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# ON AUTOMORPHISMS OF A CHARACTER RING

Dedicated to Professor Tosiro TSUZUKU

By

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## 1. Introduction

Throughout this paper $G, Z(G)$ and $C$ denote a finite group, the center of $G$ and the field of complex numbers respectively. For a finite set $S$, we denote the number of elements in $S$ by $|S|$.

Let $Irr(G)$ be the full set of irreducible $C$-characters of $G$ and $X(G)$ be the character ring of $G$. If $R$ is any subring of $C$, we write $RX(G)$ to denote the $R$-algebra of $R$-linear combinations of irreducible $C$-characters of $G$.

Suppose $G$ and $H$ are finite groups. Weidman showed that if $X(G)$ is isomorphic to $X(H)$, then $G$ and $H$ have the same character table.

In addition Saksonov proved the following theorem, which is a strengthened version of Weidman's theorem.

THEOREM 1.1. (Saksonov) *Suppose $R$ is the ring of all algebraic integers and there exists an $R$-algebra isomorphism $\phi$ from $RX(G)$ onto $RX(H)$. If $Irr(G)=\{\chi_1,\cdots,\chi_h\}$ and $Irr(H)=\{\psi_1,\cdots,\psi_h\}$, then the following holds:*

(i) *The character tables of $G$ and $H$ are the same.*

(ii) $\phi(\chi_i)=\varepsilon_i\psi_{i'}$ $(i=1,\cdots,h)$ *where the $\varepsilon_i$ are roots of unity and $i\to i'$ is a permutation.*

From now on we assume that $R$ is the ring of all algebraic integers. Then in this paper we intend to prove the following theorem.

THEOREM 1.2. *Suppose $G$ and $H$ are finite groups. Then we have*

(i) *If $u$ is a central element in $G$ and $\tau_u: RX(G)\to RX(G)$ is the map defined by $\chi\to(\chi(u)/\chi(1))\chi$ where $\chi\in Irr(G)$ and 1 is the identity element of $G$, then $\tau_u$ is an $R$-automorphism of $RX(G)$. Furthermore the map $u\to\tau_u$ is a group isomorphism of $Z(G)$ onto a subgroup $T=\{\tau_u \mid u\in Z(G)\}$ of $Aut(RX(G))$.*

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(ii) *Every R-isomorphism* $\phi: RX(G) \to RX(H)$ *is the composition of an R-isomorphism* $\theta$ *that maps* $Irr(G)$ *onto* $Irr(H)$ *with an automorphism of* $RX(H)$ *of the form* $\tau_u$ *for some element u in* $Z(H)$.

(iii) *The full group* $A = Aut(RX(G))$ *is the product of the subgroup T of part* (i) *above, which is normal, with the subgroup P consisting of those automorphisms that map* $Irr(G)$ *onto* $Irr(G)$.

## 2. Proof of Theorem 1.2

In order to prove Theorem 1.2 we prove a basic lemma concerning the roots of unity which appear in Saksonov's Theorem.

LEMMA 2.1. *Suppose for each character* $\chi$ *in* $Irr(G)$, *there is a root of unity* $\varepsilon(\chi)$ *such that each product* $\varepsilon(\chi)\chi\varepsilon(\psi)\psi$ *for* $\chi, \psi$ *in* $Irr(G)$ *is a non-negative integer linear combination of* $\varepsilon(\xi)\xi$, *as* $\xi$ *runs over* $Irr(G)$. *Then there exists u in* $Z(G)$ *such that* $\varepsilon(\chi) = \chi(u)/\chi(1)$ *for every character* $\chi$ *in* $Irr(G)$.

PROOF. If we are given $\chi$ and $\psi$ in $Irr(G)$, then we assume that

$$\chi\psi = \sum_{\xi \in Irr(G)} m_\xi \xi \text{ and } \varepsilon(\chi)\chi\varepsilon(\psi)\psi = \sum_{\xi \in Irr(G)} n_\xi \varepsilon(\xi)\xi$$

where the coefficients $m_\xi$ and $n_\xi$ are non-negative integers. Then it follows easily that $m_\xi = n_\xi$ for all characters $\xi$ in $Irr(G)$ and thus the map $\phi: \chi \to \varepsilon(\chi)\chi$ defines an automorphism of the algebra $CX(G)$. In particular the map $\phi$ permutes the primitive idempotents of this $C$-algebra (See the proof of Lemma 2.3 in [**3**]) and so it carries the characteristic class function of the identity to the characteristic class function of some other conjugacy class, say the class $K$. Therefore we have

$$(1/|G|)\sum_{\chi \in Irr(G)} \varepsilon(\chi)\chi(1)\chi = (1/|C_G(v)|)\sum_{\chi \in Irr(G)} \overline{\chi(v)}\chi$$

where $v$ is an element in $K$. It follows that for each irreducible character $\chi$ in $Irr(G)$ we have $\chi(1)\varepsilon(\chi) = |K|\chi(u)$ where $u = v^{-1}$. Applying this where $\chi$ is the principal character yields that $|K|$ is a root of unity and so $u$ is a central element in $G$. Thus for every character $\chi$ in $Irr(G)$, $\varepsilon(\chi) = \chi(u)/\chi(1)$ for some element $u$ in $Z(G)$, as claimed. Q.E.D.

PROOF OF THEOREM 1.2. (i) Suppose $u$ is a central element in $G$. Then for each character $\chi$ in $Irr(G)$ we denote by $\varepsilon(\chi)$ and $T(\chi)$ the root of unity given by $\chi(u)/\chi(1)$ and the irreducible matrix representation of $G$ which affords $\chi$ respectively. We assume further that for $\chi, \psi$ in $Irr(G)$, $\chi\psi = \sum_{\xi \in Irr(G)} m_\xi \xi$ where the $m_\xi$ are non-negative integers. Then we show $\varepsilon(\xi)=\varepsilon(\chi)\varepsilon(\psi)$ for $m_\xi \neq 0$.

Indeed $T(\chi)(u)=diag(\varepsilon(\chi),\cdots,\varepsilon(\chi))$ and $T(\psi)(u)=diag(\varepsilon(\psi),\cdots,\varepsilon(\psi))$ which have diagonals of lengths $\chi(1)$ and $\psi(1)$ respectively. Hence

$$T(\chi)(u)\otimes T(\psi)(u)=diag(\varepsilon(\chi)\varepsilon(\psi),\cdots,\varepsilon(\chi)\varepsilon(\psi))$$

where $T(\chi)\otimes T(\psi)$ is the Kronecker product of $T(\chi)$ and $T(\psi)$. Since $T(\chi)\otimes T(\psi)$ is the representation of $G$ which affords $\chi\psi$, we have $\varepsilon(\xi)=\varepsilon(\chi)\varepsilon(\psi)$ for $m_\xi \neq 0$, as claimed. Therefore we have $\varepsilon(\chi)\chi\varepsilon(\psi)\psi=\sum_{\xi\in Irr(G)} m_\xi\varepsilon(\xi)\xi$.

Thus the map $\tau_u$ defined by $\chi\to\varepsilon(\chi)\chi$ is an $R$-automorphism of $RX(G)$.

The fact that $Z(G)\cong T$ is easy to prove and so we omit its proof.

(ii) Now we can easily observe that Saksonov's result guarantees that the image of $Irr(G)$ under $\phi$ satisfies the hypotheses of Lemma 2.1 for $H$. Hence we may write $\phi(\chi_i)=\varepsilon(\psi_{i'})\psi_{i'}$, $\varepsilon(\psi_{i'})=\psi_{i'}(u)/\psi_{i'}(1)$ for some element $u$ in $Z(H)$, $(i=1,\cdots,h)$ where $Irr(G)=\{\chi_1,\cdots,\chi_h\}$, $Irr(H)=\{\psi_1,\cdots,\psi_h\}$ and $i\to i'$ is a permutation.

Therefore the map $\tau_u$ defined by $\psi\to\varepsilon(\psi)\psi$ is an $R$-automorphism of $RX(H)$ from fact (i) above. If we put $\theta=\tau_u^{-1}\phi$, then $\theta(\chi_i)=\tau_u^{-1}(\phi(\chi_i))=\psi_{i'}$, $(i=1,\cdots,h)$ and so $\theta$ maps $Irr(G)$ onto $Irr(H)$. Hence we have $\phi=\tau_u\theta$, as required.

(iii) Fact (iii) follows since fact (ii) tells us that $A=TP$ and it is clear from fact (ii) that $A$ induces a permutation action on $Irr(G)$ and $T$ is the kernel of this action. This completes the proof of the theorem. Q.E.D.

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