

ARCWISE CONNECTEDNESS OF THE COMPLEMENT IN A HYPERSPACE

By

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Abstract. The hyperspace $C(X)$ of a continuum X is always arcwise connected. In [6], S. B. Nadler Jr. and J. Quinn show that if $C(X) - \{A_i\}$ is arcwise connected for each $i = 1, 2$, then $C(X) - \{A_1, A_2\}$ is also arcwise connected. Nadler raised questions in his book [5]: *Is it still true with the two sets A_1 and A_2 replaced by n sets, n finite? What about countably many? What about a collection $\{A_\lambda : \lambda \in \Lambda\}$ which is a compact zero-dimensional subset of the hyperspace?* In this paper we prove that if $\mathcal{A} \subset C(X)$ is a closed countable subset, \mathcal{U} is an arc component of an open set of $C(X)$ and $C(X) - \{A\}$ is arcwise connected for each $A \in \mathcal{A}$, then $\mathcal{U} - \mathcal{A}$ is arcwise connected.

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1. Notation and Preliminary Lemmas

A continuum is a nonempty compact connected metric space. The letter X will always denote a nondegenerate continuum with a metric function d . Let Y be a subcontinuum of X and ε a positive number. The set $N(Y; \varepsilon)$ denotes the ε -neighborhood of Y in X , i.e., $N(Y; \varepsilon) = \{x \in X : d(x, y) < \varepsilon \text{ for some } y \in Y\}$ and Y_ε denotes the component of the closure of $N(Y; \varepsilon)$ containing Y . The hyperspace $C(X)$ of X is the space of all subcontinuum of X with the Hausdorff metric H_d defined by

$$H_d(A, B) = \inf\{\varepsilon > 0 : A \subset N(B; \varepsilon) \text{ and } B \subset N(A; \varepsilon)\}.$$

With this metric, $C(X)$ becomes a continuum. If Y is a subcontinuum of X , then

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we consider $C(Y)$ as a subspace of $C(X)$. For two subsets \mathcal{A} and \mathcal{B} of $C(X)$, let $H_d(\mathcal{A}, \mathcal{B}) = \inf \{H_d(A, B) : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$. A map is a continuous function. Any map $\mu : C(X) \rightarrow [0, 1]$ satisfying

- (1) if $A \subset B$ and $A \neq B$, then $\mu(A) < \mu(B)$,
- (2) $\mu(\{x\}) = 0$ for each $x \in X$ and $\mu(X) = 1$

is called a *Whitney map* for $C(X)$. Such a map always exists (see [7]). An *order arc* is a map $\sigma : [a, b] \rightarrow C(X)$ such that if $a \leq t_0 < t_1 \leq b$, then $\sigma(t_0) \subset \sigma(t_1)$ and $\sigma(t_0) \neq \sigma(t_1)$. It is also called an order arc from $\sigma(a)$ to $\sigma(b)$.

If A, B are distinct elements of $C(X)$, then there is an order arc from A to B if and only if $A \subset B$ (see [1]).

We often use the following lemmas which are easy to prove hence we omit their proofs.

LEMMA 1. *Let Y be a proper subcontinuum of X . If there is a subcontinuum M of X such that $M \cap Y \neq \emptyset \neq M - Y$, then for any $\varepsilon > 0$ and $y \in M \cap Y$, there is a subcontinuum N of $M \cap Y_\varepsilon$ such that $N \cap Y \neq \emptyset \neq N - Y$ and $y \in N$.*

The diameter of a subset A of X is denoted by $\delta(A)$, i.e., $\delta(A) = \sup\{d(x, y) : x, y \in A\}$.

REMARK. If \mathcal{A} is a connected subset of $C(X)$ such that $Y \in \mathcal{A}$ and $\delta(\mathcal{A}) \leq \varepsilon$, then $\mathcal{A} \subset C(Y_\varepsilon)$.

LEMMA 2. *If a subset $\{A, B, C, D\} \subset C(X)$ satisfies $A \subset B \cap C \subset B \cup C \subset D$, then $H_d(B, C) \leq H_d(A, D)$. In particular, if σ is an order arc, then $\delta(\sigma([a, b])) = H_d(\sigma(a), \sigma(b))$.*

Furthermore we need the following Krasinkiewicz-Nadler's Theorem (Theorem 3.1 of [2]).

PROPOSITION 3. *Let $\mu : C(X) \rightarrow [0, 1]$ be a Whitney map and $A_1, A_2 \in \mu^{-1}(t_0)$, where $t_0 \in [0, 1]$. Let K be a subcontinuum of $A_1 \cap A_2$. Then there is a map $\alpha : [0, 1] \rightarrow \mu^{-1}(t_0) \cap C(A_1 \cup A_2)$ such that $\alpha(0) = A_1$, $\alpha(1) = A_2$ and $K \subset \alpha(t)$ for all $t \in [0, 1]$. If $A_1 \neq A_2$, then α can be taken to be an embedding.*

In fact Theorem 3.1 of [2] is much more general, and from its proof we obtain the following lemma.

LEMMA 4. *Let $\mu : C(X) \rightarrow [0, 1]$ be a Whitney map and let A, B, C be*

subcontinua of X such that $A \cap B \supset C$. Then there is a map $\alpha : [0,1] \rightarrow \mu^{-1}(\mu(A)) \cap C(A \cup B)$ such that $\alpha(0) = A$, $\alpha(t) \supset C$ for each $t \in [0,1]$, and

if $\mu(A) \leq \mu(B)$ then $\alpha(1) \subset B$,

if $\mu(A) > \mu(B)$ then $\alpha(1) \supset B$.

In the same paper they proved (Theorem 3.5 in [2]) that:

PROPOSITION 5. *Let X be decomposable and $\mu : C(X) \rightarrow [0,1]$ a Whitney map. Then there is $s_0 \in [0,1)$ such that if $s \in [s_0,1]$, then $\mu^{-1}(s)$ is arcwise connected.*

The following proposition is Theorem 4.6 of [4].

PROPOSITION 6. *If Y is a non-degenerate proper subcontinuum of X , then the following two statements are equivalent:*

(1) $C(X) - \{Y\}$ is not arcwise connected.

(2) There is a dense subset D of Y such that if M is a subcontinuum of X satisfying $M \cap D \neq \emptyset \neq M - Y$, then $M \supset Y$.

2. Bypass Lemma

Let $K, L \in C(X)$ and $\mathcal{A} \subset C(X)$. An arc from K to L in \mathcal{A} is a map $\alpha : [a,b] \rightarrow \mathcal{A}$ such that $\alpha(a) = K$ and $\alpha(b) = L$. If α is an embedding, then we call it an embedding arc. Following is a key lemma.

LEMMA 7. *Let Y be a nondegenerate proper subcontinuum of a continuum X such that $C(X) - \{Y\}$ is arcwise connected. Let $\alpha : [0,1] \rightarrow C(X)$ be a map such that $\alpha(1) = Y$ and $\alpha(t) \in C(Y) - \{Y\}$ for each $t \in [0,1)$. Then for a given $\varepsilon > 0$, there is a map $\beta : [0,1] \rightarrow C(X) - \{Y\}$ such that $\alpha(0) = \beta(0)$, $H_d(\alpha(t), \beta(t)) < \varepsilon$ for each $t \in [0,1]$ and $\beta(1) - Y \neq \emptyset$.*

PROOF. First suppose that Y is indecomposable. Put $\varepsilon_1 = \varepsilon/3$. Since α is continuous, there is $t_0 \in [0,1)$ such that $\delta(\alpha([t_0,1])) < \varepsilon_1$. Let λ be the component of Y such that $\alpha(t_0) \subset \lambda$. By Lemma 1 and Proposition 6, there is a subcontinuum M of Y_{ε_1} such that $M - Y \neq \emptyset \neq Y - M$ and $M \cap \lambda \neq \emptyset$. We may assume that $M \cap \alpha(t_0) \neq \emptyset$. (Because let λ' be a component of Y different from λ . Since M is compact and $Y - M \neq \emptyset$, $\lambda' - M \neq \emptyset$. Thus we can replace M by $M \cup N$, where N is a continuum contained in λ such that $M \cap N \neq \emptyset \neq N \cap \alpha(t_0)$.) Let $\sigma : [t_0,1] \rightarrow C(X)$ be an order arc from $\alpha(t_0)$ to $M \cup \alpha(t_0)$. Then

$$\begin{aligned}\delta(\sigma([t_0, 1])) &= H_d(\alpha(t_0), M \cup \alpha(t_0)) \leq H_d(\alpha(t_0), Y_{\varepsilon_1}) \\ &\leq H_d(\alpha(t_0), Y) + H_d(Y, Y_{\varepsilon_1}) < 2\varepsilon_1.\end{aligned}$$

Define an arc β in $C(X)$ by

$$\beta(t) = \begin{cases} \alpha(t) & \text{if } t \in [0, t_0], \\ \sigma(t) & \text{if } t \in (t_0, 1]. \end{cases}$$

Clearly β is continuous and its image does not contain Y . If $t \in [0, t_0]$, then $H_d(\alpha(t), \beta(t)) = 0$. Suppose that $t \in (t_0, 1]$. Then since $\alpha(t_0) = \beta(t_0)$,

$$\begin{aligned}H_d(\alpha(t), \beta(t)) &\leq H_d(\alpha(t), \alpha(t_0)) + H_d(\beta(t_0), \beta(t)) \\ &\leq \delta(\alpha([t_0, 1])) + \delta(\sigma([t_0, 1])) < 3\varepsilon_1 = \varepsilon.\end{aligned}$$

For the second case, suppose that Y is decomposable. Put $\varepsilon_1 = \varepsilon/5$ and let μ be a Whitney map for $C(X)$. By Proposition 5, there is $s_0 < \mu(Y)$ such that if $s \in [s_0, \mu(Y)]$, then $\mu^{-1}(s) \cap C(Y)$ is arcwise connected. Moreover s_0 can be taken so that $\delta(\mu^{-1}([s_0, 1]) \cap C(Y)) < \varepsilon_1$. Since α is continuous, there is $t_0 \in [0, 1)$ such that $\mu(\sigma([t_0, 1])) \subset [s_0, 1]$. For simplicity, put $s_1 = \mu(\alpha(t_0))$. By Proposition 6 and Lemma 1, there is a subcontinuum M of Y_{ε_1} , such that $M - Y \neq \emptyset \neq Y - M$ and $M \cap Y \neq \emptyset$. There are two cases.

(i) Suppose there is $A \in \mu^{-1}(s_1) \cap C(Y)$ such that $A \cap M \neq \emptyset \neq Y - (A \cup M)$. Put $t_1 = (t_0 + 1)/2$ and let $\sigma_1 : [t_0, t_1] \rightarrow \mu^{-1}(s_1) \cap C(Y)$ be an arc from $\alpha(t_0)$ to A (such an arc exists since $s_0 \leq s_1 < \mu(Y)$) and $\sigma_2 : [t_1, 1] \rightarrow C(X)$ an order arc from A to $A \cup M$. Note that $\delta(\sigma_2([t_1, 1])) < 2\varepsilon_1$. Define an arc β in $C(X)$ by

$$\beta(t) = \begin{cases} \alpha(t) & \text{if } t \in [0, t_0], \\ \sigma_1(t) & \text{if } t \in (t_0, t_1], \\ \sigma_2(t) & \text{if } t \in (t_1, 1]. \end{cases}$$

Clearly β is continuous and $\beta(t) \neq Y$ for each $t \in [0, 1]$. If $t \in [0, t_0]$, then $H_d(\alpha(t), \beta(t)) = 0$. Suppose $t \in [t_0, t_1]$. Then since $\beta([t_0, t]) \subset \mu^{-1}(s_1) \cap C(Y)$,

$$\begin{aligned}H_d(\alpha(t), \beta(t)) &\leq H_d(\alpha(t), \alpha(t_0)) + H_d(\beta(t_0), \beta(t)) \\ &\leq \delta(\alpha([t_0, 1])) + \delta(\mu^{-1}(s_1) \cap C(Y)) < 2\varepsilon_1 < \varepsilon.\end{aligned}$$

Finally suppose $t \in [t_1, 1]$. Then

$$\begin{aligned}H_d(\alpha(t), \beta(t)) &\leq H_d(\alpha(t), \alpha(t_1)) + H_d(\alpha(t_1), \beta(t_1)) + H_d(\beta(t_1), \beta(t)) \\ &< \delta(\alpha([t_1, 1])) + 2\varepsilon_1 + \delta(\sigma_2([t_1, 1])) < \varepsilon_1 + 2\varepsilon_1 + 2\varepsilon_1 = 5\varepsilon_1 = \varepsilon.\end{aligned}$$

(ii) Suppose that for each $A \in \mu^{-1}(s_1) \cap C(Y)$, $A \cap M \neq \emptyset$ implies $Y \subset A \cup M$. In this case, each element of $\mu^{-1}(s_1) \cap C(Y)$ intersects M . In particular,

$\alpha(t_0) \cap M \neq \emptyset$. Considering an order arc from M to $M \cup Y$, we can enlarge M and hence we can assume $\mu(M) > s_1$. By Lemma 4, there is a map $\sigma_1 : [t_0, t_1] \rightarrow \mu^{-1}(s_1) \cap C(Y \cup M)$ from $\alpha(t_0)$ to $\sigma_1(t_1) \subset M$, where $t_1 = (t_0 + 1)/2$. Let $\sigma_2 : [t_1, 1] \rightarrow C(X)$ be an order arc from $\sigma_1(t_1)$ to M . Define an arc β in $C(X)$ by

$$\beta(t) = \begin{cases} \alpha(t) & \text{if } t \in [0, t_0], \\ \sigma_1(t) & \text{if } t \in (t_0, t_1], \\ \sigma_2(t) & \text{if } t \in (t_1, 1]. \end{cases}$$

As in case (i), β satisfies all the required conditions.

Now we prove the main lemma.

BYPASS LEMMA 8. *Let Y be a subcontinuum of X such that $C(X) - \{Y\}$ is arcwise connected and let $\alpha : [0, 1] \rightarrow C(X)$ be an arc such that $\alpha(t) = Y$ if and only if $t = 1/2$. Then for each $\varepsilon > 0$ and each a, b , where $0 \leq a < 1/2 < b \leq 1$, there is a map $\beta : [0, 1] \rightarrow C(X) - \{Y\}$ such that $\alpha(t) = \beta(t)$ for all $t \in [0, a] \cup [b, 1]$ and $H_d(\alpha(t), \beta(t)) < \varepsilon$ for all $t \in [0, 1]$.*

PROOF. If $Y = X$, then X is decomposable (by Theorem 11.4 and Corollary 11.8 of [5]). Let μ be a Whitney map for $C(X)$. By Proposition 5, there is $s_0 \in [0, 1)$ such that $\mu^{-1}(s)$ is arcwise connected for each $s \in [s_0, 1]$. Moreover s_0 can be chosen so that $\delta(\mu^{-1}[s_0, 1]) < \varepsilon/2$. Since α is continuous, there exist two numbers t_0, t_1 such that $a \leq t_0 < 1/2 < t_1 \leq b$, $\mu(\alpha(t_0)) = \mu(\alpha(t_1)) \in [s_0, 1]$ and $\delta(\alpha([t_0, t_1])) < \varepsilon/2$. Put $\mu(\alpha(t_0)) = s_1$. Then since $s_1 \in [s_0, 1]$, there is a map $\sigma : [t_0, t_1] \rightarrow \mu^{-1}(s_1)$ from $\alpha(t_0)$ to $\alpha(t_1)$. Define an arc β in $C(X) - \{Y\}$ by

$$\beta(t) = \begin{cases} \alpha(t) & \text{if } t \in [0, t_0] \cup [t_1, 1], \\ \sigma(t) & \text{if } t \in (t_0, t_1). \end{cases}$$

If $t \in (t_0, t_1)$, then

$$\begin{aligned} H_d(\alpha(t), \beta(t)) &\leq H_d(\alpha(t), \alpha(t_0)) + H_d(\sigma(t_0), \sigma(t)) \\ &\leq \delta(\alpha([t_0, t_1])) + \delta(\mu^{-1}(s_1)) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore β satisfies the required conditions.

Next suppose that Y is a proper subcontinuum of X . Put $\varepsilon_1 = \varepsilon/4$. There exist two numbers t_0, t_1 such that $a \leq t_0 < 1/2 < t_1 \leq b$ and $\delta(\alpha([t_0, t_1])) < \varepsilon_1$. Note that $\alpha([t_0, t_1]) \subset C(Y_{\varepsilon_1})$. If $\alpha([t_0, 1/2]) \subset C(Y)$, then by Lemma 7, there is a map $\sigma : [t_0, 1/2] \rightarrow C(Y_{\varepsilon_1}) - \{Y\}$ such that $\sigma(t_0) = \alpha(t_0)$, $\sigma(1/2) - Y \neq \emptyset$ and $H_d(\alpha(t), \sigma(t)) < \varepsilon_1$ for each $t \in [t_0, 1/2]$. If $\alpha([t_0, 1/2]) - C(Y) \neq \emptyset$, then put $\sigma = \alpha|_{[t_0, 1/2]}$. There is $r \in (t_0, 1/2)$ such that $\sigma(r) - Y \neq \emptyset \neq Y_{\varepsilon_1} - \sigma(r)$. Let $\tau : [r, 1/2] \rightarrow C(X)$ be

an order arc from $\sigma(r)$ to Y_{ε_1} . Define $\beta_0 : [t_0, 1/2] \rightarrow C(X)$ by

$$\beta_0(t) = \begin{cases} \sigma(t) & \text{if } t \in [t_0, r], \\ \tau(t) & \text{if } t \in (r, 1/2]. \end{cases}$$

It is easy to see that $H_d(\alpha(t), \beta_0(t)) < 4\varepsilon_1$ for each $t \in [t_0, 1/2]$.

As in the same way, we can find a map $\beta_1 : [1/2, t_1] \rightarrow C(X)$ such that $\beta_1(1/2) = Y_{\varepsilon_1}$, $\beta_1(t_1) = \alpha(t_1)$ and $H_d(\alpha(t), \beta_1(t)) < 4\varepsilon_1$ for each $t \in [1/2, t_1]$. Then the arc β defined by

$$\beta(t) = \begin{cases} \alpha(t) & \text{if } t \in [0, t_0] \cup [t_1, 1], \\ \beta_0(t) & \text{if } t \in [t_0, 1/2], \\ \beta_1(t) & \text{if } t \in [1/2, t_1] \end{cases}$$

satisfies the required conditions.

3. Arcwise Connectedness of the Complement

Let \mathcal{Y} be a closed subset of $C(X)$ such that $C(X) - \{Y\}$ is arcwise connected for each $Y \in \mathcal{Y}$. We will show that if \mathcal{Y} is a finite set, then its complement is also arcwise connected. Using this, we show that the same is fold if \mathcal{Y} is a closed countable set. If $\mathcal{A} \subset \mathcal{C}(X)$ and $\varepsilon > 0$, then we wright the ε -neighborhood of \mathcal{A} in $C(X)$ by $N(\mathcal{A}; \varepsilon)$.

THEOREM 9. *Let \mathcal{Y} be a finite subset of $C(X)$ such that $C(X) - \{Y\}$ is arcwise connected for each $Y \in \mathcal{Y}$ and let $\alpha : [0, 1] \rightarrow C(X)$ be an arc from K to L , where $K, L \in C(X) - \mathcal{Y}$. Then for each $\varepsilon > 0$, there is a map $\beta : [0, 1] \rightarrow C(X) - \mathcal{Y}$ from K to L such that $\beta([0, 1]) \subset N(\alpha([0, 1]); \varepsilon)$.*

PROOF. If $K = L$, then we can take β to be a constant map. Hence let us suppose $K \neq L$. There is an embedding $\alpha' : [0, 1] \rightarrow \alpha([0, 1])$ such that $\alpha(t) = \alpha'(t)$ for $t = 0, 1$. Therefore we can assume that α is an embedding arc and hence $\alpha^{-1}(\mathcal{Y})$ is a finite set. Let $\alpha^{-1}(\mathcal{Y}) = \{t_1, t_2, \dots, t_n\}$, where $0 < t_i < t_{i+1} < 1$ for $i = 1, 2, \dots, n-1$.

(i) Suppose $n = 1$ and without loss of generality, assume $t_1 = 1/2$. Put $\alpha(1/2) = Y$. Then $\alpha([0, 1])$ and $\mathcal{Y}_1 = \mathcal{Y} - \{Y\}$ are closed and disjoint. Put $\delta = H_d(\alpha([0, 1]), \mathcal{Y}_1)$ and $\varepsilon_1 = \min\{\varepsilon, \delta\}$. Then $\varepsilon_1 > 0$. Applying Bypass Lemma, there is a map $\beta : [0, 1] \rightarrow C(X) - \{Y\}$ from K to L such that $H_d(\alpha(t), \beta(t)) < \varepsilon_1$. By the choice of ε_1 , β satisfies the required conditions.

(ii) Suppose $k \geq 2$ and the Theorem holds for $n = k-1$. Let $\alpha^{-1}(\mathcal{Y}) = \{t_1, t_2, \dots, t_k\}$ where $0 < t_i < t_{i+1} < 1$ for $i = 1, 2, \dots, k-1$. Put $\delta = H_d(\alpha([t_0, 1]), \mathcal{Y} - \{\alpha(t_k)\})$ and

$\varepsilon_1 = \min \{\varepsilon/2, \delta\}$, where $t_0 = (t_{k-1} + t_k)/2$. Then partially applying Bypass Lemma, there is a map $\beta_1 : [0, 1] \rightarrow C(X)$ such that $\alpha|_{[0, t_0]} = \beta_1|_{[0, t_0]}$, $\alpha(1) = \beta_1(1)$, $H_d(\alpha(t), \beta_1(t)) < \varepsilon_1$ and $\beta_1([0, 1])$ does not contain $\alpha(t_k)$. Let α_1 be an embedding arc from K to L such that $\alpha_1([0, 1]) \subset \beta_1([0, 1])$. Then it is easy to see that the image of α_1 intersects at most $n - 1$ elements of \mathcal{Y} . Therefore by the inductive hypothesis, there is an arc β from K to L in $C(X) - \mathcal{Y}$ such that $\beta([0, 1]) \subset N(\alpha_1([0, 1]); \varepsilon/2)$. Hence β is a required arc.

COROLLARY 10. *Let \mathcal{F} be a closed subset of $C(X)$ and let \mathcal{A} be an arc component of $C(X) - \mathcal{F}$. If \mathcal{Y} is a finite subset of $C(X)$ such that $C(X) - \{Y\}$ is arcwise connected for each $Y \in \mathcal{Y}$, then $\mathcal{A} - \mathcal{Y}$ is arcwise connected.*

PROOF. Let K, L be arbitrary elements of $\mathcal{A} - \mathcal{Y}$. There is a map $\alpha : [0, 1] \rightarrow \mathcal{A}$ from K to L . Put $\varepsilon = (1/2)H_d(\alpha([0, 1]), \mathcal{F})$. Then $\varepsilon > 0$ and hence by Theorem 9, there is a map $\beta : [0, 1] \rightarrow C(X) - \mathcal{Y}$ from K to L such that $\beta([0, 1]) \subset N(\alpha([0, 1]); \varepsilon)$. By the definition of ε , $N(\alpha([0, 1]); \varepsilon) \cap \mathcal{F} = \emptyset$. Therefore β is an arc in $\mathcal{A} - \mathcal{Y}$ from K to L .

Let A' denote the derived set of the space A . The derived set of A of order λ is defined by

$$A^{(1)} = A', \quad A^{(n+1)} = (A^{(n)})' \quad \text{and} \quad A^{(\lambda)} = \bigcap_{n < \lambda} A^{(n)}$$

if λ is a limit ordinal (see [3]).

We say that a triple $\{\mathcal{F}, \mathcal{A}, \mathcal{Y}\}$ is *admissible* if \mathcal{F} is a closed subset of $C(X)$, \mathcal{A} is an arc component of $C(X) - \mathcal{F}$, \mathcal{Y} is a closed countable subset of $C(X)$ such that $C(X) - \{Y\}$ is arcwise connected for each $Y \in \mathcal{Y}$.

THEOREM 11. *If $\{\mathcal{F}, \mathcal{A}, \mathcal{Y}\}$ is admissible, then $\mathcal{A} - \mathcal{Y}$ is arcwise connected.*

PROOF. First observe that the least ordinal ν such that $\mathcal{Y}^{(\nu)} = \emptyset$ (such an ordinal ν exists since \mathcal{Y} does not contain perfect sets) is not a limit ordinal. Therefore there is the least ordinal λ such that $\mathcal{Y}^{(\lambda)} = \emptyset$. Denote such the ordinal λ by $\Lambda(\mathcal{Y})$. To prove the Theorem, we shall proceed by transfinite induction on $\Lambda(\mathcal{Y})$.

If $\Lambda(\mathcal{Y}) = 0$, then \mathcal{Y} is a finite set. Hence Theorem follows from Corollary 10.

Suppose that the Theorem holds for any admissible triple $\{\mathcal{F}, \mathcal{A}, \mathcal{Y}\}$ such that $\Lambda(\mathcal{Y}) < \lambda$. Let $\{\mathcal{F}, \mathcal{A}, \mathcal{Y}\}$ be an admissible triple such that $\Lambda(\mathcal{Y}) = \lambda$ and let K, L be arbitrary elements of $\mathcal{A} - \mathcal{Y}$. It is sufficient to show that there is an arc from K to L in $\mathcal{A} - \mathcal{Y}$. Since $\mathcal{Y}^{(\lambda+1)} = \emptyset$, $\mathcal{Y}^{(\lambda)}$ is a finite set. Therefore by Corollary 10,

there is a map $\alpha: [0,1] \rightarrow \mathcal{A} - \mathcal{Y}^{(\lambda)}$ from K to L . Put $\varepsilon = (1/2)H_d(\alpha([0,1]), \mathcal{Y}^{(\lambda)})$, $\mathcal{Y}_1 = \mathcal{Y} - N(\mathcal{Y}^{(\lambda)}; \varepsilon)$, $\mathcal{F}_1 = \mathcal{F} \cup N(\mathcal{Y}^{(\lambda)}; \varepsilon)$, where $N(\mathcal{Y}^{(\lambda)}; \varepsilon)$ is the closure of $N(\mathcal{Y}^{(\lambda)}; \varepsilon)$ in $C(X)$, and let \mathcal{A}_1 be the arc component of $C(X) - \mathcal{F}_1$ containing K (and hence L). Note that $\mathcal{A}_1 \subset \mathcal{A}$. The triple $\{\mathcal{F}_1, \mathcal{A}_1, \mathcal{Y}_1\}$ is admissible and $\Lambda(\mathcal{Y}_1) < \lambda$. Hence by inductive hypothesis, there is an arc from K to L in $\mathcal{A}_1 - \mathcal{Y}_1$. Since $\mathcal{A}_1 - \mathcal{Y}_1 \subset \mathcal{A} - \mathcal{Y}$ and K, L are arbitrary elements of $\mathcal{A} - \mathcal{Y}$, $\mathcal{A} - \mathcal{Y}$ is arcwise connected.

COROLLARY 12. *If \mathcal{Y} is a countable closed subset of $C(X)$ such that $C(X) - \{Y\}$ is arcwise connected for each $Y \in \mathcal{Y}$, then $C(X) - \mathcal{Y}$ is arcwise connected.*

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