ does not fork over N_0 in the sense of T^-,
- $\bar{a}$ and $\bar{b}$ are dependent over N_1 in the sense of T^-.

By lemma 4.3 and corollary 3.2(1)(2), M_1 is a-prime over M_0N_1. Hence, by lemma 2.4(3),

- $tp(\bar{b}/M_1)$ does not fork over M_0.
- $\bar{a}$ and $\bar{b}$ are dependent over M_1.

This shows that $tp(\bar{a}/M_1)$ is non-orthogonal to M_0.

(2) We use the following fact.

FACT 4.6 ([Sh, V.3.4]). *Suppose that $A \subset B$ and $p \in S(B)$ is a stationary type. Let f be an elementary mapping whose domain is B such that $f|A$ is the identity, $stp(B/A) \equiv stp(f(B)/A)$, and $stp(f(B)/B)$ does not fork over A. Then p is orthogonal to A iff p is orthogonal to $f(p)$.*

Let f be an elementary mapping whose domain is M_1 such that $f|M_0$ is the identity, $stp(M_1/M_0) \equiv stp(f(M_1)/M_0)$, and $stp(f(M_1)/M_1)$ does not fork over M_0 in the sense of T. Then, by lemma 2.4, $f|N_1$ is an elementary mapping whose domain is N_1, $f|N_0$ the identity, $stp^-(N_1/N_0) \equiv stp^-(f(N_1)/N_0)$, and $stp^-(f(N_1)/N_1)$ does not fork over N_0 in the sense of T^-. By fact 4.6, if $tp(\bar{a}/M_1)$ is non-orthogonal to M_0, then $tp(\bar{a}/M_1)$ is non-orthogonal to

$tp(f(\bar{a})/f(M_1))$. We may assume that $tp(\bar{a}/M_2)$ and $tp(f(\bar{a})/M_2)$ do not fork over M_1 and $f(M_1)$ respectively, $\bar{a}$ and $f(\bar{a})$ are dependent over M_2, where M_2 is a model containing M_1 and $f(M_1)$. By lemma 2.4, $tp^-(\bar{a}/N_2)$ and $tp^-(f(\bar{a})/N_2)$ do not fork over N_1 and $f(N_1)$ respectively, $\bar{a}$ and $f(\bar{a})$ are dependent over N_2 in the sense of T^-. Hence $tp^-(\bar{a}/N_1)$ is non-orthogonal to $tp^-(f(\bar{a})/f(N_1))$ in the sense of T^-. By fact 4.6, $tp^-(\bar{a}/N_1)$ is non-orthogonal to N_0 in the sense of T^-. □

The proof of theorem 4.1 will be completed by following two lemmas.

LEMMA 4.7. *Let T be superstable, has the $\emptyset$-reduction property over P and is minimal over P. If T has the DOP then T^- has the DOP.*

PROOF. Since T has the DOP, there are a-models $M_i (i = 0, 1, 2, 3)$ and a regular type $p = tp(\bar{a}/M_3)$ witnessing the conditions for the DOP. By the minimality of T over P, we can choose an element $b \in P^{M_3[\bar{a}]} - N_3$ such that $tp^-(b/N_3)$ is regular in the sense of T^- where $M_3[\bar{a}]$ denotes an a-prime model over $M_3\bar{a}$. We show that $N_i (i = 0, 1, 2, 3)$ and $tp(\bar{b}/N_3)$ witness the DOP of T^-. By lemma 2.4(3), N_1 and N_2 are independent over N_0. N_3 is a-prime over N_1N_2: Let N_4 be an a-model containing N_1 and N_2 in the sense of T^-. Let M_4 be an a-prime model over N_4 in the sense of T. By lemma 3.1, $P^{M_4} = N_4$. By lemma 3.2(2), we can embed M_0 into M_4 over N_0. By lemma 2.4, this embedding does not change the type of $N_1N_2M_0$. Hence we can assume that $M_0 \prec M_4$. Let $M_1' \prec M_1$ be an a-prime model over M_0N_1, then $P^{M_1'} = N_1$. By the minimality of T over P, $M_1 = M_1'$. Hence M_1 is an a-prime model over M_0N_1. Similarly, M_2 is an a-prime model over M_0N_2. Hence we can embed M_1 and M_2 into M_4 over M_0N_1 and M_0N_2 respectively. By lemma 2.4, this embedding does not change the type of M_1M_2. Hence we may assume that $M_1M_2 \subset M_4$ and can embed M_3 in M_4 over M_1M_2. Then N_3 is embedded in N_4 over N_1N_2.

$tp^-(b/N_3)$ is orthogonal to N_1 and N_2 in the sense of T^-: Since $tp(\bar{a}/M_3)$ and $tp(b/M_3)$ are dependent, $tp(b/M_3)$ is orthogonal to M_1 and M_2. By lemma 4.5, $tp^-(b/N_3)$ is orthogonal to N_1 and N_2 in the sense of T^-.

Hence T^- has the DOP. □

LEMMA 4.8. *Let T be superstable, has the $\emptyset$-reduction property over P. If T^- has the DOP then T has the DOP.*

PROOF. Since T^- has the DOP, there are a-models $N_i (i = 0, 1, 2, 3)$ and a regular type $p = tp^-(\bar{a}/N_3)$ witnessing the conditions for the DOP.

Let M_0 be an a-prime model over N_0 in the sense of T. Let M_1 and M_2 be a-prime models over M_0N_1 and M_0N_2 respectively. By lemma 2.4(3), M_1 and M_2 are independent over M_0. By lemma 4.3, $P^{M_i} = N_i (i = 0, 1, 2)$. Let M_4 be an a-prime model over M_1M_2. We may assume that $N_3 \prec N_4$ since N_3 is a-prime over N_1N_2. Let $tp^-(\bar{b}/N_4)$ be a non-forking extension of p in the sense of T^-. Then $tp^-(\bar{b}/M_4)$ is orthogonal to N_1 and N_2 in the sense of T^-. By lemma 4.5, $q = tp(\bar{b}/N_4)$ is orthogonal to M_1 and M_2. Hence $M_i (i = 0, 1, 2, 4)$ and q witness the conditions for the DOP. □

By lemmas 4.7 and 4.8, we completed the proof of theorem 4.1.

In lemma 4.7, we assumed the $\emptyset$-reduction property and the minimality. In lemma 4.8, we assumed the $\emptyset$-reduction property. The following example shows that we can not weaken these assumptions.

EXAMPLE 4.9. (1) *The following example shows that the minimality condition is necessary for lemma 4.7. Let $E_1(xy)$ and $E_2(xy)$ be crosscutting equivalence relations where the number of E_i-classes is infinite $(i = 1, 2)$ and each E_1-E_2-class is infinite. Let $L = \{P, E_1, E_2\}$ where P is contained in an E_1-E_2-class. Let $L^- = L - \{P\}$. Since the structure of N is only equality, T has the $\emptyset$-reduction property and T^- does not have the DOP. Since each E_1-E_2-class may have various infinite cardinality, T has the DOP and is not minimal over P.*

(2) *The following example shows that the $\emptyset$-reduction property is necessary for lemma 4.7. Let $E_1(xy)$ and $E_2(xy)$ be crosscutting equivalence relations where the number of E_i-classes is infinite $(i = 1, 2)$ and each E_1-E_2-class is infinite. Let $L = \{P, E_1, E_2\}$ where $P \equiv$ "$x = x$". Let $L^- = \emptyset$. Then T is minimal over P since $P^M = M$. But T does not have the $\emptyset$-reduction property over P and T^- does not have the DOP since the structure of N is only equality. T has the DOP as in* (1).

(3) *The following example shows that the $\emptyset$-reduction property is necessary for lemma 4.8. Let $E_1(xy)$, $E_2(xy)$ and $E_3(xy)$ be crosscutting equivalence relations where the number of E_i-classes is infinite $(i = 1, 2, 3)$, each E_1-E_2-class is infinite and each E_1-E_2-E_3-class is a singleton. Let $L = \{P, E_1, E_2, E_3\}$ where $P \equiv$ "$x = x$". let $L^- = \{E_1, E_2\}$. Then T does not have the DOP and the $\emptyset$-reduction property over P since the structure of M is restricted by E_3. T is minimal over P since $P^M = M$. T^- has the DOP as in* (1).

5. Deepness

In this section, we show:

THEOREM 5.1. *If T is superstable, has the $\emptyset$-reduction property over P and is minimal over P, then T is deep iff T^- is deep. The minimality condition is not necessary for if part.*

The following definitions are from [Sh].

DEFINITION 5.2 ([Sh, definition X.1.2]). *For $A \subset B \subset C$ we say $B <_A C$ if for every $\bar{c} \in C$, $tp(\bar{c}/B)$ is orthogonal to A.*

DEFINITION 5.3 ([Sh, definition X.4.1]). *Let $K = \{(M, M', \bar{a}) : tp(\bar{a}/M)$ is regular, M' is a-prime over $M\bar{a}$, and M is an a-model$\}$.*

For every member of K we define its depth, an ordinal (zero or successor but not limit) or infinity ∞, by:

(1) *$Dp(M, M', \bar{a}) \geq 0$ iff $(M, M', \bar{a}) \in K$,*

(2) *$Dp(M, M', \bar{a}) \geq \alpha + 1$ (α zero or successor) iff for some M'', $\bar{a}'$: $(M', M'', \bar{a}') \in K$, $M' <_M M''$ and $Dp(M', M'', \bar{a}') \geq \alpha$,*

(3) *$Dp(M, M', \bar{a}) \geq \delta + 1$ (δ limit) iff $Dp(M, M', \bar{a}) \geq \beta$ for $\beta < \delta$,*

(4) *$Dp(M, M', \bar{a}) = \infty$ iff for every ordinal β $Dp(M, M', \bar{a}) \geq \beta$, $Dp(M, M', \bar{a}) = \alpha$ iff $Dp(M, M', \bar{a}) \geq \alpha$ but not $Dp(M, M', \bar{a}) \geq \alpha + 1$.*

DEFINITION 5.4 ([Sh, definition X.4.2]). (1) *The depth of the theory $Dp(T)$ is $\sup\{Dp(M, M', \bar{a}) : (M, M', \bar{a}) \in K\}$ when this is finite and $\sup\{Dp(M, M', \bar{a}) : (M, M', \bar{a}) \in K\} + 1$ when this is infinite.*

(2) *We say the theory T is deep if its depth is ∞; otherwise it is shallow.*

The proof of theorem 5.1 will be completed by following two lemmas.

LEMMA 5.5. *Suppose that T is superstable, has the $\emptyset$-reduction property over P and is minimal over P. If $Dp(M, M', \bar{a}) \geq \alpha$, then there is an element $b \in N'$ such that $(N, N', b) \in K$ and $Dp(N, N', b) \geq \alpha$.*

PROOF. We prove the lemma by the induction on α.
($\alpha = 0$) If $Dp(M, M', \bar{a}) \geq 0$ then

- $tp(\bar{a}/M)$ is regular,
- M' is a-prime over $M\bar{a}$,
- M is a-model.

N and N' are a-models of T^-. By the minimality of T over P, we can choose an element $b \in N' - N$ such that $tp^-(b/N)$ is regular in the sense of T^-. $tp(b/M)$ is also regular by lemma 4.4. Hence M' is a-prime over Mb. Hence N' is a-prime over Nb in the sense of T^-: Assume that N_0 is an a-model containing N and b. Let M_0 be an a-prime model over MN_0. By lemma 4.3, $P^{M_0} = N_0$. Since M' is a-prime over Mb, we can embed M' into M_0 over Mb. Hence N' is embedded in N_0 over Nb.

($\alpha = \beta + 1$) If $Dp(M, M', \bar{a}) \geq \beta + 1$, there are a model M'' and a tuple $\bar{a}' \in \mathcal{M}$ such that

- $Dp(M', M'', \bar{a}') \in K$,
- $M' <_M M''$,
- $Dp(M', M'', \bar{a}') \geq \beta$.

As in the case $\alpha = 0$, there is an element $b \in N'$ such that $(N, N', b) \in K$ in the sense of T^-. By the induction hypothesis, there is an element $b' \in M''$ such that $(N', N'', b') \in K$ and $Dp(N, N'', b') \geq \beta$ in the sense of T^-. Hence it is sufficient to show $N' <_N N''$ in the sense of T^-. If not, there are tuples $\bar{c} \in N''$ such that $tp^-(\bar{c}/N')$ is non-orthogonal to N in the sense of T^-. By lemma 4.5, $tp(\bar{c}/M')$ is non-orthogonal to M. This is a contradiction since $M' <_M M''$.

($\alpha = \delta + 1$ where δ is limit or $\alpha = \infty$) Clear. □

Lemma 5.6. *Suppose that T is superstable,* has the $\varnothing$-reduction property over P. *If $Dp(N, N', \bar{a}) \geq \alpha$ in the sense of T^-, then there are a-models M and M' of T such that $P^M = N$, $P^{M'} = N'$ and $Dp(M, M', \bar{a}) \geq \alpha$.*

Proof. We prove the lemma by the induction on α. Let M be an a-prime model over N, and M' an a-prime model over MN'. By lemma 3.1 and lemma 4.3, $P^M = N$ and $P^{M'} = N'$.

($\alpha = 0$) If $Dp(N, N', \bar{a}) \geq 0$ in the sense of T^-, then:

- $tp^-(\bar{a}/N)$ is regular in the sense of T^-,
- N' is a-prime over $N\bar{a}$ in the sense of T^-,
- N is an a-model in the sense of T^-.

$tp(\bar{a}/M)$ is regular by lemma 4.4. Since M' is a-prime over MN', it is a-prime over $M\bar{a}$: Assume that M_0 is an a-model containing M and a. We can embed N' into N_0 over Na. By lemma 2.4(4), we know that this embedding does not change the type of MN'. Hence we can embed M' into M_0 over MN'.

($\alpha = \beta + 1$) If $Dp(N, N', \bar{a}) \geq \beta + 1$, there are a model N'' and a tuple $\bar{a}' \in P^{\mathscr{M}}$ such that:

- $(N', N'', \bar{a}') \in K$ in the sense of T^-,
- $N' <_N N''$ in the sense of T^-,
- $Dp(N', N'', \bar{a}') \geq \beta$ in the sense of T^-.

Let M'' be an a-prime model over $M'N''$. By lemma 4.3, $P^{M''} = N''$. As in the case $\alpha = 0$, we can show that $(M', M'', \bar{a}') \in K$. $Dp(M', M'', \bar{a}') \geq \beta$ by the induction hypothesis. We show $M' <_M M''$. If not, there is a type over M which is not orthogonal to $tp(M''/M')$. Let $tp(\bar{c}/M')$ be a non-forking extension of the type such that $\bar{c}$ and M'' are dependent over M'. Since M'' is a-prime over $M'N''$, $\bar{c}$ and N'' are dependent over M'. Hence $tp(N''/M')$ is non-orthogonal to M. By lemma 4.5, $tp^-(N''/N')$ is non-orthogonal to N in the sense of T^-. This is a contradiction since $N' <_N N''$.

($\alpha = \delta + 1$ where δ is limit or $\alpha = \infty$) Clear. □

By lemma 5.5 and lemma 5.6, $Dp(T) = Dp(T^-)$. Hence the proof of theorem 5.1 was completed.

6. Countable stable theories

In lemma 3.1, we showed that for any a-model N of T^-, there is an a-model M of T such that $P^M = N$. In this section, we sow that if T is countable and stable then for any model N of T^- there is a model M of T with $P^M = N$.

DEFINITION 6.1. *Let $A \subset B$. We say that B is locally atomic over A if for any $\bar{c} \in B$ and a formula $\varphi(\bar{x}\bar{y})$, there is a formula $\psi(\bar{x}) \in p$ such that $\psi(\bar{x})$ isolates $p|_\varphi$ where $p = tp(\bar{c}/A)$ and $p|_\varphi = \{\varphi(\bar{x}\bar{b}) : \varphi(\bar{x}\bar{b}) \in p\}$.*

The following fact is essential for theorem 6.3 below.

FACT 6.2 ([Sh, IV.3.1]). *Let T be countable and stable. For any set A, there is a locally atomic model over A.*

THEOREM 6.3. *Suppose that T is countable, stable and has the $\varnothing$-reduction property. Let N be a model of T^- and M' a locally atomic model over N. Then $N' = N$.*

PROOF. If $N' \neq N$, then we can choose $a \in N' - N$. Since M' is locally atomic over N, there is a formula $\varphi(x\bar{b}) \in p = tp(a/N)$ such that $\varphi(x\bar{b})$ isolates

$p|_{x \neq y}$ in the sense of T. By the $\emptyset$-reduction property, there is an L^--formula $\psi(x\bar{y})$ such that $(\forall x\bar{y} \in P)[\varphi(x\bar{y}) \leftrightarrow \psi^P(x\bar{y})]$ holds. Since $M' \models \varphi(a\bar{b})$, we can see that $N' \models (\exists x)\psi(x\bar{b})$. Hence $N \models (\exists x)\psi(x\bar{b})$. Let $c \in N$ be a witness of $\psi(x\bar{b})$. Then c realizes $p|_{x \neq y}$ by the choice of φ and ψ. This is a contradiction since $x \neq c \in p|_{x \neq y}$. □

The following example shows that the countable condition is necessary for theorem 6.3.

EXAMPLE 6.4. *There is a stable uncountable theory T and a model N of T^- such that no model M of T satisfies $P^M = N$.*

Let $L = \{P, c_i(i < \omega), F_\eta(\eta \in 2^\omega), R_\eta(\eta \in 2^{<\omega})\}, L^- = L - \{P\}$ *and* $M = \langle M; N, c_i^M(i < \omega), F_\eta^M(\eta \in 2^\omega),\ R_\eta^M(\eta \in 2^{<\omega})\rangle$ *where*

(i) $N = \{c_i^M : i < \omega\} \cup \{a\}$,

(ii) $M = N \cup \{b_\eta : \eta \in 2^\omega\}$,

(iii) F_η^M *is a function from* $M - N$ *to* N,

(iv) $F_\eta^M(b_\nu) = c_i^M \Leftrightarrow \eta|i = \nu|i$ *and* $\eta(i) \neq \nu(i)$,

(v) $F_\eta^M(b_\nu) = a \Leftrightarrow \eta = \nu$,

(vi) $R_\eta^M(b_\nu) \Leftrightarrow \eta$ *is an initial segment of* ν.

Then T is stable and has the $\emptyset$-reduction property over P since any definable set in N is definable by c_is. Let $N' = \{c_i^M : i < \omega\}$, then N' is a model of T^-. But there is no model M' such that $P^{M'} = N'$ because $tp(a)$ does not have the realization in N'.

The following example shows that the stability is necessary for theorem 6.3.

EXAMPLE 6.5. *There is a countable unstable theory T and a model N of T^- such that no model M of T satisfies $P^M = N$.*

Let $L = \{P, R, U_i(i = 1,2,3)\}$, $L^- = L - P$ *and* $M = \langle M; N, R^M, U_i^M(i = 1,2,3)\rangle$ *where*

(i) $M = U_1^M \cup U_2^M \cup U_3^M$ *where* U_3^M *is the set of all functions from* U_1^M *to* U_2^M,

(ii) $N = U_1^M \cup U_2^M$,

(iii) $R^M(x,y,f) \Leftrightarrow x \in U_1^M \wedge y \in U_2^M \wedge f \in U_3^M \wedge f(x) = y$,

(iv) U_1^M *and* U_2^M *are countable.*

Then T is a countable theory with the $\emptyset$-reduction property over P since the structure of M depends on U_3^M. Let N' be a model of T^- such that $|U_1^{N'}| \neq |U_2^{N'}|$. But there is no model M' such that $P^{M'} = N'$ because $|U_1^M| = |U_2^M|$ holds in every model M of T.

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