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MORPHISMS OF INVERSE SYSTEMS REQUIRE MESHES

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Abstract. Resolutions of spaces can be viewed as special inverse systems, which behave very much like inverse systems behave in the compact case. T. Watanabe defined a category of polyhedral resolutions and showed that the limit functor defines a natural equivalence between this category and the category of topologically complete spaces. In order to develop his theory he had to consider gauged inverse systems, i.e., inverse systems whose terms are endowed with certain coverings, called meshes. This paper is devoted to the question if one can develop an analogous theory for usual (nongauged) inverse systems. An example is exhibited, which suggests a negative answer.[1]

1. Introduction

In this paper we consider *inverse systems* $\boldsymbol{X}=(X_a, p_{aa'}, A)$, indexed by *directed sets* A. Each term is a topological space X_a and $p_{aa'}: X_{a'} \to X_a$ is a mapping, defined whenever $a \leq a'$. If $a \leq a' \leq a''$, then $p_{aa'}p_{a'a''} = p_{aa''}$ and $p_{aa} = \mathrm{id}$. We say that $\boldsymbol{X}$ is *cofinite* if A is cofinite, i.e., every element of A has only finitely many predecessors. $\boldsymbol{X}$ is *polyhedral* if every term X_a is a polyhedron endowed with the CW-topology. With every system $\boldsymbol{X}$ is associated its limit space $X = \lim \boldsymbol{X}$, as well as a collection $\boldsymbol{p}=(p_a)$ of canonical mappings $p_a : X \to X_a$, satisfying condition

$$p_{aa'}p_{a'} = p_a, a \leq a'.$$

We write $\boldsymbol{p}: X \to \boldsymbol{X}$.

The notion of resolution was introduced and studied by several authors (P. Bacon [1], K. Morita [9], [10], [11], S. Mardešić [2], [3]. Also see [4], [6]). In

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[1] *Key words:* Inverse system, resolution, mapping of inverse systems, approximate mapping of inverse systems.

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order to recall it, we need some notation. If $\mathcal{V}$ is a covering of Y and $f,g:X\to Y$ are mappings, then $(f,g)\prec\mathcal{V}$ means that, every $x\in X$ admits a $V\in\mathcal{V}$ such that $f(x),g(x)\in V$. For coverings $\mathcal{U},\mathcal{U}'$ of X, we write $\mathcal{U}'\prec\mathcal{U}$, if $\mathcal{U}'$ refines $\mathcal{U}$. By $\mathrm{Cov}(X)$ we denote the set of all normal coverings of the space X. For paracompact spaces, normal coverings coincide with open coverings. In particular, this is the case for polyhedra endowed with the CW-topology. If $A\subseteq X$ and $\mathcal{U}\in\mathrm{Cov}(X)$, then the star of A with respect to $\mathcal{U}$ is the set $\mathrm{st}(A,\mathcal{U})=\cup\{U\in\mathcal{U}|A\cap U\neq\emptyset\}\subseteq X$. We also define $\mathrm{st}(\mathcal{U})=\{\mathrm{st}(U',\mathcal{U})|U'\in\mathcal{U}\}\in\mathrm{Cov}(X)$.

A *resolution of a space* X consists of an inverse system $\boldsymbol{X}$ and of a collection of (canonical) mappings $\boldsymbol{p}=(p_a):X\to\boldsymbol{X}$, satisfying (1). Moreover, for any polyhedron P and any $\mathcal{V}\in\mathrm{Cov}(P)$, the following two condition must be satisfied:

(R1) $$(\forall f:X\to P)(\exists a\in A)(\exists g:X_a\to P)$$
$$(gp_a,f)\prec\mathcal{V}.$$

(R2) $$(\exists\mathcal{V}'\in\mathrm{Cov}(P))(\forall a\in A)(\forall g,g':X_a\to P)$$
$$(gp_a,g'p_a)\prec\mathcal{V}'\Rightarrow(\exists a'\geq a)(gp_{aa'},g'p_{aa'})\prec\mathcal{V}.$$

If $\boldsymbol{X}$ is an inverse system formed by compact Hausdorff spaces X_a, then its limit X is also a Hausdorff compact space and the canonical mappings $p_a:X\to X_a$ satisfy conditions (R1) and (R2), i.e., $\boldsymbol{p}=(p_a):X\to\boldsymbol{X}$ is a resolution. On the other hand, if $\boldsymbol{p}:X\to\boldsymbol{X}$ is a resolution consisting of completely regular spaces X_a and the space X is topologically complete (e.g., paracompact), then $\boldsymbol{p}$ is a limit of $\boldsymbol{X}$. Therefore, resolutions can be viewed as special cases of inverse limits.

If $\boldsymbol{X}$ and $\boldsymbol{Y}=(Y_b,q_{bb'},B)$ are inverse systems of spaces, indexed by directed sets A, B respectively, then a *mapping of systems* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ consists of a function $f:B\to A$ and of mappings $f_b:X_{f(b)}\to Y_b$, $b\in B$, having the property that, whenver $b\leq b'$, there exists an $a\geq f(b),f(b')$, such that

$$f_bp_{f(b)a}=q_{bb'}f_{b'}p_{f(b')a}. \tag{2}$$

A mapping of systems $\boldsymbol{f}$ induces a unique *limit mapping* $f=\lim\boldsymbol{f}:X\to Y$, satisfying the condition

$$f_bp_{f(b)}=q_bf,b\in B. \tag{3}$$

(Using the same letter f for $f:B\to A$ and $f:X\to Y$ should cause no confusion.)

A *resolution of a mapping* $f:X\to Y$ consists of resolutions of spaces

$\boldsymbol{p}: X \to \boldsymbol{X}$ and $\boldsymbol{q}: Y \to \boldsymbol{Y}$ and of a mapping of systems $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ such that (3) holds. The resolution $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ is *polyhedral* if $\boldsymbol{p}$ and $\boldsymbol{q}$ are polyhedral resolutions.

Every topological space X and every mapping $f: X \to Y$ admit polyhedral resolutions ([1], Theorem 3.2, [3], Theorem 11). However, there are simple examples of mappings $f: X \to Y$ and of polyhedral resolutions $\boldsymbol{p}: X \to \boldsymbol{X}, \boldsymbol{q}: Y \to \boldsymbol{Y}$, such that there exists no mapping of systems $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$, satisfying condition (3) (see, e.g., [12]).

The only way out of this difficulty is to consider approximate mappings instead of mappings of systems. In order to define this notion, T. Watanabe [13] enriched the structure of an inverse system by requiring that each term X_a of the system $\boldsymbol{X}$ is endowed with a normal covering $\mathcal{U}_a$, called the *mesh* at a. Meshes are subject to the following requirement.

(A) $(\forall a \in A)(\forall \mathcal{U} \in \mathrm{Cov}(X_a))(\exists a' \geq a)(\forall a'' \geq a')$

$$\mathcal{U}_{a''} \prec p_{aa''}^{-1}(\mathcal{U}).$$

We refer to such systems $\mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ as *gauged systems* and we denote them by script characters (continuing to use bold characters for usual inverse systems). By definition, the *limit of a gauged system* $\mathcal{X} = (X_a, \mathcal{U}_a, p_{aa'}, A)$ is the limit X of the *associated system* $\boldsymbol{X} = (X_a, p_{aa'}, A)$. Similarly, $\boldsymbol{p} = (p_a): X \to \mathcal{X}$ is a *gauged resolution* of X if $\boldsymbol{p} = (p_a): X \to \boldsymbol{X}$ is a resolution of X, and $\boldsymbol{f} = (f, f_a): \mathcal{X} \to \mathcal{Y}$ is a mapping of gauged systems if $\boldsymbol{f} = (f, f_a): \boldsymbol{X} \to \boldsymbol{Y}$ is a mapping of systems.

An inverse system $\boldsymbol{X} = (X_a, p_a, A)$ is said to *admit meshes* provided there exists a family of coverings $\{\mathcal{U}_a | a \in A\}, \mathcal{U}_a \in \mathrm{Cov}(X_a)$, such that $\mathcal{X} = (X_a, \mathcal{U}_a, p_a, A)$ is a gauged system. Such a family of coverings is called *admissible*. There exist usual inverse systems, which do not admit meshes. Necessary and sufficient conditions for the existence of admissible meshes have been studied in [7]. A simple example of a system which does not admit meshes is the system $\boldsymbol{X} = (X_n, p_{nn'}, \boldsymbol{N})$, where $X_n = \boldsymbol{R}$, for each $n \in \boldsymbol{N}$, and all the bonding mappings $p_{nn'}$ are identity mappings ([7], Example 5).

A *gauged approximate mapping* $\boldsymbol{f}: \mathcal{X} \to \mathcal{Y} = (Y_b, \mathcal{V}_b, q_{bb'}, B)$ is a collection $\boldsymbol{f} = \{f, f_b | b \in B\}$ consisting of a function $f: B \to A$ and of mappings $f_b: X_{f(b)} \to Y_b, b \in B$, such that the following condition holds:

(AM) $(\forall b \leq b')(\exists a \geq f(b), f(b'))(\forall a' \geq a)$,

$$(q_{bb'} f_{b'} p_{f(b')a'}, f_b p_{f(b)a'}) \prec \mathrm{st}(\mathcal{V}_b).$$

Note that this definition, due to Watanabe [13 §2] uses meshes of $\mathcal{Y}$ to measure the discrepancy from commutativity of the diagrams formed by the mappings $f_b, f_{b'}, b \le b'$, and the relevant bonding mappings. In fact, this is the primary reason for introducing meshes.

Watanabe showed ([13], Lemma 7.3, also see [8], Theorem 5.8) that, for topologically complete spaces Y_b, every approximate mapping $\boldsymbol{f}: \mathcal{X} \to \mathcal{Y}$ admits unique mapping $f: X \to Y$ between the limit spaces, which satisfies the following condition

$$\text{(LAM)} \quad (\forall b \le B)(\forall \mathcal{V} \in \operatorname{Cov}(Y_b))(\exists b' \ge b)(\forall b'' \ge b'),$$
$$(q_{bb''} f_{b''} p_{f(b'')}, q_b f) \prec \mathcal{V}.$$

The mapping $\boldsymbol{f}$ is called the limit of $\boldsymbol{f}$. If $f: X \to Y$ is a mapping $\boldsymbol{p}: X \to \mathcal{X}, \boldsymbol{q}: Y \to \mathcal{Y}$ are resolutions and $\boldsymbol{f}: \mathcal{X} \to \mathcal{Y}$ is a gauged approximate mapping satisfying (LAM), we say that $\boldsymbol{f}$ is a *gauged approximate resolution* of f.

One of the main results of Watanabe's theory is his *approximate expansion theorem* ([13], Theorem (4.3)), which asserts that, for an arbitrary gauged resolution $p: X \to \mathcal{X}$ and a cofinite polyhedral gauged resolution $\boldsymbol{q}: Y \to \mathcal{Y}$, every mapping $f: X \to Y$ admits a gauged approximate mapping $\boldsymbol{f}: \mathcal{X} \to \mathcal{Y}$, which satisfies condition (LAM).

The main purpose of this paper is to show that, for usual (nongauged) inverse systems, an analogous result does not exist. In order to state our result precisely, we introduce the following definition.

DEFINITION 1. *Let $f: X \to Y$ be a mapping and let $\boldsymbol{p}: X \to \boldsymbol{X}$ and $\boldsymbol{q}: Y \to \boldsymbol{Y}$ be resolutions. A collection $\boldsymbol{f} = (f, f_b)$, consisting of a function $f: B \to A$ and of mappings $f_b: X_{f(b)} \to Y_b, b \in B$, is said to be an approximate expansion of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$, providied the condition* (LAM) *is satisfied.*

THEOREM 1. *There exists a mapping $f: X \to Y$ and there exist polyhedral resolutions $\boldsymbol{p}: X \to \boldsymbol{X}$ and $\boldsymbol{q}: Y \to \boldsymbol{Y}$, such that $\boldsymbol{Y}$ is cofinite and there does not exist an approximate expansion $\boldsymbol{f} = (f, f_b)$ of f with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$.*

We see in this theorem a proof for our claim that, in order to develop a satisfactory theory of morphisms between inverse systems, meshes are indispensable.

2. Counterexample to the nongauged expansion theorem

In this section we exhibit an example, which proves the above theorem. For this purpose we need a simple lemma.

LEMMA 1. *Let C be the Cantor set, let $I=[0,1]$ be the unit interval and let $h:C\to I$ be a mapping onto I. There exists an inverse sequence $\boldsymbol{C}=(C_n,r_{nn'},N)$, consisting of finite discrete spaces C_n and of surjective mappings $r_{nn'}$, and there exist surjective mappings $r_n:C\to C_n, n\in N$, such that $\boldsymbol{r}=(r_n):C\to \boldsymbol{C}$ is the inverse limit of $\boldsymbol{C}$. Moreover, for any $n\in N$ and any mapping $g:C_n\to I$, there exist an open covering $\mathcal{W}$ of I and a point $z\in C$ such that, for each $W\in\mathcal{W}$, either $h(z)\notin W$ or $gr_n(z)\notin W$.*

PROOF. The first assertion is well-known and we omit its proof. To prove the second assertion, note that $g(C_n)$ is a finite set. Hence, there exists a point $t\in I\setminus(g(C_n)\cup\{0,1\})$ and there exists an $\varepsilon>0$ such that $(t-\varepsilon,t+\varepsilon)\subseteq I$ and $(t-\varepsilon,t+\varepsilon)\cap g(C_n)=\emptyset$. Put $\mathcal{W}=\{[0,t),(t-\varepsilon,t+\varepsilon),(t,1]\}$ and choose as z any point from C such that $h(z)=t$. Clearly, $\mathcal{W}$ and z have the desired properties.

Construction of the example. For each $m\in N$, let $C^m=C\times\{m\}, I^m=I\times\{m\}$, $h^m=h\times 1:C^m\to I^m, \boldsymbol{C}^m=(C_n^m,r_{nn'}^m,N)$ and $\boldsymbol{r}^m=(r_n^m):C^m\to\boldsymbol{C}^m$ be copies of C, I, h, $\boldsymbol{C}$ and $\boldsymbol{r}$, respectively (see Lemma 1). We define X, Y and f as disjoint sums

$$X=\coprod_{m\in N}C^m, Y=\coprod_{m\in N}I^m, f=\coprod_{m\in N}h^m. \tag{4}$$

We define $\boldsymbol{q}=(q_m):Y\to\boldsymbol{Y}=(Y_m,q_{mm'},N)$, by putting $Y_m=Y, q_{mm'}=\mathrm{id}, q_m=\mathrm{id}$. Clearly, $\boldsymbol{q}$ is a cofinite polyhedral resolution of Y. We define $\boldsymbol{p}=(p_a):X\to\boldsymbol{X}=(X_a,p_{aa'},A)$ as follows. $A=\boldsymbol{N}^N$ is the set of all sequences in N, ordered by putting $a=(a_1,a_2,\cdots)\le a'=(a_1',a_2',\cdots)$, provided $a_i\le a_i'$, for all $i\in N$.

$$X_a=\coprod_{m\in N}C_{a_m}^m, \tag{5}$$

$$p_{aa'}|C_{a_m'}^m=r_{a_ma_m'}^m:C_{a_m'}^m\to C_{a_m}^m\subseteq X_a, \tag{6}$$

$$p_a|C^m=r_{a_m}^m:C^m\to C_{a_m}^m\subseteq X_a. \tag{7}$$

That $\boldsymbol{p}$ is indeed a resolution of X is easily seen by verifying conditions (B1) and (B2) of [6, I, §6, Theorem 5], which are equivalent to conditions (R1) and (R2). This verification was performed in [5, Theorem 6]. Note that each X_a is a countable discrete space, hence, it is a polyhedron.

Now consider any function $f:N\to A$ and any collection of mappings

$f_m : X_{f(m)} \to Y, m \in N$. We must prove that condition (LAM) is not satisfied. Since all $q_{mm'}, q_m$ are identity mappings, (LAM) assumes the following simpler form

$$(\forall m \in N)(\forall \mathcal{V} \in \mathrm{Cov}(Y))(\exists m' \geq m)(\forall m'' \geq m')$$

$$(f_{m''} p_{f(m'')}, f) \prec \mathcal{V}. \tag{8}$$

Consequently, it suffices to show that, there exists an open covering $\mathcal{V}$ of Y such that $(f_m p_{f(m)}, f) \prec \mathcal{V}$ fails, for each $m \in N$, or equivalently, there exists a $V \in \mathrm{Cov}(Y)$ and there exist points $x^m \in X, m \in N$, such that, for each $m \in N$, the following condition holds:

(α) For each $V \in \mathcal{V}$, either $f(x^m) \notin V$ or $f_m p_{f(m)}(x^m) \notin V$.

In order to achieve this, it suffices to define, for each $m \in N$, an open covering $\mathcal{V}^m$ of I^m and a point $x^m \in C^m \subseteq X$ having the following property:

(β) For each $V \in \mathcal{V}^m$, either $h^m(x^m) \notin V$ or $f_m r^m_{f(m)}(x^m) \notin V$.

Indeed, one can then define $\mathcal{V}$ as the union of all collections $\mathcal{V}^m, m \in N$. Since the sets I^m are open in Y, $\mathcal{V}$ is an open covering of Y. Moreover, condition (α) is satisfied. To see this, consider a $V \in \mathcal{V}$. If V belongs to $\mathcal{V}^m$, then by (β), either $h(x^m) = h^m(x^m) \notin V$ or $f_m p_{f(m)}(x^m) = f_m r^m_{f(m)}(x^m) \notin V$. If $V \notin \mathcal{V}^m$, then $V \subseteq I^{m'}$, $m' \neq m$, and since $I^m \cap I^{m'} = \emptyset$ and $h(x^m) = h^m(x^m) \in I^m$, we conclude that $h(x^m) \notin V$.

In defining the covering $\mathcal{V}^m$ and the point $x^m \in C^m$, for a given $m \in N$, we distinguish two cases. $(i) f_m(C^m_{f(m)}) \subseteq I^m$. In this case we apply Lemma 1 to $h = h^m : C^m \to I^m, r^m : C^m \to C^m, n = f(m) \in N$ and $g = f_m | C^m_{f(m)} : C^m_{f(m)} \to I^m$. We obtain a covering $\mathcal{V}^m$ of I^m and a point $x^m \in C^m$ such that, for each $V \in \mathcal{V}^m$, either $h^m(x^m) \notin V$ or $f_m r^m_{f(m)}(x^m) \notin V$. (ii) There exists a point $y \in C^m_{f(m)}$ such that $f_m(y) \notin I^m$. In this case we take for $\mathcal{V}^m$ the covering which consists of the set $V = I^m$ alone and we choose for $x^m \in C^m$ a point such that $r^m_{f(m)}(x^m) = y$. Then $f_m r^m_{f(m)}(x^m) \notin I^m = V$. Hence, in both cases $\mathcal{V}^m$ and x^m satisfy (β).

REMARK 1. It is a consequence of Watanabe's approximate expansion theorem for gauged systems, that in the above example the system $\boldsymbol{Y}$ does not admit admissible meshes.

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