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THE SUM OF CONSECUTIVE FRACTIONAL PARTS

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1. Introduction

If we sum any h consecutive terms in the sequence

$$\{\theta\}, \{2\theta\}, \{3\theta\}, \ldots,$$

what are reasonable bounds for the sum? Of course, it is at least as large as 0 and no more than h. But that is far too too rough. Thus we consider the following problem:

What is the least upper bound and the greatest lower bound of

$$B_h(\theta) = \sum_{i=1}^{h} \{(N+i)\theta\}?$$

Here, θ is a given irrational number, and h is some positive integer. The variable N is restricted to the non-negative integers.

In the case of just one fractional part the bounds are known. As remarked in [2] we have

THEOREM 1. *Let the continued fraction expansion of θ be*

$$\theta = [a_0, a_1, a_2, \ldots],$$

and denote by $q_n (= a_n q_{n-1} + q_{n-2})$ the denominator of the nth convergent

$$[a_0, a_1, \ldots, a_n].$$

Then if q is an integer satisfying $0 < q < q_n$,

$$\begin{cases} \{q_{n-1}\theta\} \le \{q\theta\} \le \{(q_n - q_{n-1})\theta\} & \text{when } n \text{ is odd;} \\ \{(q_n - q_{n-1})\theta\} \le \{q\theta\} \le \{q_{n-1}\theta\} & \text{when } n \text{ is even.} \end{cases}$$

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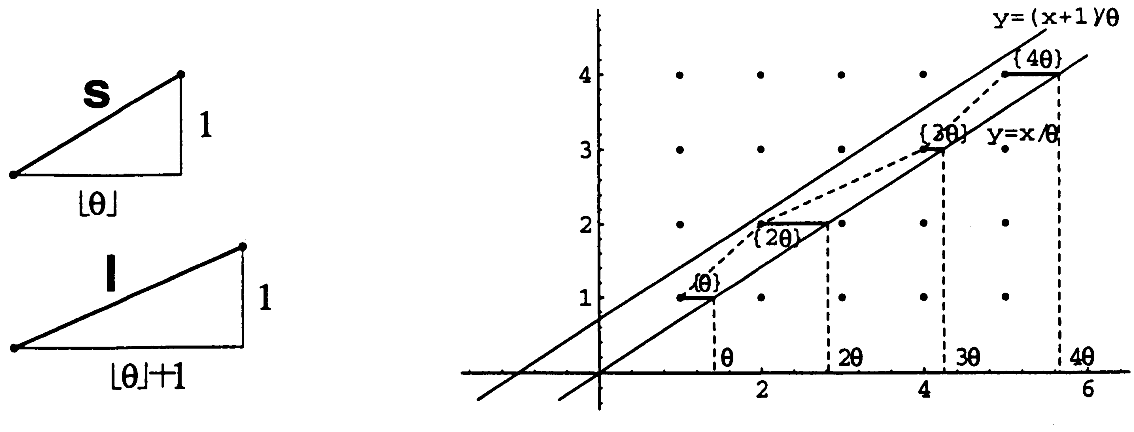

Figure 1

Then, when $h \geq 2$, how can we express the exact bounds for $B_h(\theta)$? In this paper we answer this question. Similar but different problems of this type can be found in e.g. [3] or [4]. In that paper inhomogeneous sequences are treated. However, our discussion includes the inhomogeneous case as one can easily see. That is, our $B_h(\theta)$ is the same as

$$\sum_{i=1}^{h}\{i\theta+\phi\} \quad \text{when } i\theta+\phi \neq m \quad \textit{for all } i \in \boldsymbol{N} \text{ and } m \in \boldsymbol{Z}.$$

The main ideas about the geometrical aspects come from [1] and [5].

2. The case $h = 2$

Suppose $\theta > 1$ from hereon. The other cases are similar.

Given the lines $y = x/\theta$ and $y = (x+1)/\theta$, then for $i = 1, 2, \dots, h$ the $\{(N+i)\theta\}$ can be considered as segments parallel to the x-axis, from the integer point $\lfloor (N+i)\theta \rfloor$, between the two lines, to $(N+i)\theta$ on the line $y = x/\theta$. Then, the slope of the segment which connects two consecutive integer points $(N+i)\theta$ and $(N+i+1)\theta$ is either $1/\lfloor\theta\rfloor$ or $1/(\lfloor\theta\rfloor+1)$. We denote these two kinds of slanted segment by s or l, respectively. And we call these slanted segments *forms*.

Let $h = 2$. Then $B_2(\theta)$ takes its minimum value when s or l is nearest to $y = x/\theta$. Hence, this form is .s or l., where . means that the left side of s or the right side of l touches the line $y = x/\theta$. Precisely speaking, this segment never touches because θ is irrational; but, we refer to the situation in that way for convenience. Then, the minimum value of $B_2(\theta)$ is $\{\theta\}$ or $1 - \{\theta\}$, respectively. We call these forms with dot, *patterns*.

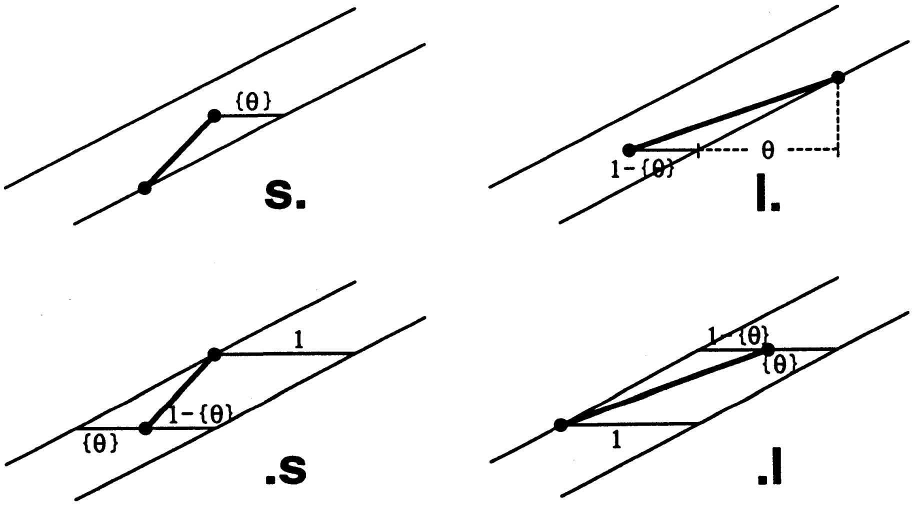

Figure 2

Similarly, $B_2(\theta)$ takes its maximum value when s or l is nearest to $y=(x+1)/\theta$. Hence, this pattern is s. or .l, where . means that the right side of s or the left side of l touches the line $y=(x+1)/\theta$. Then, the maximum value is $2-\{\theta\}$ or $\{\theta\}+1$, respectively.

Therefore, we get the following result:

THEOREM 2. *Let θ be irrational. When $\{\theta\}<1/2$, the pattern is* .s *and* s. *and*

$$\{\theta\}<B_2(\theta)<2-\{\theta\},$$

whereas when $\{\theta\}>1/2$, the pattern is l. *and* .l *and*

$$1-\{\theta\}<B_2(\theta)<\{\theta\}+1.$$

One can obtain the results of the cases $h=3,4,\dots$ in similar ways although they become more complicated. For example, when $h=3$, consider the patterns in Figure 3. But, we omit the results of small h because they are not of great importance in many applications.

3. The general case

Let $0<\{\theta\}<1/2$. If $\{\theta\}$ is changed to $1-\{\theta\}$, the corresponding cases with $\{\theta\}>1/2$ are easily obtained. Moreover, it is enough to find only a minimum for $B_h(\theta)$, because the maximum can be obtained by subtracting that minimum from h. That is clear on observing that the horizontal distance between $y=x/\theta$ and $y=(x+1)/\theta$ is 1.

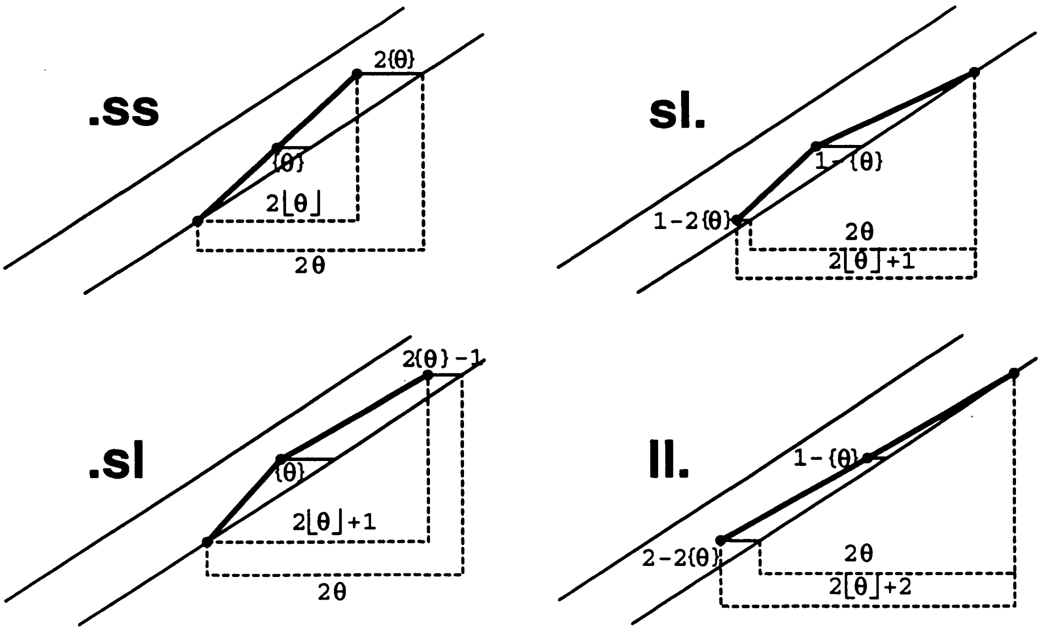

Figure 3

Before considering the general case, we shall translate the patterns onto the points of the ordinary number line. For a given positive integer h, there are h points $(N+1)\{\theta\}, (N+2)\{\theta\}, \dots, (N+h)\{\theta\}$ on the number line, and there are $h-1$ intervals each with length $\{\theta\}$. Each interval corresponds to one s or l. The case s refers to an interval between two consecutive integer points and l refers to an interval arching over an integer point.

A dot . indicates that the point is nearest to the integer in h points. For example, .ssls means that the first point $(N+1)\theta$ is nearest to the integer, that is, $\{(N+1)\theta\}$ is least. The case ssl.s means that the fourth point, the point on the right of the interval arching over an integer, is nearest to the integer. On the other hand, slss. means that the last point $(N+5)\{\theta\}$ is nearest to the integer, that is, $\{(N+5)\theta\}$ is largest. s.lss means that the second point, the point on the left of the interval arching over an integer, is nearest to the integer.

Under this translation, it is easily checked that each pattern gives the minimum or maximum value shown in the theorems above.

Now, consider the general case. It isn't easy to state the results for the general case, but when $\{\theta\} < 2/h$ (thus, when $\{\theta\} > 1 - 2/h$ too) there is good regularity. One can easily see that the pattern is $.\underbrace{\mathrm{ss}\cdots\mathrm{s}}_{h-1}$ when $\{\theta\} < 1/h$, and that as $\{\theta\}$ grows, one l moves forward from the tail to the middle of the pattern, and a dot appears on the tail or head of the pattern in turn.

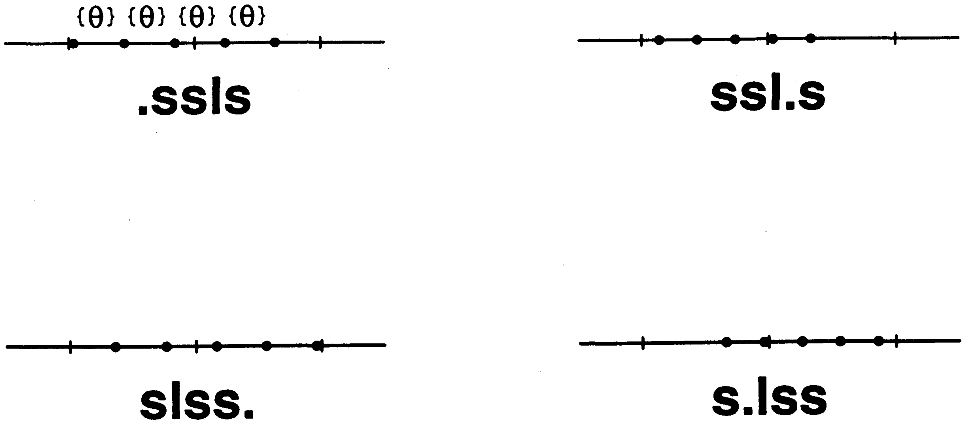

Figure 4

That leads to the following theorem:

Theorem 3. *Let θ be irrational, and take $h \geq 2$. Then when $\{\theta\} < 1/h$,*

$$\underbrace{.\mathrm{ss}\cdots\mathrm{s}}_{h-1} \qquad \frac{h(h-1)}{2}\{\theta\} < B_h(\theta) < h - \frac{h(h-1)}{2}\{\theta\}.$$

When $\frac{1}{h-i}\frac{h-1}{h} < \{\theta\} < \frac{1}{h-i}$, $i = 1, 2, \ldots, h - \left\lceil \frac{h}{2} \right\rceil$,

$$\underbrace{\mathrm{s}\cdots\mathrm{s}}_{h-i-1}\,1.\,\underbrace{\mathrm{s}\cdots\mathrm{s}}_{i-1} \qquad h - i - \frac{h(h-2i+1)}{2}\{\theta\} < B_h(\theta) < \frac{h(h-2i+1)}{2}\{\theta\} + i.$$

When $\frac{1}{h-i} < \{\theta\} < \frac{1}{h-i-1}\frac{h-1}{h}$, $i = 1, 2, \ldots, h - \left\lfloor \frac{h}{2} \right\rfloor - 1$,

$$.\underbrace{\mathrm{s}\cdots\mathrm{s}}_{h-i-1}\,1\,\underbrace{\mathrm{s}\cdots\mathrm{s}}_{i-1} \qquad \frac{h(h-1)}{2}\{\theta\} - i < B_h(\theta) < h - \frac{h(h-1)}{2}\{\theta\} + i.$$

Proof. When $\{\theta\} < 1/(h-1)$, since $0 < (h-1)\{\theta\} < 1$, the consecutive $h-1$ intervals which each have length $\{\theta\}$, that is the h points

$$(N+1)\{\theta\}, (N+2)\{\theta\}, \ldots, (N+h)\{\theta\},$$

can be arranged between two consecutive integers.

First, when $\{\theta\} < 1/h$, consider the case when these h points are located in the position $\underbrace{.\mathrm{ss}\cdots\mathrm{s}}_{h-1}$ on the number line. We shift these points to the right by ε

with $0 < \varepsilon < 1$. Let h' be an integer satisfying $(h'-1)\{\theta\} + \varepsilon < 1 < h'\{\theta\} + \varepsilon$. Unless there is no such h', clearly $h(h-1)\{\theta\}/2$ is least. For if such an h' exists, then

$$\begin{aligned} B_h(\theta) &= (h'\{\theta\} + \varepsilon - 1) + ((h'+1)\{\theta\} + \varepsilon - 1) + \cdots + ((h-1)\{\theta\} + \varepsilon - 1) \\ &\quad + ((h'-1)\{\theta\} + \varepsilon) + ((h'-2)\{\theta\} + \varepsilon) + \cdots + \varepsilon \\ &> 0 + \{\theta\} + \cdots + (h - h' - 1)\{\theta\} \\ &\quad + (h-1)\{\theta\} + (h-2)\{\theta\} + \cdots + (h-h')\{\theta\} = h(h-1)\{\theta\}/2, \end{aligned}$$

showing that $h(h-1)\{\theta\}/2$ is least.

Next, we shift h points from the position $.\underbrace{\mathrm{ss}\cdots\mathrm{s}}_{h-1}$ to the left by $k\{\theta\} + \rho$ with $k = 0, 1, \ldots$ and $0 \le \rho < \{\theta\} < 1/h$ beyond the integer point. Then,

$$\begin{aligned} B_h(\theta) &= (1 - k\{\theta\} - \rho) + (1 - (k-1)\{\theta\} - \rho) + \cdots + (1 - \{\theta\} - \rho) + (1 - \rho) \\ &\quad + (\{\theta\} - \rho) + (2\{\theta\} - \rho) + \cdots + ((h-k-1)\{\theta\} - \rho) \\ &= \frac{h(h-1)}{2}\{\theta\} + k(1 - h\{\theta\}) + (1 - h\rho) > \frac{h(h-1)}{2}\{\theta\}. \end{aligned}$$

Thus, $h(h-1)\{\theta\}/2$ is least.

When $1/h < \{\theta\} < 1/(h-1)$, consider the case when the h points are located in the position $\underbrace{\mathrm{s}\cdots\mathrm{s}}_{h-2}1.$ on the number line. Similarly, we can prove $h - 1 - h(h-1)\{\theta\}/2$ is least whichever we shift the points to the right or to the left.

The other cases are proved in a similar way.

The case when $\{\theta\}$ is very close to $1/2$ can be easily imagined. It is expressed as:

THEOREM 4. *Let θ be irrational with $(h-1)/(2h) < \{\theta\} < 1/2$, h an integer with $h \ge 2$. Then if h is odd,*

$$\underbrace{\mathrm{s}1\cdots\mathrm{s}1}_{h-1}. \quad \frac{(h+1)(h-1)}{4} - \frac{h(h-1)}{2}\{\theta\} < B_h(\theta) < h - \frac{(h+1)(h-1)}{4} + \frac{h(h-1)}{2}\{\theta\}.$$

If h is even,

$$\underbrace{\mathrm{s}1\cdots\mathrm{s}1}_{h-2}.\mathrm{s} \quad \frac{h(h-2)}{4} - \frac{h(h-3)}{2}\{\theta\} < B_h(\theta) < -\frac{h(h-6)}{4} + \frac{h(h-3)}{2}\{\theta\}.$$

PROOF. It suffices to prove the minimum value of $B_h(\theta)$. Put

$$b_h(\theta) = \frac{(h+1)(h-1)}{4} - \frac{h(h-1)}{2}\{\theta\}.$$

First, let h be odd, and shift h points to the left by ε. Remark that

$$0 < 1 - 2\{\theta\} < 2 - 4\{\theta\} < \cdots < \frac{h-1}{2} - (h-1)\{\theta\}$$
$$< 1 - \{\theta\} < 2 - 3\{\theta\} < \cdots < \frac{h-1}{2} - (h-2)\{\theta\} < 1.$$

When $(i-1)(1-2\{\theta\}) < \varepsilon < i(1-2\{\theta\})$ for $i = 1, 2, \dots, (h-1)/2$,

$$B_h(\theta) - b_h(\theta) = i - h\varepsilon > i - ih(1-2\{\theta\}) > 0.$$

When $(h-1)/2 - (h-1)\{\theta\} < \varepsilon < 1 - \{\theta\}$,

$$B_h(\theta) - b_h(\theta) = \frac{h+1}{2} - h\varepsilon > \frac{h+1}{2} - h(1-\{\theta\}) > 0.$$

When $(i'-1) - (2i'-3)\{\theta\} < \varepsilon < i' - (2i'-1)\{\theta\}$ for $i' = 2, 3, \dots, (h-1)/2$,

$$B_h(\theta) - b_h(\theta) = \frac{h-1}{2} + i' - h\varepsilon > \frac{h-1}{2} + i' - i'h + (2i'-1)h\{\theta\} > 0.$$

When $(h-1)/2 - (h-2)\{\theta\} < \varepsilon < 1$,

$$B_h(\theta) - b_h(\theta) = h - h\varepsilon > 0.$$

Shift h points to the right by ρ. In any case of

$$0 < \rho < 1 - ((h-1)/2 - (h-2)\{\theta\}),$$

$$1 - ((h-2i+1)/2 - (h-2i)\{\theta\}) < \rho < 1 - ((h-2i-1)/2 - (h-2i-2)\{\theta\})$$
$$\text{for} \quad i = 1, 2, \dots, (h-3)/2 \quad \text{with } h \geq 5,$$

$$1 - (1 - \{\theta\}) < \rho < 1 - ((h-1)/2 - (h-1)\{\theta\}),$$

$$1 - ((h-1)/2 - (i'-1) - (h+1-2i')\{\theta\})$$
$$< \rho < 1 - ((h-1)/2 - i' - (h-1-2i')\{\theta\})$$
$$\text{for} \quad i' = 1, 2, \dots, (h-3)/2 \quad \text{with } h \geq 5$$

$$\text{or} \quad 1 - (1 - 2\{\theta\}) < \rho < 1,$$

we can similarly check that $B_h(\theta) - b_h(\theta) > 0$.

Next, let h be even with $h \geq 4$. The case $h = 2$ follows from Theorem 2. Now put

$$b_h(\theta) = \frac{h(h-2)}{4} - \frac{h(h-3)}{2}\{\theta\}.$$

Noting

$$0 < 1 - 2\{\theta\} < 2 - 4\{\theta\} < \cdots < \frac{h-2}{2} - (h-2)\{\theta\} < \{\theta\}$$
$$< 1 - \{\theta\} < 2 - 3\{\theta\} < \cdots < \frac{h-2}{2} - (h-3)\{\theta\} < 1,$$

we can show $B_h(\theta) - b_h(\theta) > 0$. The proof is similar to the odd case.

Lastly, we pose two conjectures and some open problems.

CONJECTURE 1. *For a given integer h with $h \geq 2$, there are $2\sum_{i=1}^{h-1}\phi(i)$ cases. Here, $\phi(i)$ is Euler's function, that is, the number of positive integers less than and prime to i.*

CONJECTURE 2. *For a given integer h with $h \geq 2$, when*

$$\frac{i}{h} < \{\theta\} < \frac{i+1}{h} \quad (i = 0, 1, \ldots, h-1),$$

the pattern which gives the bounds for $B_h(\theta)$ consists of $(h-i-1)$ s's and i1's.

REMARK. When $i = 0, 1$ (therefore, $i = h-1, h-2$ also), the result is clear from Theorem 3.

1 What are the bounds for $B_h(\theta)$ when $1/\lceil h/2 \rceil < \theta < (h-1)/2h$ (h: even) or $(h-1)/(\lfloor h/2 \rfloor h) < \theta < (h-1)/2h$ (h: odd)?

2 Farey series appears in the case of $\{\theta\}$. What are the other fractions? How can we decide those generally? For example, the case $h = 8$ shows

$$0, \frac{1}{8}, \frac{1}{7}, \frac{7}{48}, \frac{1}{6}, \frac{7}{40}, \frac{1}{5}, \frac{7}{32}, \frac{1}{4}, \frac{9}{32}, \frac{2}{7}, \frac{7}{24}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{17}{40}, \frac{3}{7}, \frac{7}{16}, \frac{1}{2}.$$

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