

# COUNTABLE PRODUCT OF FUNCTION SPACES HAVING $p$ -FRÉCHET-URYSOHN LIKE PROPERTIES

By

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**Abstract.** We exhibit in this article some classes of spaces for which properties  $\gamma$  and  $\gamma_p$  are countable additive, and we prove that for some type of spaces and ultrafilters  $p \in \omega^*$ ,  $\gamma$  is equivalent to  $\gamma_p$ . We obtain: (1) If  $\{X_n\}_{n < \omega}$  is a sequence of metrizable locally compact spaces with  $\gamma_p$  ( $p \in \omega^*$ ), then  $\prod_{n < \omega} C_\pi(X_n)$  is a  $FU(p)$ -space; (2)  $C_\pi(X)$  is a Fréchet-Urysohn (resp.,  $FU(p)$ ) space iff  $C_\pi(F(X))$  has the same property, where  $F(X)$  is the free topological group generated by  $X$ ; (3) For a locally compact metrizable and non countable space  $X$ ,  $C_\pi(X)$  is a Fréchet-Urysohn (resp.,  $FU(p)$ ) space iff  $C_\pi(L_\pi(X))$  is Fréchet-Urysohn (resp.,  $FU(p)$ ), where  $L_\pi(X)$  is the dual space of  $C_\pi(X)$ ; (4) For every Čech-complete space  $X$  and every  $p \in \omega^*$  for which  $\mathbf{R}$  does not have  $\gamma_p$ ,  $C_\pi(X)$  is Fréchet-Urysohn iff  $C_\pi(X)$  is a  $FU(p)$ -space. Also we give some results concerning  $P$ -points in  $\omega^*$  related with  $p$ -Fréchet-Urysohn property and topological function spaces.

## 0. Introduction

In [GN], [G], [Mc], [Py], the authors studied the properties needed in a space  $X$  in order to have the Fréchet-Urysohn property in the space  $C_\pi(X)$  of continuous functions from  $X$  to the real line  $\mathbf{R}$  considered with the pointwise convergence topology. They gave conditions in  $X$  in terms of cover properties. In [McN<sub>1</sub>] the next general result was proved:

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THEOREM 0.1. *Let  $X$  be a space and let  $\mathcal{A}$  be a hereditarily closed, compact network. Then the following are equivalent*

- (a)  $C_{\mathcal{A}}(X)$  is a Fréchet-Urysohn space;
- (b)  $X$  satisfies property  $\mathcal{A}\gamma$ .

In [GT<sub>1</sub>] and [GT<sub>2</sub>] the authors considered the more general concepts of WFU( $M$ )-spaces and SFU( $M$ )-spaces where  $M \subset \omega^*$ . They proved:

THEOREM 0.2. ([GT<sub>2</sub>]) *Let  $X$  be a space and let  $\emptyset \neq M \subset \omega^*$ . The following statements are equivalent*

- (a)  $C_{\pi}(X)$  is a SFU( $M$ )-space (resp., WFU( $M$ )-space);
- (b)  $X$  has property  $S\gamma_M$  (resp.,  $W\gamma_M$ ).

It is also noted in these articles (see for example [GT<sub>2</sub>, 3.2.3]) that Fréchet-Urysohn, WFU( $M$ ) and SFU( $M$ ) properties with  $\emptyset \neq M \subset \omega^*$  are not finite multiplicative, even in  $C_{\pi}$ -spaces.

On the other hand, in the generalization of the Fréchet-Urysohn property in terms of ultrafilters, in particular the concepts of  $p$ -Fréchet-Urysohn property, arises a rich variety of properties that could be very different from the original as we can appreciate in the following two theorems.

THEOREM 0.3. ([GN]) (a) *If  $C_{\pi}(X)$  is a Fréchet-Urysohn space, then  $X$  is zero-dimensional.*

(b) ([GN]) *If  $C_{\pi}(A)$  is Fréchet-Urysohn where  $A \subset \mathbf{R}$ , then  $A$  is of strong measure zero.*

These results are in contrast to the following theorem.

THEOREM 0.4. ([GT<sub>1</sub>]) *If  $X^n$  is Lindelöf for every  $n < \omega$  and  $w(X) \leq 2^{\omega}$ , then there is  $p \in \omega^*$  such that  $C_{\pi}(X)$  is an FU( $p$ )-space.*

In particular,  $C_{\pi}(\mathbf{R})$  is an FU( $p$ )-space for some  $p \in \omega^*$  and is far from being a Fréchet-Urysohn space.

In this article we will analyze when the Fréchet-Urysohn like properties are countable productive in function spaces  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  (Sections 4, 5 and 6). In particular, as a main result, we prove that  $C_{\pi}(X)$  is an FU( $p$ )-space iff  $\prod_{n < \omega} C_{\pi}(X^n)$  has the same property, obtaining some interesting Corollaries. On the other hand, we will find some class  $\mathcal{C}$  of ultrafilters  $p \in \omega^*$  for which the FU( $p$ ) property is very similar to that of Fréchet-Urysohn. Besides, we will show that

for a Čech-complete space  $X$  and  $p \in \mathcal{C}$ , “ $C_\pi(X)$  is Fréchet-Urysohn” is equivalent to “ $C_\pi(X)$  is an  $FU(p)$ -space” (Section 7). In Section 3 we will study some generalizations of Telgarsky’s games defined in [Te]. Also we will obtain some generalizations of Theorems proved in [GT<sub>1</sub>],[GT<sub>2</sub>] and [McN<sub>1</sub>] (Section 2 and Theorems 5.10, 6.5 and 8.6). In the last section we determine some conditions for which a space  $X$  has a compactification  $\kappa X$  with a Fréchet-Urysohn like function space. We are going to consider all these problems in the general frame of spaces  $C_{\mathcal{A},\mathcal{U}}(X,Y)$  where  $\mathcal{A}$  is a closed network of  $X$  and  $\mathcal{U}$  is a compatible uniformity of the space  $Y$ .

### 1. Preliminaries

The letters  $X, Y, Z, \dots$  will denote Tychonoff spaces. The set of natural numbers and its Stone-Čech compactification will be denoted by  $\omega$  and  $\beta\omega$  respectively, and  $\omega^* = \beta\omega \setminus \omega$  is the collection of the non-principal ultrafilters on  $\omega$ . If  $f: \omega \rightarrow \beta\omega$ , then  $\bar{f}: \beta\omega \rightarrow \beta\omega$  will denote the Stone extension of  $f$ . If  $X$  is a space and  $x \in X$ ,  $N(x)$  will be the set of neighborhoods of  $x$  in  $X$ .  $\mathcal{K}(X)$  or simply  $\mathcal{K}$  is the set of compact subsets of  $X$  and  $\mathcal{F}(X)$  or  $\mathcal{F}$  will denote the set of finite subsets of  $X$ , and, as usual,  $\mathcal{P}(X)$  is the collection of subsets of  $X$ . For a collection  $\mathcal{C} = \{X_\lambda : \lambda \in \Lambda\}$  of spaces,  $\coprod_{\lambda \in \Lambda} X_\lambda$  will be the free topological sum of spaces in  $\mathcal{C}$ .

The *Rudin-Keisler (pre)-order* in  $\omega^*$  is defined as follows: for  $p, q \in \omega^*$ ,  $p \leq_{RK} q$  if there is  $f: \omega \rightarrow \omega$  such that  $\bar{f}: (q) \rightarrow p$ . If  $p \leq_{RK} q$  and  $q \leq_{RK} p$ , then we say that  $p$  and  $q$  are RK-equivalent (in symbols,  $p \approx_{RK} q$ ). It is not difficult to verify that  $p \approx_{RK} q$  iff there is a permutation  $\sigma$  of  $\omega$  such that  $\bar{\sigma}(q) = p$ . The *type* of  $p \in \omega^*$  is the set  $T(p)$  of all RK-equivalent ultrafilters of  $p$ . Observe that the Rudin-Keisler pre-order in  $\omega^*$  is an order in  $\{T(p) : p \in \omega^*\}$ .

A collection  $\mathcal{A}$  of closed subsets of a space  $X$  is a *network* if for every  $x \in X$  and every  $V \in \mathcal{N}(x)$  there is  $A \in \mathcal{A}$  such that  $x \in A \subset V$ .

If  $X$  and  $Y$  are two spaces,  $C(X, Y)$  is the collection of continuous functions from  $X$  to  $Y$ , and for  $A \subset X$  and  $B \subset Y$  we set  $[A, B] = \{f \in C(X, Y) : f(A) \subset B\}$ . When  $\mathcal{A}$  is a network we can consider the topology  $\tau_{\mathcal{A}}$  in  $C(X, Y)$  generated by  $\varphi = \{[A, B] : A \in \mathcal{A} \text{ and } B \subset Y \text{ is open}\}$  as a sub-base. The pair  $(C(X, Y), \tau_{\mathcal{A}})$  is denoted by  $C_{\mathcal{A}}(X, Y)$ ,  $\tau_{\mathcal{F}}$  is the pointwise convergence topology and  $C(X, Y)$  with this topology is denoted by  $C_\pi(X, Y)$ . When  $\mathcal{A} = \mathcal{K}(X)$ , then we obtain the compact-open topology in  $C(X, Y)$  and we write  $C_K(X, Y)$ . If  $\mathcal{A}$  is a *compact* network (every  $A \in \mathcal{A}$  is compact) on  $X$ , then  $C_{\mathcal{A}}(X, Y)$  is a Tychonoff space. Observe also that for every network  $\mathcal{A}$  on  $X$ ,  $\tau_{\mathcal{A}} = \tau_{\mathcal{B}}$  where  $\mathcal{B}$  is the family of

finite unions of elements of  $\mathcal{A}$ ; this is why we will identify each network and the collection of finite unions of its elements with the same symbol without explicit mention.

When we consider a compatible uniformity  $\mathcal{U}$  in  $Y$  and  $\mathcal{A}$  is a network on  $X$ , we can define a uniformity  $\hat{\mathcal{U}}$  in  $C(X, Y)$  as follows: for each  $U \in \mathcal{U}$  and each  $A \in \mathcal{A}$  we put  $\hat{U}(A) = \{(f, g) \in C(X, Y) \times C(X, Y) : (f(x), g(x)) \in U \text{ for every } x \in A\}$ . We will denote by  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  the space of continuous functions from  $X$  to  $Y$  endowed with the topology  $\tau_{\mathcal{A}, \mathcal{U}}$  generated by the uniformity  $\hat{\mathcal{U}} = \{\hat{U}(A) : U \in \mathcal{U}, A \in \mathcal{A}\}$ . For  $f \in C(X, Y)$ ,  $U \in \mathcal{U}$  and  $A \in \mathcal{A}$ ,  $\hat{U}(A)(f) = \{g \in C(X, Y) : (g(x), f(x)) \in U \text{ for all } x \in A\}$  is a canonical neighborhood of  $f$  in  $\tau_{\mathcal{A}, \mathcal{U}}$ . If  $X \in \mathcal{A}$  we simply write  $C_{\mathcal{U}}(X, Y)$  instead of  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$ . By  $(Y, \mathcal{U})$  we will mean that  $Y$  is a space and  $\mathcal{U}$  is a compatible uniformity on  $Y$ .

For every network  $\mathcal{A}$  we have that  $\tau_{\pi} \subset \tau_{\mathcal{A}}$ , and if  $\mathcal{A}$  is a compact network on  $X$  and  $\mathcal{U}$  is a compatible uniformity of  $Y$ , then  $\tau_{\mathcal{A}} \subset \tau_{\mathcal{A}, \mathcal{U}}$ . If, in addition,  $\mathcal{A}$  is *hereditarily closed* (every closed subset of an element of  $\mathcal{A}$  belongs to  $\mathcal{A}$ ), then  $\tau_{\mathcal{A}} \subset \tau_{\mathcal{A}, \mathcal{U}}$ .

The notion of a  $p$ -limit of a sequence in a space  $X$  for  $p \in \omega^*$  was introduced by Bernstein in [B]: For a sequence  $(x_n)_{n < \omega}$  in  $X$ , the point  $x \in X$  is a  $p$ -limit of  $(x_n)_{n < \omega}$  (in symbols,  $x = p\text{-}\lim x_n$ ) if for each  $V \in \mathcal{N}(x)$ ,  $\{n < \omega : x_n \in V\} \in p$ . This definition suggests the following generalizations of the concepts of Fréchet-Urysohn property.

DEFINITION 1.1. Let  $\emptyset \neq M \subset \omega^*$  and let  $X$  be a space.

(1) (Kocinac [Ko])  $X$  is a  $WFU(M)$ -space if for every  $A \subset X$  and  $x \in \text{cl } A$ , there are  $p \in M$  and a sequence  $(x_n)_{n < \omega}$  in  $A$  such that  $x = p\text{-}\lim x_n$ .

(2) (Kocinac [Ko])  $X$  is a  $SFU(M)$ -space if for  $A \subset X$  and  $x \in \text{cl } A$ , there is a sequence  $(x_n)_{n < \omega}$  in  $A$  such that  $x = p\text{-}\lim x_n$  for every  $p \in M$ .

Observe that, for  $p \in \omega^*$ ,  $WFU(\{p\})\text{-space} = SFU(\{p\})\text{-space}$ ; in this case, we simply say  $FU(p)$ -space (this concept was discovered by Comfort and Savchenko independently). We remark that for a space  $X$  we have: (a)  $X$  has countable tightness if and only if  $X$  is a  $WFU(\omega^*)$ -space; and (b)  $X$  is a  $SFU(\omega^*)$ -space if and only if  $X$  is Fréchet-Urysohn.

DEFINITION 1.2. A space  $Y$  is a *strictly Fréchet-Urysohn space* if for every sequence  $(F_n)_{n < \omega}$  of subsets of  $Y$  and every  $y \in \bigcap_{n < \omega} \text{cl}_Y F_n$ , there exists  $y_n \in F_n$  for each  $n < \omega$ , such that  $y = \lim y_n$ .

**2.  $p$ -Fréchet-Urysohn property in  $C_{\mathcal{A}, \mathcal{B}}$ -spaces**

We say that a collection  $\mathcal{G} \subset \mathcal{P}(X)$  is an  $\mathcal{A}$ -cover of  $X$  with  $\mathcal{A} \subset \mathcal{P}(X)$  if for every  $A \in \mathcal{A}$  there is  $G \in \mathcal{G}$  such that  $A \subset G$ . If  $\mathcal{A}$  is the set of finite subsets of  $X$ ,  $\mathcal{G}$  is called an  $\omega$ -cover.

DEFINITION 2.1. Let  $\mathcal{A}$  and  $\mathcal{B}$  be two networks on  $X$  and  $\emptyset \neq M \subset \omega^*$ .

(1) A space  $X$  satisfies property  $(\mathcal{A}, \mathcal{B})-W\gamma_M$  if for every open  $\mathcal{A}$ -cover  $\mathcal{G}$  of  $X$  there is a sequence  $(G_n)_{n < \omega}$  in  $\mathcal{G}$  and there is  $p \in M$  such that

$$X = \mathcal{B} - \lim_p G_n$$

where this last expression means that for each  $B \in \mathcal{B}$ ,  $\{n < \omega : B \subset G_n\} \in p$ .

(2) A space  $X$  satisfies property  $(\mathcal{A}, \mathcal{B})-S\gamma_M$  if for every open  $\mathcal{A}$ -cover  $\mathcal{G}$  of  $X$  there is a sequence  $(G_n)_{n < \omega}$  in  $\mathcal{G}$  such that

$$x = \mathcal{B} - \lim_p G_n$$

for every  $p \in M$ .

(3) A space  $X$  satisfies property  $(\mathcal{A}, \mathcal{B})-\gamma$  if  $X$  satisfies  $(\mathcal{A}, \mathcal{B})-S\gamma_{\omega^*}$  or equivalently, if for every open  $\mathcal{A}$ -cover  $\mathcal{G}$  there is a sequence  $(G_n)_{n < \omega}$  in  $\mathcal{G}$  such that  $X = \mathcal{B} - \lim G_n$  that is, every  $B \in \mathcal{B}$  belongs to  $G_n$  for every  $n$  bigger than a natural number.

(4) A space  $X$  satisfies property  $(\mathcal{A}, \mathcal{B})-\varepsilon$  if for every open  $\mathcal{A}$ -cover  $\mathcal{G}$  of  $X$  we can find a countable  $\mathcal{B}$ -subcover of  $\mathcal{G}$ .

(5) For a space  $X$  and a network  $\mathcal{A}$  on  $X$ , the least cardinal  $\alpha$  for which every open  $\mathcal{A}$ -cover of  $X$  has an  $\mathcal{A}$ -subcover of cardinality  $\alpha$  is denoted by  $\mathcal{A}L(X)$ . We say that a space  $X$  is  $\mathcal{A}$ -Lindelöf if  $\mathcal{A}L(X) = \aleph_0$ .

(6) For a cardinal number  $\mathcal{K}$  and a network  $\mathcal{A}$  on a space  $X$  we will say that  $X$  is  $\mathcal{K}$ - $\mathcal{A}$ -bounded if  $X$  is infinite and every subset of  $X$  of cardinality  $\leq \mathcal{K}$  is contained in some element of  $\mathcal{A}$ . A space  $X$  is  $\mathcal{K}$ -bounded if  $X$  is  $\mathcal{K}$ - $\mathcal{A}$ -bounded.

(7) Let  $\mathcal{A}$  be a network on  $X$ . We say that  $X$  is  $\mathcal{A}$ -hemicompact if there is a countable collection  $\mathcal{A}'$  of  $\mathcal{A}$  which is an  $\mathcal{A}$ -cover. A space  $X$  is hemicompact if  $X$  is  $\aleph_0$ -hemicompact.

If  $\mathcal{A} = \mathcal{B}$  we only write  $\mathcal{A}S\gamma_M$ ,  $\mathcal{A}\gamma$ ,  $\mathcal{A}\varepsilon$  etc. and when  $\mathcal{A} = \mathcal{B}$  is  $\mathcal{F}$  or  $\mathcal{K}$  we simply write  $S\gamma_M$ ,  $W\gamma_M$ ,  $\gamma$ ,  $\varepsilon$ , and  $kS\gamma_M$ ,  $kW\gamma_M$ ,  $k\gamma$ ,  $k\varepsilon$  respectively. If  $M = \{p\}$  for some  $p \in \omega^*$ , then  $(\mathcal{A}, \mathcal{B})-\gamma_p$  will be the equivalent properties  $(\mathcal{A}, \mathcal{B})-S\gamma_{\{p\}}$  and  $(\mathcal{A}, \mathcal{B})-W\gamma_{\{p\}}$ . Observe that every  $\mathcal{A}$ -hemicompact space satisfies  $\mathcal{A}\gamma$ .

The following summarizes some basic relations between the properties listed in Definition 2.1. (The proof of (4) and (6) below are similar to those given for Theorem 1.5 and Lemma 2.2 in [GT<sub>1</sub>] respectively; see also 2.3.3. (5) is a consequence of (4)).

**THEOREM 2.2.** *Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  and  $\mathcal{D}$  be networks on  $X$  and let  $\emptyset \neq M, N \subset \omega^*$ . Then,*

(1) For  $p \in M: (\mathcal{A}, \mathcal{B}) - \gamma \Rightarrow (\mathcal{A}, \mathcal{B}) - S\gamma_M \Rightarrow (\mathcal{A}, \mathcal{B}) - \gamma_p \Rightarrow (\mathcal{A}, \mathcal{B}) - W\gamma_M \Rightarrow (\mathcal{A}, \mathcal{B}) - \varepsilon$ .

(2) If  $\mathcal{A}$  is compact then,  $(\mathcal{A}, \mathcal{B}) - \varepsilon \Rightarrow X$  has  $\varepsilon$ .

(3) If  $\mathcal{A}$  refines  $\mathcal{C}$  and  $\mathcal{D}$  refines  $\mathcal{B}$ :

$(\mathcal{A}, \mathcal{B}) - \gamma \Rightarrow (\mathcal{C}, \mathcal{D}) - \gamma; (\mathcal{A}, \mathcal{B}) - S\gamma_M \Rightarrow (\mathcal{C}, \mathcal{D}) - S\gamma_M; (\mathcal{A}, \mathcal{B}) - W\gamma_M \Rightarrow (\mathcal{C}, \mathcal{D}) - W\gamma_M$ , and  $(\mathcal{A}, \mathcal{B}) - \varepsilon \Rightarrow (\mathcal{C}, \mathcal{D}) - \varepsilon$ .

(4) Let  $p, q \in \omega^*$  with  $p \leq_{RK} q$ , let  $f: \omega \rightarrow \omega$  be such that  $\bar{f}(q) = p$ , and suppose that  $X = \mathcal{B} - \lim_p G_n$ . Then,  $X = \mathcal{B} - \lim_q F_n$  where  $F_n = G_{f(n)}$ .

(5) If for every  $p \in M$  there is  $q \in N$  such that  $p \leq_{RK} q$ , then

$$(\mathcal{A}, \mathcal{B}) - W\gamma_M \Rightarrow (\mathcal{A}, \mathcal{B}) - W\gamma_N.$$

(6) Let  $p \in \omega^*$  and let  $X$  be a space with  $\mathcal{A}\varepsilon$ . Then,  $X$  has  $(\mathcal{A}, \mathcal{B}) - \gamma_p$  iff for every countable open  $\mathcal{A}$ -cover  $\{B_n : n < \omega\}$  of  $X$  there is  $q \in \omega^*$  such that  $q \leq_{RK} p$  and  $X = \mathcal{B} - \lim_q B_n$ .

**REMARKS AND NOTATIONS 2.3.** (1) Observe that if  $\mathcal{C}$  and  $\mathcal{D}$  are networks on  $X$  and  $Z$  respectively,  $F \subset X$  and  $f: X \rightarrow Y$  is a continuous function from  $X$  onto  $Y$ , then  $\mathcal{C}_F = \{C \cap F : C \in \mathcal{C}\}$ ,  $f(\mathcal{C}) = \{\text{cl } f(C) : C \in \mathcal{C}\}$ ,  $\mathcal{C}^n = \{C_1 \times \dots \times C_n : C_i \in \mathcal{C} \text{ and } n < \omega\}$  and  $\mathcal{C} \times \mathcal{D} = \{C \times D : C \in \mathcal{C} \text{ and } D \in \mathcal{D}\}$  are networks on  $F$ ,  $Y$ ,  $X^n$  and  $X \times Z$  respectively.

(2) If  $\mathcal{A}$  is a network and  $X = \mathcal{A} - \lim_p G_n$  for some sequence  $(G_n)_{n < \omega}$  of subsets of  $X$ , then  $X = \bigcup_{n \in B} G_n$  for each  $B \in p$ . In fact, for every  $x \in X$  there is an  $A \in \mathcal{A}$  containing  $x$ . Then,  $x \in G_j$  for every  $j \in B \cap \{n < \omega : A \subset G_n\}$ .

(3) If  $X$  has  $\mathcal{A}\gamma_p$  and  $\mathcal{G} = \{G_n : n < \omega\}$  is an open  $\mathcal{A}$ -cover of  $X$ , then there is  $q \leq_{RK} p$  such that  $X = \mathcal{A} - \lim_q G_n$ : In fact, there is a sequence  $(G_{n_j})_{j < \omega}$  in  $\mathcal{G}$  such that  $X = \mathcal{A} - \lim_p G_{n_j}$ . Define  $f: \omega \rightarrow \omega$  by  $f(j) = n_j$  for each  $j < \omega$ , and take  $q = \bar{f}(p)$ .

(4) Every second countable space  $X$  satisfies  $\mathcal{A}\varepsilon$  for every compact network  $\mathcal{A}$  on  $X$ .

(5) In that which follows  $\vartheta$  will denote one of the elements in the set of

properties  $\{S\gamma_M : \emptyset \neq M \subset \omega^*\} \cup \{W\gamma_M : \emptyset \neq M \subset \omega^*\} \cup \{\varepsilon\}$ . In particular  $\vartheta$  may be the property  $\gamma$ .

**THEOREM 2.4.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two networks on  $X$  and suppose that  $X$  satisfies  $(\mathcal{A}, \mathcal{B})-\vartheta$ . Then,*

- (1)  $F \subset X$  has  $(\mathcal{A}_F, \mathcal{B}_F)-\vartheta$  if  $F$  is closed.
- (2) If  $F = \bigcup_{n < \omega} F_n \subset X$  where  $F_i$  is closed for every  $i < \omega$ ,  $\{F_n : n < \omega\}$  is a  $\mathcal{B}_F$ -cover and  $\mathcal{A}$  refines  $\mathcal{B}$ , then  $F$  has  $(\mathcal{A}_F, \mathcal{B}_F)-\vartheta$ .
- (3)  $Y$  has  $(f(\mathcal{A}), f(\mathcal{B}))- \vartheta$  if  $f : X \rightarrow Y$  a continuous function from  $X$  onto  $Y$  and, either  $\mathcal{B}$  is compact or  $Y$  is normal or  $f$  is closed.
- (4)  $X^j$  has  $(\mathcal{A}^j, \mathcal{B}^j)-\vartheta$  for every  $j < \omega$  if  $\mathcal{A}$  is a compact network on  $X$ .

**PROOF.** We will only prove the assertions in 2, 3 (assuming that  $Y$  is normal) and 4 when  $\vartheta = S\gamma_M(\emptyset \neq M \subset \omega^*)$ , the rest of the proofs are analogous.

(2) We may suppose that  $F_1 \subset F_2 \subset \dots$ . If  $\mathcal{G}$  is an open  $\mathcal{A}_F$ -cover of  $F$ , then  $\mathcal{G}$  is an open  $\mathcal{A}_{F_n}$ -cover of  $F_n$  for every  $n < \omega$ . Because of (1), for each  $n < \omega$  there is a countable  $\mathcal{B}_{F_n}$ -subcover  $\mathcal{G}_n$  of  $\mathcal{G}$ . Since  $\{F_n : n < \omega\}$  is a  $\mathcal{B}_F$ -cover,  $\bigcup_{n < \omega} \mathcal{G}_n$  is a countable  $\mathcal{B}_F$ -subcover of  $\mathcal{G}$ . That means that  $F$  has  $(\mathcal{A}_F, \mathcal{B}_F)-\varepsilon$ , and hence we can suppose, without loss of generality, that  $\mathcal{G}$  is countable:  $\mathcal{G} = \{G_n : n < \omega\}$ . The collection  $\mathcal{U} = \{U_n = G_n \cup (X \setminus F_n) : n < \omega\}$  is an open  $\mathcal{A}$ -cover of  $X$ , so there is a sequence  $(U_{n_j})_{j < \omega}$  in  $\mathcal{U}$  such that  $X = \mathcal{B}-\lim_p U_{n_j}$  for all  $p \in M$ . Therefore, again using the fact that the set of  $F_n$ 's is a  $\mathcal{B}_F$ -cover, we obtain that  $F = \mathcal{B}_F-\lim_p G_{n_j}$  for every  $p \in M$ .

(3) Let  $\mathcal{G}$  be an open  $f(\mathcal{A})$ -cover of  $Y$ . For each  $A \in \mathcal{A}$ , there exists  $G_A \in \mathcal{G}$  such that  $\text{cl } f(A) \subset G_A$ . Since  $Y$  is normal. We can find for each  $A \in \mathcal{A}$  an open subset  $H_A$  of  $Y$  satisfying  $\text{cl } f(A) \subset H_A \subset \text{cl } H_A \subset G_A$ . The collection  $\mathcal{H} = \{H_A : A \in \mathcal{A}\}$  is an open  $f(\mathcal{A})$ -cover of  $Y$ . Thus,  $\mathcal{F} = \{f^{-1}(H) : H \in \mathcal{H}\}$  is an open  $\mathcal{A}$ -cover of  $X$ . So, there exists a sequence  $(A_i)_{i \in \mathbb{N}}$  in  $\mathcal{A}$  for which  $X = \mathcal{B}-\lim_p f^{-1}(H_{A_i})$  for every  $p \in M$ . Therefore,  $Y = f(\mathcal{B})-\lim_p G_{A_i}$  for every  $p \in M$ .

(4) Let  $\mathcal{G}$  be an open  $\mathcal{A}^j$ -cover of  $X^j$ . For each  $A \in \mathcal{A}$  there is a  $G \in \mathcal{G}$  such that  $A^j \subset G$ . Since  $A$  is compact, there is an open subset  $U_A$  for which  $A^j \subset (U_A)^j \subset G$ . The family  $\mathcal{G}' = \{(U_A)^j : A \in \mathcal{A}\}$  is an open  $\mathcal{A}^j$ -cover of  $X^j$  that refines  $\mathcal{G}$  (we are assuming that  $\mathcal{A}$  is closed under finite unions). Since  $X$  has  $(\mathcal{A}, \mathcal{B})-S\gamma_M$ , there is a sequence  $(A_n)_{n < \omega}$  in  $\mathcal{A}$  such that  $X = \mathcal{B}-\lim_p U_{A_n}$  for every  $p \in M$ . Since  $\mathcal{B}$  is closed under finite unions we obtain that  $X^j = \mathcal{B}^j-\lim_p (U_{A_n})^j$ . ■

For a network  $\mathcal{A}$  on  $X$ , a compatible uniformity  $\mathcal{U}$  in  $Y$ , a sequence  $(f_n)_{n < \omega}$  in  $C(X, Y)$ , a function  $f \in C(X, Y)$  and a  $p \in \omega^*$  the symbols:  $(\mathcal{A}, \mathcal{U})\text{-}\lim_p f_n = f$ ,  $(\mathcal{A}, \mathcal{U})\text{-}\lim f_n = f$ ,  $\mathcal{A}\text{-}\lim_p f_n = f$  and  $\mathcal{A}\text{-}\lim f_n = f$  mean that  $(f_n)_{n < \omega}$   $p$ -converge (resp., converge) in  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  (resp.,  $C_{\mathcal{A}}(X, Y)$ ).

The least cardinality of a base for a uniformity  $\mathcal{U}$  is called the *weight* of  $\mathcal{U}$  and is denoted by  $w(\mathcal{U})$ .

In the next result we obtain some relations between the tightness of  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  and  $C_{\mathcal{A}}(X, Y)$  and the  $\mathcal{A}$ -Lindelöf degree of  $X$ . Its proof can be achieved by using similar ideas to those developed in the proof of Theorem 4.7.1 in [McN<sub>1</sub>].

**THEOREM 2.5.** *Let  $\mathcal{A}$  be a network in a space  $X$ , and let  $\mathcal{U}$  be a compatible uniformity in a space  $Y$ . Then,*

- (1)  $\mathcal{A}L(X) \leq \min\{t(C_{\mathcal{A}}(X, Y)), t(C_{\mathcal{A}, \mathcal{U}}(X, Y))\}$  if  $\mathcal{A}$  is compact and  $Y$  contains a non trivial path;
- (2)  $t(C_{\mathcal{A}, \mathcal{U}}(X, Y)) \leq \mathcal{A}L(X) \cdot w(\mathcal{U})$ ;
- (3)  $t(C_{\mathcal{A}}(X, Y)) \leq \mathcal{A}L(X) \cdot \min\{\mathcal{X}(y, Y) : y \in Y\}$  if  $C_{\mathcal{A}}(X, Y)$  is homogeneous where  $\mathcal{X}(y, Y)$  is the character of  $y$  in  $Y$ .

As a consequence of the previous result we have:

**THEOREM 2.6.** *Let  $\mathcal{A}$  be a compact network in a space  $X$  and let  $Y$  be a space with a non-trivial path having  $\mathcal{U}$  as a compatible uniformity. Then,*

- (1) *If  $Y$  is metrizable then,  $X$  has  $\mathcal{A}\mathcal{E} \Leftrightarrow C_{\mathcal{A}, \mathcal{U}}(X, Y)$  has countable tightness;*
- (2) *If  $C_{\mathcal{A}}(X, Y)$  is homogeneous and  $Y$  is first countable, then  $X$  has  $\mathcal{A}\mathcal{E} \Leftrightarrow C_{\mathcal{A}}(X, Y)$  has countable tightness.*

For a  $p \in \omega^*$ ,  $\|\{q \in \omega^* : p \leq_{RK} q\}\| = 2^{2^p}$ ; hence using Theorem 2.2.6 we obtain the following result that generalizes (and its proof is similar to) Theorem 2.3 in [GT<sub>1</sub>].

**THEOREM 2.7.** *Let  $\mathcal{A}$  be a Lindelöf network of  $X$ . If  $X$  has  $\mathcal{A}\mathcal{E}$  and  $w(X) \leq 2^\omega$ , then there is  $M \subset \omega^*$  of cardinality  $= 2^{2^\omega}$  such that  $X$  has  $\mathcal{A}\gamma_q$  for every  $q \in M$ .*

In order to prove the main Theorem of this section we need some lemmas.

**LEMMA 2.8.** *Let  $\mathcal{A}$  be a network on  $X$  and let  $\mathcal{U}$  be a compatible uniformity of a space  $Y$ . Let  $\emptyset \neq \Phi \subset C_{\mathcal{A}, \mathcal{U}}(X, Y)$ ,  $f \in \text{cl}_{C_{\mathcal{A}, \mathcal{U}}(X, Y)} \Phi$  and  $U \in \mathcal{U}$ . Then*

$\mathcal{G}(\Phi, f, U) = \{\text{coz}_{f,U}(g) : g \in \Phi\}$  is an open  $\mathcal{A}$ -cover of  $X$ , where  $\text{coz}_{f,U}(g) = \{x \in X : (f(x), g(x)) \in U\}$ .

PROOF. Let  $A \in \mathcal{A}$  and consider the neighborhood  $\hat{U}(A)(f)$  of  $f$ . By assumption there is  $g \in \hat{U}(A)(f) \cap \Phi$ . Then, we have that  $A \subset \text{coz}_{f,U}(g)$ . ■

LEMMA 2.9. Let  $\mathcal{A}$  be a compact network on  $X$  and  $(Y, \mathcal{U})$  be a space with a non trivial path  $\sigma : [0, 1] \rightarrow Y$ . Let  $\mathcal{H}$  be an open  $\mathcal{A}$ -cover of  $X$  with  $X \notin \mathcal{H}$  and let  $U \in \mathcal{U}$  such that  $(\sigma(0), \sigma(1)) \notin U$ . Then,  $f_0 \in (\text{cl}_{C_{\mathcal{A}, \mathcal{U}}(X, Y)} \Phi) \setminus \Phi$ , where  $\Phi = \Phi(\mathcal{H}, f_0, U) = \{g \in C(X, Y) : \text{coz}_{f_0, U}(g) \subset H \text{ for some } H \in \mathcal{H}\}$  and  $f_0$  is the constant function  $\sigma(0)$  from  $X$  to  $Y$ .

PROOF. Since  $X \notin \mathcal{H}$ ,  $f_0 \notin \Phi$ . Now let  $\hat{V}(A)(f_0) \in \mathcal{N}(f_0)$  where  $V \in \mathcal{U}$  and  $A \in \mathcal{A}$ . By assumption, there is  $H \in \mathcal{H}$  containing  $A$ . We take a continuous function  $t : X \rightarrow [0, 1]$  for which  $t(A) = \{0\}$  and  $t(X \setminus H) = \{1\}$ . If  $g = \sigma \circ t$ , then  $g \in \Phi \cap \hat{V}(A)(f_0)$ . ■

LEMMA 2.10. Let  $p \in \omega^*$ ,  $\mathcal{U}$  be a compatible uniformity of a space  $Y$  and  $\mathcal{A}$  be a network on a space  $X$ . Let  $(f_n)_{n < \omega}$  be a sequence in  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$ . Then,  $X = \mathcal{A}\text{-}\lim_p \text{coz}_{f,U} f_n$  for every  $U \in \mathcal{U}$  if and only if  $f = (\mathcal{A}, \mathcal{U})\text{-}\lim_p f_n$ .

PROOF. ( $\Rightarrow$ ) Consider a canonical neighborhood  $\hat{U}(A)(f)$  of  $f$ . By assumption we have that  $\{n < \omega : A \subset \text{coz}_{f,U} f_n\} \in p$ . Then,  $\{n < \omega : f_n \in \hat{U}(A)(f)\} \in p$ . That is  $f = (\mathcal{A}, \mathcal{U})\text{-}\lim_p f_n$ .

( $\Leftarrow$ ) Let  $A \in \mathcal{A}$  and  $U \in \mathcal{U}$ . We know that  $\{n < \omega : f_n \in \hat{U}(A)(f)\} \in p$ , so  $\{n < \omega : A \subset \text{coz}_{f,U}(f_n)\} \in p$ . ■

LEMMA 2.11. Let  $p \in \omega^*$ ,  $\mathcal{A}$  be a network on a space  $X$  and  $(Y, d)$  a metric space with a compatible uniformity  $\mathcal{U}$ . Let  $(f_n)_{n < \omega}$  be a sequence of elements belonging to  $C(X, Y)$  and  $f \in C(X, Y)$ , and let  $(\varepsilon_n)_{n < \omega}$  be a sequence of positive real numbers. If for every  $\varepsilon > 0$  we have that  $\{n < \omega : \varepsilon_n < \varepsilon\} \in p$  and

$$X = \mathcal{A}\text{-}\lim_p \text{coz}_{f, \varepsilon_n} f_n,$$

then  $f = (\mathcal{A}, \mathcal{U})\text{-}\lim_p f_n$

PROOF. The set  $W(A, f, \varepsilon) = \{g \in C(X, Y) : d(g(x), f(x)) < \varepsilon \forall x \in A\}$  is a canonical neighborhood of  $f$  in  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$ . By assumption,  $\{n < \omega : f_n \in W(A, f, \varepsilon_n)\} \cap \{n < \omega : \varepsilon_n < \varepsilon\} \in p$ , so  $\{n < \omega : f_n \in W(A, f, \varepsilon)\} \in p$ . Therefore,  $f = (\mathcal{A}, \mathcal{U})\text{-}\lim_p f_n$ . ■

The following theorem is a consequence of the previous lemmas and its proof is similar to that of Theorem 2.13 in [GT<sub>1</sub>].

**THEOREM 2.12.** *Let  $\emptyset \neq M \subset \omega^*$ , Let  $\mathcal{A}$  be a compact network on a non  $\omega$ - $\mathcal{A}$ -bounded space  $X$  and let  $(Y, \mathcal{U})$  be a metrizable space with a non trivial path. Then,  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is a WFU( $M$ )-space (resp., SFU( $M$ )-space) if and only if  $X$  has  $\mathcal{A}W\gamma_M$  (resp.,  $\mathcal{A}S\gamma_M$ ). In particular,  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is a Fréchet-Urysohn space if and only if  $X$  has  $\mathcal{A}\gamma$ .*

**PROOF.**  $\Rightarrow$ ) Let  $\mathcal{G}$  be an open  $\mathcal{A}$ -cover of  $X$  such that  $X \notin \mathcal{G}$ . Let  $\sigma: [0, 1] \rightarrow Y$  be a path satisfying  $\sigma(0) \neq \sigma(1)$  and let  $U \in \mathcal{U}$  be such that  $(\sigma(0), \sigma(1)) \notin U$ . Define  $f_0: X \rightarrow Y$  by  $f_0(x) = \sigma(0)$  for every  $x \in X$ . If  $\Phi = \Phi(\mathcal{G}, f_0, \mathcal{U})$ , then  $f_0 \in (\text{cl}_{C_{\mathcal{A}, \mathcal{U}}(X, Y)} \Phi) \setminus \Phi$  (Lemma 2.9). Since  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is a WFU( $M$ )-space (resp., SFU( $M$ )-space), there is a sequence  $(f_n)_{n < \omega}$  in  $\Phi$  such that  $f_0 = \mathcal{A}\text{-}\lim_p f_n$  for a  $p \in M$  (resp., for every  $p \in M$ ). For each  $n < \omega$ , choose  $G_n \in \mathcal{G}$  for which  $\text{coz}_{f_0, U}(f_n) \subset G_n$ . We claim that  $X = \mathcal{A}\text{-}\lim_p G_n$ : in fact, fix  $A \in \mathcal{A}$ . Since  $f_0 = \mathcal{A}\text{-}\lim_p f_n$ ,  $\{n < \omega : f_n \in \hat{U}(A)(f_0)\} \in p$ ; thus,  $\{n < \omega : A \subseteq \text{coz}_{f_0, U}(f_n)\} \in p$ . Therefore,  $\{n < \omega : A \subset G_n\} \in p$ .

$\Leftarrow$ ) Let  $d$  be a compatible metric in  $(Y, \mathcal{U})$ . Let  $\Phi \subset C_{\mathcal{A}, \mathcal{U}}(X, Y)$  and suppose that  $f \in (\text{cl}_{C_{\mathcal{A}, \mathcal{U}}(X, Y)} \Phi) \setminus \Phi$ . If  $X$  is finite, then  $\mathcal{A} = \mathcal{F}$  and  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is the first countable space  $Y^n$  for some  $n < \omega$ . Now, suppose that  $X$  is infinite. We take  $Z = \{x_n : n < \omega\}$  such that  $x_n \neq x_m$  if  $n < m < \omega$  and  $Z$  is not contained in any  $A \in \mathcal{A}$ . Let  $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots$  be a sequence of positive real numbers converging to 0. For each  $n < \omega$ ,  $\mathcal{G}_n = \mathcal{G}(\Phi, f, \varepsilon_n)$  is an open  $\mathcal{A}$ -cover of  $X$  (Lemma 2.8). Define, for each  $n < \omega$ ,  $\mathcal{H}_n = \{G \setminus \{x_n\} : G \in \mathcal{G}_n\}$  and  $\mathcal{H} = \bigcup_{n < \omega} \mathcal{H}_n$ . It happens that  $\mathcal{H}$  is an open  $\mathcal{A}$ -cover of  $X$ . Since  $X$  has  $\mathcal{A}W\gamma_M$  (resp.,  $\mathcal{A}S\gamma_M$ ), there is a sequence  $(H_j)_{j < \omega}$  in  $\mathcal{H}$  such that  $X = \mathcal{A}\text{-}\lim_p H_j$  for a  $p \in M$  (resp., for every  $p \in M$ ). For each  $j < \omega$  there are  $f_j \in \Phi$ ,  $n_j < \omega$  and  $G_j \in \mathcal{G}_{n_j}$  such that  $H_j = G_j \setminus \{x_{n_j}\}$  and  $G_j = \text{coz}_{f_0, \varepsilon_{n_j}} f_j$ . Thus  $X = \mathcal{A}\text{-}\lim_p (\text{coz}_{f_0, \varepsilon_{n_j}} f_j)$ . Suppose that there is  $\varepsilon > 0$  such that  $\{j < \omega : \varepsilon < \varepsilon_{n_j}\} \in p$ . Then, there is  $m < \omega$  for which  $B = \{j < \omega : n_j = m\} \in p$ , hence  $X = \bigcup_{j \in B} H_j$  (see Remark 2.3.2) and  $H_j \in \mathcal{H}_m$  for every  $j \in B$ , which is a contradiction since  $x_m \notin H_j$  for every  $j \in B$ . Therefore,  $\{j < \omega : \varepsilon_{n_j} < \varepsilon\} \in p$  for all  $\varepsilon > 0$ . Now the conclusion is obtained from Lemma 2.10. ■

**REMARK 2.13.** Observe that the sufficiency in Theorem 2.12 was proved only using the facts that  $X$  is not an  $\omega$ - $\mathcal{A}$ -bounded space and  $Y$  is a metrizable space, and in the proof of the necessity we only used the hypothesis regarding  $Y$  and  $\mathcal{A}$ .

COROLLARY 2.14. *Let  $\mathcal{A}$  be a network on a  $\mathcal{A}$ -hemicompact space  $X$ . If  $(Y, \mathcal{U})$  is metrizable, then  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is Fréchet-Urysohn. In particular, for every hemicompact (resp., countable) space  $X$ ,  $C_k(X, Y)$  (resp.,  $C_\pi(X, Y)$ ) is a Fréchet-Urysohn space.*

PROOF. If  $X \in \mathcal{A}$ , then  $\tau_{\mathcal{A}, \mathcal{U}}$  is the uniform topology, so  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is metrizable. If  $X \notin \mathcal{A}$ , then  $X$  is not an  $\omega$ - $\mathcal{A}$ -bounded space, and is easy to see that  $X$  has  $\mathcal{A}\gamma$ . Then,  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is a Fréchet-Urysohn space. ■

PROBLEM 2.15. For every  $\sigma$ -compact space  $X$  and every metrizable space  $Y$ , is  $C_k(X, Y)$  Fréchet-Urysohn?

THEOREM 2.16. *Let  $\emptyset \neq M \subset \omega^*$ , let  $\mathcal{A}$  be a compact network on a non  $\omega$ - $\mathcal{A}$ -bounded space  $X$  and let  $(Y, \mathcal{U})$  be a metrizable space with a non trivial path. Then,  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is an SFU( $M$ )-space (resp., WFU( $M$ )-space) if and only if  $(C_{\mathcal{A}, \mathcal{U}}(X, Y))^\omega$  is an SFU( $M$ )-space (resp., WFU( $M$ )-space).*

PROOF. Let  $D$  be a countable discrete space and let  $\mathcal{C} = \mathcal{F}(D)$  be the set of finite subsets of  $D$ .  $\mathcal{C}$  is a compact network on  $D$  and  $D$  is  $\mathcal{C}$ -hemicompact. Because of Theorem 2.12 we know that  $X$  has  $\mathcal{A}S\gamma_M$ , and using the result in Lemma 5.10 below,  $X \times D$  has  $(\mathcal{A} \times \mathcal{C}) - S\gamma_M$  (see 2.3.1) Applying again 2.12, we obtain that  $C_{(\mathcal{A} \times \mathcal{C}), \mathcal{U}}(X \times D, Y) \cong (C_{\mathcal{A}, \mathcal{U}}(X, Y))^\omega$  is an SFU( $M$ )-space. ■

For a network  $\mathcal{A}$  on a space  $X$  we say that  $X$  has *strictly*  $\mathcal{A}\gamma$  if for each sequence  $(\mathcal{G}_n)_{n < \omega}$  of open  $\mathcal{A}$ -covers of  $X$ , there exists  $G_n \in \mathcal{G}_n$  for each  $n < \omega$ , such that  $X = \mathcal{A} - \lim G_n$ .

Following arguments similar to those used in [GN, pag 155] it is possible to prove that for every network  $\mathcal{A}$  on a non  $\omega$ - $\mathcal{A}$ -bounded space  $X$ ,  $\mathcal{A}\gamma$  is equivalent to strict  $\mathcal{A}\gamma$  property. Moreover, using Lemmas 2.8 and 2.11 and making some changes in the proof of  $(\gamma') \Rightarrow$  (iii) in [GN, pag 156] we also obtain:

LEMMA 2.17. *Let  $X$  be a space and  $\mathcal{A}$  be a network on  $X$ . Let  $(Y, \mathcal{U})$  be a metrizable space. If  $X$  satisfies strictly  $\mathcal{A}\gamma$ , then  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is a strictly Fréchet-Urysohn space.*

Hence we obtain the following result (see Theorem 0.1).

THEOREM 2.18. *Let  $\mathcal{A}$  be a compact network on a non  $\omega$ - $\mathcal{A}$ -bounded space*

$X$  and let  $(Y, \mathcal{U})$  be a metrizable space with a non trivial path. Then the following conditions are equivalent.

- (a)  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is a strictly Fréchet-Urysohn space;
- (b)  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is a Fréchet-Urysohn space;
- (c)  $X$  has  $\mathcal{A}\gamma$ .
- (d)  $X$  satisfies strictly  $\mathcal{A}\gamma$ .

We cannot obtain a similar result when we consider the more general properties that we are taking into account in this paper. In fact,  $\mathbf{R}$  satisfies  $\gamma_p$  for some  $p \in \omega^*$  (Theorem 0.4) but  $\mathbf{R}$  does not satisfy strictly  $\gamma_p$  for any  $p \in \omega^*$  (see [GT<sub>2</sub>, Corollary 2.5]).

### 3. Topological Games

In this Section we are going to analyze some topological games which generalize those defined by Telgarski in [Te] and we will relate this theory with properties  $\mathcal{A}\mathcal{S}\gamma_M$  and  $\mathcal{A}\mathcal{W}\gamma_M$ .

For a space  $(X, \tau)$  and two closed networks  $\mathcal{A}$  and  $\mathcal{B}$  on  $X$ , and for  $\emptyset \neq M \subset \omega^*$  we define the following games  $G(\mathcal{A}, \mathcal{B}, X)$ ,  $G^1(\mathcal{A}, \mathcal{B}, X)$ ,  $G_{WM}(\mathcal{A}, \mathcal{B}, X)$  and  $G_{SM}(\mathcal{A}, \mathcal{B}, X)$ : There are two players I and II. They alternately choose elements belonging to a sequence in  $\mathcal{P}(X)$  so that each player knows  $\mathcal{A}, \mathcal{B}, (X, \tau)$ , and the  $n$  first choices made by both players when the  $n + 1$  element of  $\mathcal{P}(X)$  has to be chosen. Player I chooses first and each of his choices belongs to  $\mathcal{A}$ . If  $A_n$  is the  $n$  choice of I, then the  $n$  choice of II is an open set  $G_n$  such that  $A_n \subset G_n$ . A *play* in  $G$ , where  $G = G(\mathcal{A}, \mathcal{B}, X)$  (resp.,  $G = G^1(\mathcal{A}, \mathcal{B}, X)$ ;  $G = G_{WM}(\mathcal{A}, \mathcal{B}, X)$ ;  $G = G_{SM}(\mathcal{A}, \mathcal{B}, X)$ ), is a sequence  $P = (A_1, G_1, A_2, G_2, \dots, A_n, G_n, \dots)$  such that for each  $n < \omega$   $A_n \subset G_n$ . I wins  $P$  if for each  $B \in \mathcal{B}$  there is  $n < \omega$  such that  $B \subset G_n$  (resp.,  $X = \mathcal{B} - \lim G_n$ ; there is  $p \in M$  such that  $X = \mathcal{B} - \lim_p G_n$ ; for every  $p \in M$ ,  $X = \mathcal{B} - \lim_p G_n$ ). II wins  $P$  if I does not win  $P$ . A finite sequence  $(E_m)_{m < k}$  in  $\mathcal{P}(X)$  is *admissible* for the game  $G$  if there is a play  $P$  in  $G$  such that  $P = (E_1, \dots, E_m, \dots)$ . A function  $s$  is a *strategy* for I (resp., II) in  $G$  if the domain of  $s$  consists of admissible sequences  $(E_1, \dots, E_n)$  with  $n$  even (resp., odd) and with values in  $\mathcal{P}(X)$  such that for each  $(E_1, \dots, E_n)$  in the domain of  $s$ ,  $(E_1, \dots, E_n, s(E_1, \dots, E_n))$  is an admissible sequence. A strategy  $s$  is said to be *winning* for I (resp., II) in  $G$  if I (resp., II) using  $s$  wins each play of  $G$ .  $I(G)$  (resp.,  $II(G)$ ) denotes the set of all winning strategies of I (resp., II) in  $G$ .

When  $\mathcal{B} = \mathcal{A}$  we denote  $G(\mathcal{A}, \mathcal{B}, X)$ ,  $G^1(\mathcal{A}, \mathcal{B}, X)$ ,  $G_{WM}(\mathcal{A}, \mathcal{B}, X)$  and  $G_{SM}(\mathcal{A}, \mathcal{B}, X)$  by  $G(\mathcal{A}, X)$ ,  $G^1(\mathcal{A}, X)$ ,  $G_{WM}(\mathcal{A}, X)$  and  $G_{SM}(\mathcal{A}, X)$  respectively, and

by  $G(X)$ ,  $G^1(X)$ ,  $G_{WM}(X)$  and  $G_{SM}(X)$  if  $\mathcal{A} = \mathcal{B} = \mathcal{F} = \{F \subset X : F \text{ is finite}\}$ . Also, if  $M = \{p\}$  for some  $p \in \omega^*$ , we write  $G_p(\mathcal{A}, \mathcal{B}, X)$  instead of  $G_{W\{p\}}(\mathcal{A}, \mathcal{B}, X) = G_{S\{p\}}(\mathcal{A}, \mathcal{B}, X)$ .

**THEOREM 3.1.** *Let  $\phi \neq M \subset \omega^*$  and let  $\mathcal{A}$  and  $\mathcal{B}$  be two closed networks in  $X$ . Then,  $I(G^1(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset \Leftrightarrow I(G_{SM}(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset \Leftrightarrow I(G_{WM}(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset \Leftrightarrow I(G(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset$ .*

**PROOF.** It is not difficult to see that  $I(G^1(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset \Rightarrow I(G_{SM}(\mathcal{A}, \mathcal{B}, X)) \Rightarrow I(G_{WM}(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset \Rightarrow I(G(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset$ . So, we have only to prove that  $I(G(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset \Rightarrow I(G^1(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset$ . We give a similar proof to that given for Theorem 1 in [GN]: Assume that  $s$  is a winning strategy for I in  $G(\mathcal{A}, \mathcal{B}, X)$ . We now give a winning strategy  $t$  for I in  $G^1(\mathcal{A}, \mathcal{B}, X)$ . Let  $t(\emptyset) = A_1 = s(\emptyset) \in \mathcal{A}$  be the first choice of I. If  $(A_1, G_1, \dots, A_n, G_n)$  have been chosen, let  $t(A_1, G_1, \dots, A_n, G_n) = A_{n+1}$  be defined as follows: for each subsequence  $1 \leq i_1 < i_2 < \dots < i_j \leq n$  we put

$$A(i_1, \dots, i_j) = s(A_{i_1}, G_{i_1}, A_{i_2}, G_{i_2}, \dots, A_{i_j}, G_{i_j}).$$

Let  $A_{n+1} = \bigcup \{A(i_1, \dots, i_j) : 1 \leq i_1 < \dots < i_j \leq n\}$ .

We claim that the play  $P = (A_1, G_1, \dots, A_n, G_n, \dots)$  is a win for I in  $G^1(\mathcal{A}, \mathcal{B}, X)$ . In fact, if it were not, there would be  $B \in \mathcal{B}$  and a sequence  $1 \leq i_1 < \dots < i_n < \dots$  such that  $B$  would not be a subset of  $G_{i_n}$  for every  $n < \omega$ . But  $(A_{i_1}, G_{i_1}, \dots, A_{i_n}, G_{i_n}, \dots)$  is a win for I in  $G(\mathcal{A}, \mathcal{B}, X)$  because of the way  $t$  was defined. So, for some  $j$ ,  $B \subset G_{i_j}$ , which is a contradiction. ■

**THEOREM 3.2.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two networks of a space  $X$ . Then,  $I(G(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset$  implies that  $X$  satisfies  $(\mathcal{A}, \mathcal{B}) - \gamma$ .*

**PROOF.** If  $X$  does not have  $(\mathcal{A}, \mathcal{B}) - \gamma$ , then there is an open  $\mathcal{A}$ -cover  $\mathcal{G}$  of  $X$  such that for every sequence  $G_1, \dots, G_n, \dots$  of  $\mathcal{G}$  there is  $B \in \mathcal{B}$  which is not contained in a subsequence  $G_{i_1}, \dots, G_{i_j}, \dots$  of  $(G_n)_{n < \omega}$ . Thus, II has a winning strategy for  $G^1(\mathcal{A}, \mathcal{B}, X)$ . In fact, for each choice  $A_n \in \mathcal{A}$  of I, II has to take  $G_n \in \mathcal{G}$  such that  $A_n \subset G_n$ . So,  $I(G(\mathcal{A}, \mathcal{B}, X)) = I(G^1(\mathcal{A}, \mathcal{B}, X)) = \emptyset$ . ■

The proof of the next Theorem is basically the same to that given for Theorem 4.7 in [Te].

**THEOREM 3.3.** *Let  $(X_n)_{n < \omega}$  be a sequence of disjoint spaces and for each*

$n < \omega$  let  $\mathcal{A}_n, \mathcal{B}_n$  be two networks on  $X_n$  such that  $I(G(\mathcal{A}_n, \mathcal{B}_n, X_n)) \neq \emptyset$  for all  $n < \omega$ . Then,  $I(G(\mathcal{A}, \mathcal{B}, X)) \neq \emptyset$  where  $X$  is the sum of spaces  $X_n$  and  $\mathcal{A} = \bigcup_{n < \omega} \mathcal{A}_n$ ,  $\mathcal{B} = \bigcup_{n < \omega} \mathcal{B}_n$ .

#### 4. Countable Product of Fréchet-Urysohn $C_{\mathcal{A}, \mathcal{U}}$ -spaces.

DEFINITION 4.1. (1) (Telgarsky) For a network  $\mathcal{A}$  in a space  $X$  we say that  $X$  is  $\mathcal{A}$ -scattered if for every closed set  $E$  in  $X$  there is a point  $x \in E$  and a neighborhood  $V$  of  $x$  such that  $\text{Cl}_x V \cap E \in \mathcal{A}$ .

(2) A network  $\mathcal{A}$  in a space  $X$  is said to be  $\gamma$ -real if  $[0,1]$  has  $f(\mathcal{A})\gamma$  for some  $f \in C(X, [0,1])$ .

An example of a non  $\gamma$ -real network on a sequential space  $X$  is the set of countable closed subsets of  $X$ . In fact, as was remarked by Gerlitz and Nagy ([GN, pag 157]), if  $(G_n)_{n < \omega}$  is a sequence of subsets of  $[0,1]$  having Lebesgue measure  $\leq 1/2$ , then  $G_n$  has Lebesgue measure  $\leq 1/2$ .

THEOREM 4.2. Let  $X$  be a Čech-complete space, and let  $\mathcal{A}$  be a non  $\gamma$ -real network on  $X$ . If  $X$  has  $\mathcal{A}\gamma$  then  $X$  is  $\mathcal{A}$ -Lindelöf and scattered.

PROOF. We know that  $\mathcal{A}\gamma \Rightarrow \mathcal{A}\varepsilon$ , and if  $X$  is not scattered then we can find a compact subset  $K$  of  $X$  that may be continuously mapped onto  $[0,1]$ . So,  $[0,1]$  has  $f(\mathcal{A})\gamma$  for some  $f \in C(X, [0,1])$  (Theorem 2.4 and 3.2.J in [En]), which is a contradiction. ■

In the next result we put Theorem 9.3 in [Te] and Theorem 3.2 together.

THEOREM 4.3. Let  $X$  be a space and  $\mathcal{A}$  be a network on  $X$ . Then,  $X$  is Lindelöf and  $\mathcal{A}$ -scattered  $\Rightarrow I(G(\mathcal{A}, X)) \neq \emptyset \Rightarrow X$  has  $\mathcal{A}\gamma$ .

Therefore, using Theorems 2.12, 3.3 and 4.3 we conclude the following

COROLLARY 4.4. Let  $\{X_n : n < \omega\}$  be a countable set of Lindelöf spaces, and let, for each  $n < \omega$ ,  $\mathcal{A}_n$  be a compact network on  $X_n$ . If for each  $i < \omega$ ,  $X_i$  is  $\mathcal{A}_i$ -scattered, then, for every sequence  $((Y_n, \mathcal{U}_n))_{n < \omega}$  of metrizable spaces, we have that

$$C_{\mathcal{A}_1, \mathcal{U}_1}(X_1, Y_1) \times \cdots \times C_{\mathcal{A}_n, \mathcal{U}_n}(X_n, Y_n) \times \cdots$$

is a Fréchet-Urysohn space.

PROOF. In fact, the free topological sum  $X = \coprod_{n < \omega} X_n$  has  $\mathcal{A}\gamma$  where  $\mathcal{A}$  is the collection of finite unions of elements of  $\bigcup_{n < \omega} \mathcal{A}_n$ . Since  $\mathcal{A}$  is compact,  $X$  is not an  $\omega$ - $\mathcal{A}$ -bounded space. Hence, for every metrizable space  $(Y, \mathcal{U})$ ,  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is a Fréchet-Urysohn space (Theorem 2.12). We conclude our proof by observing that  $\prod_{i < \omega} C_{\mathcal{A}_i, \mathcal{U}_i}(X_i, Y_i)$  is homeomorphic to a closed subspace of  $C_{\mathcal{A}, \mathcal{U}}(X, \prod_{i < \omega} Y_i)$  where  $\mathcal{U}$  is the Cartesian product of the uniformities  $\{\mathcal{U}_i\}_{i < \omega}$ . ■

As a consequence of Theorem 4 in [GN] and in virtue of Theorems 4.2 and 4.3 we obtain: (recall that  $\mathcal{F}(X) = \mathcal{F}$  denotes the family of finite subsets of  $X$ )

THEOREM 4.5. *Let  $X$  be a Čech-complete space and let  $\mathcal{A}$  be a non  $\gamma$ -real Lindelöf network containing  $\mathcal{F}$ . Then the following are equivalent*

- (a)  $X$  has  $\gamma$ ;
- (b)  $X$  satisfies  $\mathcal{A}\gamma$ ;
- (c)  $X$  is a Lindelöf scattered space.

PROOF. (a)  $\Leftrightarrow$  (c) is Gerlitz-Nagy's Theorem. Because of  $\mathcal{F} \subset \mathcal{A}$ , scattered implies  $\mathcal{A}$ -scattered and so (c)  $\Rightarrow$  (b) is a consequence of Theorem 4.3. Finally, (b)  $\Rightarrow$  (c) because of Theorem 4.2 and the fact that an  $\mathcal{A}$ -Lindelöf space, where every element of  $\mathcal{A}$  is Lindelöf, is a Lindelöf space. ■

Theorems 2.12 and 4.5 give us the following:

COROLLARY 4.6. *Let  $X$  be a Čech-complete space, let  $\mathcal{A}$  containing  $\mathcal{F}$  be a compact non  $\gamma$ -real network on a non  $\omega$ - $\mathcal{A}$ -bounded space  $X$ , and let  $(Y, \mathcal{U})$  be a metrizable space with a non trivial path. Then  $C_\pi(X)$  is Fréchet-Uryson if and only if  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$  is Fréchet-Urysohn.*

COROLLARY 4.7. *Let  $X_n$  be a Čech-complete space satisfying  $\gamma$  for every  $n < \omega$ . Then  $\prod_{n < \omega} X_n$  has  $\gamma$ .*

PROOF. Every space  $X_n$  is Lindelöf and scattered, so  $Z = \prod_{n < \omega} X_n$  is Čech-complete ([En, pag 198]) Lindelöf and scattered. Therefore  $Z$  has  $\gamma$ . ■

On account of Corollary 4.7 it follows that if  $X = \bigcup_{n < \omega} X_n$  where  $X_n$  is a Čech-complete space with  $\gamma$  for every  $n < \omega$ , then  $X$  has  $\gamma$ : in fact,  $\bigcup_{n < \omega} X_n$  is a continuous image of  $\prod_{n < \omega} X_n$ .

As a consequence of Theorem 4.5, the space of ordinals  $[0, \alpha]$  has  $\gamma$ , so for

every ordinal number  $\alpha$  with countable cofinality,  $[0, \alpha)$  also has  $\gamma$ . It is possible to prove even more:  $I(G([0, \alpha])) \neq \emptyset$  and  $[0, \alpha)$  is an  $\aleph_0$ -simple space, that is, for every continuous function  $f$  from  $[0, \alpha)$  into a space of weight  $\leq \aleph_0$  the cardinality of  $f([0, \alpha))$  does not exceed  $\aleph_0$ . Hence, if  $\text{cof } \alpha \leq \aleph_0$ , then  $C_\pi([0, \alpha))$  is a Fréchet-Urysohn strongly  $\aleph_0$ -monolithic space (see [Ar<sub>2</sub>]). In general  $[0, \alpha)$  is not a Lindelöf space, but using some similar techniques to those developed in [G, ps. 260–262] it is possible to prove that for every ordinal number  $\alpha$ ,  $[0, \alpha)$  has the Gerlits property  $\phi$  (it is important to recall here that  $\phi + \varepsilon = \gamma$ ).

Since  $\gamma$  is preserved under finite powers and closed subsets, then a product  $X \times Y$  has  $\gamma$  iff  $X \amalg Y$  satisfies  $\gamma$ . So, we obtain:

**COROLLARY 4.8.** *Let  $X_i$  be a  $\sigma$ -Čech-complete space satisfying  $\gamma$  for every  $1 \leq i \leq n$ . Then,  $X_1 \times \cdots \times X_n$  has  $\gamma$ .*

**THEOREM 4.9.** *Let  $X_n$  be a  $\sigma$ -Čech-complete space such that  $C_\pi(X_n)$  is a Fréchet-Urysohn space for every  $n < \omega$ . Then  $\prod_{n < \omega} C_\pi(X_n)$  is a Fréchet-Urysohn space.*

**PROOF.** The space  $\prod_{n < \omega} X_n$  is the union of a countable collection of Čech-complete subspaces; hence it is the continuous image of a free topological sum of a countable collection of Čech-complete spaces. So, using 4.7 and Theorem 2.4, we have that  $\prod_{n < \omega} X_n$  has  $\gamma$ . Thus,  $\prod_{n < \omega} C_\pi(X_n)$  is a Fréchet-Urysohn space. ■

Observe that this result can be generalized to  $C_{\mathcal{A}, \mathcal{U}}(X, Y)$ -spaces by using Corollary 4.6.

Nogura [N] proved that if  $(X_n)_{n < \omega}$  is a sequence of spaces such that  $X_1 \times \cdots \times X_n$  is a strictly Fréchet-Urysohn space for every  $n < \omega$ , then  $\prod_{i < \omega} X_i$  is strictly Fréchet-Urysohn (we express this saying that strictly Fréchet-Urysohn property is an *almost countable productive property*). So, Theorem 2.18 produce the following:

**THEOREM 4.10.** *For each  $i < \omega$  let  $\mathcal{A}_i$  be a compact network on a non  $\omega$ - $\mathcal{A}_i$ -bounded space  $X_i$ , and let  $(Y, \mathcal{U})$  be a metrizable space with a non trivial path. Then, for every  $n < \omega$   $\prod_{1 \leq i \leq n} C_{\mathcal{A}_i, \mathcal{U}}(X_i, Y)$  is Fréchet-Urysohn iff  $\prod_{i < \omega} C_{\mathcal{A}_i, \mathcal{U}}(X_i, Y)$  is Fréchet-Urysohn.*

**COROLLARY 4.11.** *Let  $(X_n)_{n < \omega}$  be a sequence of spaces. Then, for every  $n < \omega$ ,  $\prod_{1 \leq i \leq n} C_\pi(X_i)$  is Fréchet-Urysohn iff  $\prod_{i < \omega} C_\pi(X_i)$  is a Fréchet-Urysohn space.*

**5. The dual space  $L_\pi(X)$  and the free topological groups  $A(X)$  and  $F(X)$ .**

In the following theorems we are going to use the notations established in 2.3.5.

**THEOREM 5.1.** *Let  $\mathcal{A}$  be a compact network on a space  $X$ . Suppose that  $X$  is a non  $\omega$ - $\mathcal{A}$ -bounded space. Then,  $X$  has  $\mathcal{A}\vartheta$  if and only if  $Z = \prod_{n<\omega} X^n$  satisfies  $(\bigcup_{n<\omega} \mathcal{A}^n)\vartheta$ . Moreover, if  $\vartheta = \varepsilon$ , the assertion is true for every space  $X$ .*

**PROOF.**  $\Leftarrow$ ) is a consequence of Theorem 2.4.

$\Rightarrow$ ) Let  $\mathcal{B} = \bigcup_{n<\omega} \mathcal{A}^n$ . If  $\vartheta = \varepsilon$ , we can prove, following similar arguments to those given in the proof of Theorem 2.4.4, that for every  $n < \omega$ ,  $X \amalg \dots \amalg X^n$  has  $\mathcal{B}_n\varepsilon$  where  $\mathcal{B}_n$  is the collection of finite unions of elements belonging to  $\bigcup_{i=1}^n \mathcal{A}^i$ . So, if  $\mathcal{G}$  is an open  $\mathcal{B}$ -cover, there is for each  $n < \omega$  a countable collection  $\mathcal{G}_n$  of  $\mathcal{G}$  which is a  $\mathcal{B}_n$ -cover. Then,  $\bigcup_{n<\omega} \mathcal{G}_n$  is a countable  $\mathcal{B}$ -subcover of  $\mathcal{G}$ .

Now, suppose that  $\vartheta = S\gamma_M$  for some  $\emptyset \neq M \subset \omega^*$  (for the other possible values of  $\vartheta$  the proof is analogous). If  $X$  is finite, then  $Z$  is countable and discrete, hence  $\mathcal{B} = \mathcal{F}$  and we obtain the desired conclusion since every countable space has  $\gamma$ . If  $X$  is infinite, let  $N = \{x_1, \dots, x_n, \dots\} \subset X$  such that  $x_i \neq x_j$  if  $i \neq j$  and  $N$  is not contained in any element of  $\mathcal{A}$ . Let  $\mathcal{G}$  be an open  $\mathcal{B}$ -cover of  $Z$ . For each  $n < \omega$  we put  $\mathcal{H}_n = \{V \subset X : V \text{ is open and } V \amalg V^2 \amalg \dots \amalg V^n \subset G \text{ for some } G \in \mathcal{G}\}$ . We define  $\hat{\mathcal{H}}_n = \{V \setminus \{x_n\} : V \in \mathcal{H}_n\}$ . We claim that the collection  $\hat{\mathcal{H}} = \bigcup_{n<\omega} \hat{\mathcal{H}}_n$  is an open  $\mathcal{A}$ -cover of  $X$ . In fact, let  $A \in \mathcal{A}$  and let  $s = \min\{n < \omega : x_n \notin A\}$ ; there is  $G \in \mathcal{G}$  such that  $F = A \amalg A^2 \amalg \dots \amalg A^s \subset G$ . Since  $A$  is compact there is an open subset  $V$  of  $X$  such that  $F \subset \prod_{1 \leq i \leq s} V^i \subset G$ . Hence  $A \subset V \setminus \{x_s\}$ . So  $\hat{\mathcal{H}}$  is an  $\mathcal{A}$ -cover of  $X$ . By assumption, there is  $(\hat{H}_j)_{j<\omega}$  in  $\hat{\mathcal{H}}$  such that  $X = \mathcal{A}\text{-lim}_p \hat{H}_j$  for every  $p \in M$  where  $\hat{H}_j = V_j \setminus \{x_{n_j}\} \in \hat{\mathcal{H}}_{n_j}$ . For each  $j < \omega$  there is  $G_j \in \mathcal{G}$  satisfying:

$$V_j \amalg (V_j)^2 \amalg \dots \amalg (V_j)^{n_j} \subset G_j$$

Observe that  $\{n_j : j < \omega\}$  must be a cofinal increasing sequence of  $\omega$  because if there is  $m < \omega$  such that  $n_j \leq m$  for every  $j < \omega$ , and if  $A \in \mathcal{A}$  is such that  $\{x_1, \dots, x_m\} \subset A$ , no  $\hat{H}_j$  contains  $A$ , which is a contradiction.

Now we claim that  $Z = \mathcal{B}\text{-lim}_p G_n$  for every  $p \in M$ . In fact, let  $B \in \mathcal{B}$ . There is  $K \in \mathcal{A}$  and there is  $n < \omega$  such that  $B \subset \prod_{1 \leq i < n} K^i$ . Since  $X = \mathcal{A}\text{-lim}_p \hat{H}_j$

for every  $p \in M$ ,  $\{k < \omega : K \subset V_j \setminus \{x_{n_j}\}\} \in p$  for all  $p \in M$ . Thus,  $\{k < \omega : K \subset V_j \setminus \{x_{n_j}\}\} \cap \{k < \omega : n_j > n\} \subset \{j < \omega : B \subset G_j\} \in p$  for every  $p \in M$ . ■

The following two Corollaries are consequences of Theorems 2.12 and 5.1.

**COROLLARY 5.2.** *Let  $\emptyset \neq M \subset \omega^*$  let  $X$  be a space and  $Y$  be metrizable with a non trivial path. Then,  $C_\pi(X, Y)$  is a SFU( $M$ )-space (resp., WFU( $M$ )-space) if and only if  $\prod_{n < \omega} C_\pi(X^n, Y)$  is a SFU( $M$ )-space (resp., WFU( $M$ )-space). In particular,  $C_\pi(X, Y)$  is Fréchet-Urysohn if and only if  $\prod_{n < \omega} C_\pi(X^n, Y)$  is a Fréchet-Urysohn space.*

**COROLLARY 5.3.** *Let  $\emptyset \neq M \subset \omega^*$ , let  $X$  be a non  $\omega$ -bounded space and let  $Y$  be a metrizable space with a non trivial path. Then,  $C_k(X, Y)$  is a SFU( $M$ )-space (resp., WFU( $M$ )-space) if and only if  $\prod_{n < \omega} C_k(X^n, Y)$  is a SFU( $M$ )-space (resp., WFU( $M$ )-space). In particular,  $C_k(X, Y)$  is Fréchet-Urysohn if and only if  $\prod_{n < \omega} C_k(X^n, Y)$  is a Fréchet-Urysohn space.*

There are spaces  $Y$  having a compact network  $\mathcal{A}$  for which  $Y$  has  $\mathcal{A}\vartheta$  but  $Y^\omega$  does not have  $(\bigcup_{n < \omega} \mathcal{A}^n)\vartheta$ . A trivial example is  $Y = \{0, 1\}$ ,  $\mathcal{A} = \mathcal{F}$ , and  $\vartheta = \gamma_p$  with  $p \in \omega^*$  such that  $\mathbf{R}$  does not satisfy  $\gamma_p$ . For topological function spaces this can also happen, even when the base space is compact, in fact, C. Laflamme proved that it is consistent with ZFC that there is a  $P$ -point  $p \in \omega^*$  for which  $\mathbf{R} = C_\pi(\{x\})$  has  $\gamma_p$ ; because of Theorem 6.5 below  $C_\pi(\{x\})^\omega = \mathbf{R}^\omega$  does not satisfy  $\gamma_p$ . The natural question now is: Does  $C_\pi(X^\omega)$  have  $\gamma_p$  if  $C_\pi(X)$  does? When  $X$  is a compact zero-dimensional space the answer to this question is affirmative:

**COROLLARY 5.4.** *Let  $p \in \omega^*$  and let  $X$  be a compact zero-dimensional space. Then  $C_\pi(X)$  has  $\gamma_p$  if and only if  $C_\pi(X^\omega)$  has  $\gamma_p$ .*

**PROOF.**  $\Rightarrow$ ) It is a consequence of Theorem 5.1 and Proposition IV.8.2 in [Ar<sub>1</sub>].

$\Leftarrow$ )  $C_\pi(X)$  is homeomorphic to a closed subset of  $C_\pi(X)$ . In fact,  $\text{pr}^\# : C_\pi(X) \rightarrow C_\pi(X^\omega)$  is a closed embedding where  $\text{pr} : X^\omega \rightarrow X$  is one of the projections (see [Ar<sub>1</sub>]). ■

For a space  $X$ , we will denote by  $F(X, e)$  (resp.,  $A(X, e)$ ) the free topological (resp., Abelian) group generated by  $X$  with a distinguished point  $e \in X$  (see

[Gr]). We know that  $F(X, e)$  and  $A(X, e)$  are continuous images of the space  $\prod_{n < \omega} (X^n \times \{-1, 1\}^n)$ . Furthermore,  $X$  is homeomorphic to a closed subset of both  $F(X, e)$  and  $A(X, e)$ , so we obtain, from Theorem 5.1, the following:

**THEOREM 5.5.** *Let  $\mathcal{A}$  be a compact network on a non  $\omega$ - $\mathcal{A}$ -bounded space  $X$  and let  $e \in X$ . Then, the following are equivalent*

- (a)  $X$  has  $\mathcal{A}\vartheta$ .
- (b)  $A(X, e)$  has  $\mathcal{B}\vartheta$ .
- (c)  $F(X, e)$  has  $\mathcal{B}\vartheta$ .

where  $\mathcal{B} = \{A_1^{\varepsilon_1} \cdots A_n^{\varepsilon_n} : A_i \in \mathcal{A}, \varepsilon_i \in \{-1, 1\}, 1 \leq i \leq n \text{ and } n < \omega\}$ .

**PROOF.**  $\Rightarrow$ ) Let  $\{-1, 1\}$  be considered with the discrete topology and set  $Z = \prod_{n < \omega} (X^n \times \{-1, 1\}^n)$ . The map  $f : Z \rightarrow F(X, e)$  (resp.,  $f : Z \rightarrow A(X, e)$ ) defined by  $f(x_1, \dots, x_n, \varepsilon_1, \dots, \varepsilon_n) = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$  is an onto continuous function. If  $X$  has  $\mathcal{A}\vartheta$  then  $X \times \{-1, 1\}$  satisfies  $(\mathcal{A} \times \mathcal{F}) - \vartheta$  (Theorem 2.4.4). Because of Theorem 5.1 we have that  $Z$  has  $[\bigcup_{n < \omega} (\mathcal{A} \times \mathcal{F})^n] - \vartheta$ . So by Theorem 2.4.3,  $F(X, e)$  (resp.,  $A(X, e)$ ) has  $\mathcal{B}\vartheta$ .

$\Leftarrow$ ) We may consider  $X$  as a closed subset of  $F(X, e)$  and  $A(X, e)$ . On the other hand,  $\mathcal{A} \subset \mathcal{B}_X$  and for every  $A_1, \dots, A_n \in \mathcal{A}$  and every  $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ ,  $(A_1^{\varepsilon_1} \cdots A_n^{\varepsilon_n}) \cap X \subset A_1 \cup \dots \cup A_n \cup A$  where  $e \in A \in \mathcal{A}$ . Now the conclusion follows from Theorem 2.2.3. ■

**COROLLARY 5.6.** *Let  $\emptyset \neq M \subset \omega^*$ , let  $Y$  be a metrizable space and  $X$  a space with  $e \in X$ . Then, the following are equivalent*

- (a)  $C_\pi(X, Y)$  is a SFU(M)-space (resp., WFU(M)-space).
- (b)  $C_\pi(A(X, e), Y)$  is a SFU(M)-space (resp., WFU(M)-space).
- (c)  $C_\pi(F(X, e), Y)$  is a SFU(M)-space (resp., WFU(M)-space).

*In particular,  $C_\pi(X, Y)$  is Fréchet-Urysohn  $\Leftrightarrow C_\pi(A(X, e), Y)$  is a Fréchet-Urysohn space  $\Leftrightarrow C_\pi(F(X, e), Y)$  is a Fréchet-Urysohn space.*

In order to have a similar result for the compact-open topology it is enough that every compact subset of  $F(X)$  (resp.,  $A(X)$ ) is contained in an  $A_1^{\varepsilon_1} \cdots A_n^{\varepsilon_n}$  with  $A_i$  compact for every  $i < n$ . If  $X$  is a  $k_\omega$ -space (A space is said to be  $k_\omega$  if it is the union of an increasing sequence of compact subspaces having their weak topology), then  $F(X)$  is also a  $k_\omega$ -space ([Or]), so  $C_k(F(X))$  is completely metrizable (see [St] and [McN<sub>2</sub>]).

We say that a subset  $A$  of a topological group  $G$  generates  $G$  if  $A$  contains the identity of  $G$ , algebraically generates  $G$  and has the finest topology compatible with both the group structure and the original topology on  $A$ . Ordmen and Smith-Thomas ([OS-T]) proved that every topological group  $G$  generated by a  $k_\omega$ -space

$A$  is a quotient image of  $F(A)$ , so:

**THEOREM 5.7.** *Let  $\phi \neq M \subset \omega^*$ . Let  $G$  be a topological group generated by a  $k_\omega$ -space  $A$ . Then,*

(a) *If  $A$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ), then  $C_\pi(G)$  is a SFU( $M$ )-space (resp., WFU( $M$ )-space).*

(b)  *$C_k(G)$  is completely metrizable.*

**PROOF.** (a) If  $A$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ), then  $F(A)$  does too (Theorem 5.5). Besides,  $G$  is a continuous image of  $F(A)$ , so  $G$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ).

(b) As was noted above,  $F(A)$  is a  $k_\omega$ -space, hence, since  $G$  is a quotient image of  $F(A)$ ,  $G$  is a  $k_\omega$ -space. Then  $C_k(G)$  is a completely metrizable space. ■

The dual space  $L_\pi(X)$  of  $C_\pi(X)$ , with the pointwise convergence topology, is a continuous image of the space  $\prod_{n < \omega} (X^n \times \mathbf{R}^n)$ , so we also obtain from Theorem 5.1 the following result.

**THEOREM 5.8.** *Let  $\mathcal{A}$  be a compact network on a space  $X$  and let  $\mathcal{C}$  be a compact network on  $\mathbf{R}$ . Then,*

$$\mathbf{R} \times X \text{ has } (\mathcal{C} \times \mathcal{A}) - \vartheta \Rightarrow L_\pi(X) \text{ has } \mathcal{B}\vartheta \Rightarrow X \text{ has } \mathcal{A}\vartheta$$

where

$$\mathcal{B} = \{C_1 A_1 + \dots + C_n A_n : A_i \in \mathcal{A}, C_i \in \mathcal{C} \text{ for every } 1 \leq i \leq n \text{ and } n < \omega\}.$$

**PROOF.**  $\mathbf{R} \times X$  is a non  $\omega$ - $(\mathcal{C} \times \mathcal{A})$ -bounded space; so  $\prod_{n < \omega} (\mathbf{R}^n \times X^n)$  satisfies  $\mathcal{D}\vartheta$  where  $\mathcal{D} = \bigcup_{n < \omega} (\mathcal{C}^n \times \mathcal{A}^n)$  (Theorem 5.1.). The function  $f : \prod_{n < \omega} (\mathbf{R}^n \times X^n) \rightarrow L_\pi(X)$  defined by  $f(r_1, \dots, r_n, x_1, \dots, x_n) = r_1 x_1 + r_2 x_2 + \dots + r_n x_n$  is continuous. Since  $\mathcal{D}$  is compact we can use Theorem 2.4 and deduce that  $L_\pi(X)$  has  $f(\mathcal{D}) - \vartheta$ . It remains to note that  $f(\mathcal{D}) = \mathcal{B}$ .

Besides, if  $L_\pi(X)$  has  $\mathcal{B}\vartheta$ , then  $X$  has  $\mathcal{B}_x\vartheta$ . Observe that  $\mathcal{A} \subset \mathcal{B}_x$  and for every  $n < \omega$  and every  $(C_1, \dots, C_n, A_1, \dots, A_n) \in \mathcal{C}^n \times \mathcal{A}^n$ ,  $(C_1 A_1 + \dots + C_n A_n) \cap X \subset A_1 \cup \dots \cup A_n \in \mathcal{A}$ . Using 2.2.3 we conclude that  $X$  has  $\mathcal{A}\vartheta$ . □

The complete circle of implications in the last Theorem can be obtained when  $X$  is a non  $\omega$ -bounded space and  $\mathcal{A} = \mathcal{H}$  by applying the following two lemmas. The first one is trivial and the second one is a generalization of Theorem 3.3 in [GT<sub>2</sub>].

**LEMMA 5.9.** *Let  $\mathcal{A}$  be a compact network on  $X$  and let  $\mathcal{C}$  be a network on  $Y$*

containing  $Y$ . If  $X$  has  $\mathcal{A}\vartheta$ , then  $X \times Y$  has  $(\mathcal{A} \times \mathcal{C}) - \vartheta$ .

LEMMA 5.10. Let  $\mathcal{A}$  be a compact network on a non  $\omega$ - $\mathcal{A}$ -bounded space  $X$ . Let  $\mathcal{C}$  be a compact network on a  $\mathcal{C}$ -hemicompact space  $Y$ . If  $X$  satisfies  $\mathcal{A}\vartheta$ , then  $X \times Y$  has  $(\mathcal{A} \times \mathcal{C}) - \vartheta$ . Moreover, if  $\vartheta = \varepsilon$ , the assertion is true for every space  $X$ .

PROOF. We prove the lemma when  $\vartheta = S\gamma_M$  where  $\emptyset \neq M \subset \omega^*$ , the rest of the assertions can be proved in a similar way. Because of the previous Lemma we have only to prove the case when  $Y \notin \mathcal{C}$ . Let  $(Y_n)_{n < \omega}$  be a strictly increasing sequence of elements in  $\mathcal{C}$  witnessing the  $\mathcal{C}$ -hemicompactness of  $Y$ . If  $X$  is finite, then  $\mathcal{A} = \mathcal{F}$  and  $X \times Y$  is a countable union of elements belonging to  $\mathcal{A} \times \mathcal{C}$ , hence  $X \times Y$  has  $(\mathcal{A} \times \mathcal{C}) - S\gamma_M$ . Suppose now that  $X$  is infinite and let  $F = \{x_n : n < \omega\} \subset X$  be such that  $x_i \neq x_j$  if  $i \neq j$  and  $F$  is not contained in any element of  $\mathcal{A}$ . Let  $\mathcal{G}$  be an open  $(\mathcal{A} \times \mathcal{C})$ -cover of  $X \times Y$ . For each  $n < \omega$  we put  $\mathcal{H}_n = \{V \subset X : V \text{ is open and there is } G \in \mathcal{G} \text{ such that } V \times Y_n \subset G\}$ . It happens that  $\mathcal{H}' = \bigcup_{n < \omega} \mathcal{H}'_n$  where  $\mathcal{H}'_n = \{V \setminus \{x_n\} : V \in \mathcal{H}_n\}$  is an open  $\mathcal{A}$ -cover of  $X$ . So, there is a sequence  $(H'_j)_{j < \omega}$  in  $\mathcal{H}'$  such that  $X = \mathcal{A}\text{-lim}_p H'_j$  for every  $p \in M$ , where  $H'_j = V_j \setminus \{x_{n_j}\}$  for some open set  $V_j$  of  $X$  and some natural number  $n_j$ . As was noted in the proof of Theorem 5.1, the set  $\{n_j : j < \omega\}$  is a cofinal increasing sequence in  $\omega$ . For each  $j < \omega$  there is  $G_j \in \mathcal{G}$  such that  $V_j \times Y_{n_j} \subset G_j$ . It is possible now to prove that  $X \times Y = (\mathcal{A} \times \mathcal{C})\text{-lim}_p G_j$  for every  $p \in M$ . ■

For  $y \in L_\pi(X)$  we define  $l(y)$  as follows. If  $y = \bar{0}$  is the zero element of  $L_\pi(X)$ , then  $l(y) = 0$ . If  $y \neq \bar{0}$ , then  $l(y) = \min \{n < \omega : \text{there are } x_1, \dots, x_n \in X \text{ and } r_1, \dots, r_n \in \mathbf{R} \text{ such that } y = r_1x_1 + \dots + r_nx_n\}$ . Let  $L_{\pi,n}(X) = \{y \in L_\pi(X) : l(y) \leq n\}$ .

Observe that for  $\vartheta \in \{S\gamma_M, W\gamma_M : \emptyset \neq M \subset \omega^*\} \cup \{\varepsilon\}$ ,  $X$  and  $Y$  spaces,  $F \subset X$  closed and  $f : X \rightarrow Y$  a continuous onto function, we have (see Theorem 2.4) (i)  $X$  has  $\mathcal{K}(X)\vartheta$  implies  $F$  has  $\mathcal{K}(F)\vartheta$ ; (ii)  $X$  has  $\mathcal{K}(X)\vartheta$  implies  $X^n$  has  $\mathcal{K}(X^n)\vartheta$ ; and (iii)  $X$  has  $\mathcal{K}(X)\vartheta$  implies  $Y$  satisfies  $\mathcal{K}(Y)\vartheta$  if  $f$  is perfect. Besides, if  $X$  is locally compact, then  $X$  has  $\mathcal{K}\vartheta$  iff  $X$  has  $\mathcal{A}\vartheta$  where  $\mathcal{A} = \{CIU : U \text{ is open and } CIU \text{ is compact}\}$ . These remarks and the previous Lemmas imply

COROLLARY 5.11. Let  $X$  be a locally compact non  $\omega$ -bounded space and let  $Y$  be a metric space with a non trivial path. Then,  $C_k(X, Y)$  is a Fréchet-Urysohn space iff  $C_k(L_{\pi,n}(X), Y)$  is Fréchet-Urysohn for a  $0 < n < \omega$  (hence, for every  $0 < n < \omega$ ).

PROOF.  $\Rightarrow$ ) Let  $0 < n < \omega$  be fixed.  $\mathbf{R}$  is hemicompact and  $X$  has  $\mathcal{K}\gamma$  and is

not  $\omega$ -bounded; so  $\mathbf{R} \times X$  has  $\mathcal{H}\gamma$  (Lemma 5.10). Because of Proposition 1 in [P], Theorem 5.1 and the remarks made above we conclude that  $L_{\pi,n}(X)$  has  $\mathcal{H}\gamma$ . Since  $X$  is a closed non  $\omega$ -bounded subspace of  $L_{\pi,n}(X)$ , this is not  $\omega$ -bounded, so  $C_k(L_{\pi,n}(X), Y)$  is Fréchet-Urysohn (Theorem 2.12).

$\Leftrightarrow$ ) Because of Theorem 2.12,  $L_{\pi,n}(X)$  has  $\mathcal{H}\gamma$  (see 2.13). Since  $X$  is homeomorphic to a closed subset of  $L_{\pi,n}(X)$ , then  $X$  has  $\mathcal{H}\gamma$  (Theorem 2.4). On the other hand,  $X$  is not  $\omega$ -bounded, so  $C_k(X, Y)$  is a Fréchet-Urysohn space. ■

**PROBLEM 5.12.** Does  $L_\pi(X)$  satisfy  $\mathcal{H}\gamma$  if  $X$  does?

For metrizable spaces we have:

**COROLLARY 5.13.** *Let  $\emptyset \neq M \subset \omega^*$  and let  $X$  be a non-countable metrizable either locally compact or  $\sigma$ -compact space. Then,  $X$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ) if and only if  $L_\pi(X)$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ).*

**PROOF.** If  $X$  has a  $S\gamma_M$ , then  $X$  is  $\sigma$ -compact. So, there is a compact metrizable non countable space satisfying  $S\gamma_M$ . This implies that  $X \times \mathbf{R}$  has  $S\gamma_M$  (Theorem 6.1 below). Now we have to apply Theorems 5.8 and 2.4. ■

This result is not true if  $X$  is countable because in that case  $X$  has  $\gamma$  but  $L_\pi(X)$  never satisfies this property. In fact,  $\mathbf{R} \setminus \{0\}$  is a continuous image of  $L_{\pi,1}(X)$  which is a closed subspace of  $L_\pi(X)$ .

## 6. The countable product of $p$ -Fréchet-Urysohn $C_{\mathcal{A}, \mathcal{U}}$ -spaces.

In this section we will give some results about the property  $\gamma_p$  ( $p \in \omega^*$ ) in a free topological sum; that is, we will find sufficient conditions in order to obtain the  $p$ -Fréchet-Urysohn property in a countable product of function spaces where each factor is an  $FU(p)$ -space.

**THEOREM 6.1.** *Let  $\emptyset \neq M \subset \omega^*$ . The following statements are equivalent*

- (1)  $\mathbf{R}$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ).
- (2) There is a compact metrizable and non countable space  $X$  with  $S\gamma_M$  (resp.,  $W\gamma_M$ ).
- (3) Every  $\sigma$ -compact metrizable space  $X$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ).
- (4) There is a Čech-complete non scattered space satisfying  $S\gamma_M$  (resp.,  $W\gamma_M$ ).

**PROOF.** (1)  $\Leftrightarrow$  (3) was basically proved in [GT<sub>1</sub>]; (3)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (4)

are trivial.

(2)  $\Rightarrow$  (1) Since  $X$  is second countable and non countable, there is  $\emptyset \neq F \subset X$  which is perfect. Furthermore,  $F$  is completely metrizable because  $X$  is. Then  $F$  contains a copy of the Cantor set ([En, 4.5.5]). Using [GT<sub>1</sub>, Theorem 3.18] we conclude that  $\mathbf{R}$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ).

(4)  $\Rightarrow$  (1) Every Čech-complete non scattered space contains a compact subset which can be continuously mapped onto  $[0,1]$  (see 2.4. and Theorem 3.18 in [GT<sub>1</sub>]). ■

As a consequence of the previous Theorem we have:

**THEOREM 6.2.** *Let  $\emptyset \neq M \subset \omega^*$ , let  $X_n$  be a metrizable  $\sigma$ -compact space and  $Y_n$  be a metrizable space for every  $n < \omega$ . If either: (i) every  $X_i$  is countable, or (ii) there is  $i < \omega$  for which  $X_i$  is non countable,  $Y_i$  has a non trivial path and  $C_\pi(X_i, Y_i)$  is a SFU(M)-space (resp., WFU(M)-space), then*

$$\prod_{n < \omega} C_\pi(X_n, Y_n) \text{ is a SFU(M)-space (resp., WFU(M)-space).}$$

**PROOF.** If every  $X_i$  ( $i < \omega$ ) is countable, then  $X = \prod_{n < \omega} X_n$  is countable and so  $X$  has  $\gamma$ . If this is not the case, since there exists a compact metrizable and non countable subspace of  $X_i$  with  $S\gamma_M$  (resp.,  $W\gamma_M$ ) (Theorem 2.12), the metrizable  $\sigma$ -compact space  $\prod_{n < \omega} X_n$  also has  $S\gamma_M$  (resp.,  $W\gamma_M$ ) (Theorem 6.1). Then, in both cases, because of Theorem 2.12,  $\prod_{n < \omega} C_\pi(X_n, \prod_{n < \omega} Y_n) \cong \prod_{n, m < \omega} C_\pi(X_n, Y_m)$  is a SFU(M)-space (resp., WFU(M)-space). Therefore, its closed subset  $\prod_{n < \omega} C_\pi(X_n, Y_n)$  also has this property. ■

**COROLLARY 6.3.** *Let  $\emptyset \neq M \subset \omega^*$  and let  $X_i$  be a metrizable either locally compact or  $\sigma$ -compact space having  $S\gamma_M$  (resp.,  $W\gamma_M$ ) for  $1 \leq i \leq n$ . Then  $X_1 \times \dots \times X_n$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ).*

**PROOF.** The space  $Z = X_1 \prod \dots \prod X_n$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ) and  $\prod_{i=1}^n X_i$  is a closed subset of  $Z^n$ . ■

When we are considering compact metrizable spaces we can strengthen Corollary 6.3 as follows.

**THEOREM 6.4.** *Let  $\emptyset \neq M \subset \omega^*$  and let  $(X_n)_{n < \omega}$  be a sequence of compact metrizable spaces. If there is  $i < \omega$  for which  $X_i$  is a non countable space satisfying  $S\gamma_M$  (resp.,  $W\gamma_M$ ), then  $\prod_{n < \omega} X_n$  has  $S\gamma_M$  (resp.,  $W\gamma_M$ ).*

PROOF. In fact,  $\prod_{n<\omega} X_n$  is a compact metrizable space; we only have to apply Theorem 6.1. ■

This last result cannot be generalized to metric locally compact spaces. In fact,  $\mathbf{R}^\omega$  does not satisfy  $\gamma_p$  if  $p \in \omega^*$  is a  $P$ -point in  $\omega^*([GT_1])$ , but as was proved by C. Laflamme, it is consistent with ZFC that there exists a  $P$ -point  $p \in \omega^*$  such that  $\mathbf{R}$  has  $\gamma_p$ .

The result concerning  $\mathbf{R}^\omega$  and  $P$ -points can be generalized as follows.

**THEOREM 6.5.** *Let  $p \in \omega^*$  be a  $P$ -point in  $\omega^*$ , and let  $X_n$  be a non countably compact space for every  $n < \omega$ . Then,  $\prod_{n<\omega} X_n$  does not satisfy  $\gamma_p$ .*

PROOF. Suppose that  $X = \prod_{n<\omega} X_n$  satisfies  $\gamma_p$ . For each  $i < \omega$  let  $Z_i = \{z_1^i, z_2^i, \dots, z_n^i, \dots\}$  be a countable closed and discrete subset of  $X_i$  ( $z_s^i \neq z_t^i$  if  $s \neq t$ ). If  $x \in X_i$ , we take  $G_x^i \in \mathcal{N}(x)$  such that  $|G_x^i \cap Z_i| \leq 1$ . For each pair of natural numbers  $i, n$  let  $\mathcal{G}_n^i$  be the collection of the unions of  $\leq n$  elements belonging to  $\mathcal{G}^i = \{G_x^i : x \in X_i\}$ . Put  $\mathcal{F}_n = \{\pi_n^{-1}(G) : G \in \mathcal{G}_n^i\}$  and  $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$  where  $\pi_n$  is the projection from  $X = \prod_{n<\omega} X_n$  onto  $X_n$ .  $\mathcal{F}$  is an open  $\omega$ -cover of  $X$ , so there is a sequence  $(F_n)_{n<\omega}$  in  $\mathcal{F}$  for which  $X = \lim_p F_n$ .

The collection  $\{A_j = \{n < \omega : F_n \in \mathcal{F}_j\} : j < \omega\}$  is a partition of  $\omega$ . We claim that  $A_j \notin p$  for every  $j < \omega$ . In fact, suppose that there is  $j < \omega$  such that  $A_j \in p$ . For each  $m < \omega$  we define  $f_m \in X$  as follows:  $\pi_n(f_m) = z_m^n$ . Let  $S(m) = \{n < \omega : f_m \in F_n\}$ . Since  $X = \lim_p F_n$ ,  $S(m) \in p$ . Hence, there exists  $n_0 \in A_j \cap (\bigcap_{1 \leq i \leq j+1} S(i)) \in p$ . Then  $f_1, \dots, f_{j+1} \in F_{n_0} = \pi_j^{-1}(G)$  where  $G \in \mathcal{G}_j^j$ . This means that  $z_1^j, \dots, z_{j+1}^j \in G$ , but this is a contradiction because, by construction, no element of  $\mathcal{G}_j^j$  can contain  $j+1$  elements of  $Z_j$ . Thus  $A_j \notin p$  for every  $j < \omega$ . Since  $p$  is a  $P$ -point, there is  $A \in p$  such that  $|A \cap A_j| < \omega$  for every  $j < \omega$ . Besides,  $X = \bigcup_{n \in A} F_n$ . If  $A_j \neq \emptyset$ , we set  $A \cap A_j = \{n(j, 0), \dots, n(j, r_j)\}$ . Then, we have that  $X = \bigcup \{ \bigcup_{i \leq r_j} F_{n(j,i)} : j < \omega \text{ and } A_j \neq \emptyset \}$ . For each  $j < \omega$  with  $A_j \neq \emptyset$  and for each  $i \leq r_j$  let  $G_{n(j,i)} \in \mathcal{G}_j^j$  which satisfies  $\pi_j^{-1}(G_{n(j,i)}) = F_{n(j,i)}$ . Take  $x = (x_j)_{j<\omega} \in X$  where  $x_j \in X_j \setminus \bigcup_{i \leq r_j} G_{n(j,i)}$  if  $A_j \neq \emptyset$ , and  $x_j \in X_j$  otherwise. It happens that  $x \notin \bigcup \{ \bigcup_{i \leq r_j} F_{n(j,i)} : j < \omega \text{ and } A_j \neq \emptyset \}$ , which is a contradiction. ■

**COROLLARY 6.6.** *Let  $X$  be a normal and non countably compact space and let  $p \in \omega^*$  be a  $P$ -point. Then  $C_\pi(C_\pi(X))$  is not an  $FU(p)$ -space.*

PROOF. By assumption we can take a countable discrete and closed subset  $D$

of  $X$ . Since  $X$  is normal and  $D$  is closed,  $C_\pi(D)$  is a continuous image of  $C_\pi(X)$ . If  $C_\pi(X)$  has  $\gamma_p$ , so does  $C_\pi(D) \cong \mathbf{R}^D$  which is in contradiction with Theorem 6.5.

■

We have already seen that  $\mathbf{R}^\omega$  does not satisfy  $\gamma_p$  if  $p$  is a  $P$ -point in  $\omega^*$ . In [GT<sub>1</sub>, Theorem 3.20] the authors gave a list of necessary and sufficient conditions in order to guarantee that  $\mathbf{R}^\omega$  has  $\gamma_p$  ( $p \in \omega^*$ ). As a consequence of these remarks we have (see 2.3.5):

**THEOREM 6.7.** *The following statements are equivalent.*

(a)  $\mathbf{R}^\omega$  satisfies  $\vartheta$  (resp.,  $\mathcal{H}\vartheta$ ).

(b) Every Čech-complete space of countable weight has  $\vartheta$  (resp.,  $\mathcal{H}\vartheta$ ).

**PROOF.** We have only to prove (a)  $\Rightarrow$  (b). If  $\mathbf{R}^\omega$  has  $\vartheta$  (resp.,  $\mathcal{H}\vartheta$ ) then  $J(\omega)^\omega \times [0,1]^\omega$  satisfies  $\vartheta$  where  $J(\omega)$  is the hedgehog of spininess  $\omega$  ([GT<sub>1</sub>]) (resp.,  $\mathcal{H}\vartheta$ , Theorem 2.4.4 and remarks before Corollary 5.11). On the other hand, every second countable and Čech-complete space  $X$  can be considered as a closed subset of  $J(\omega)^\omega \times [0,1]^\omega$  ([So]). So  $X$  has  $\vartheta$  (resp.,  $\mathcal{H}\vartheta$ ). ■

**PROBLEM 6.8.** Is  $p$ -Fréchet-Urysohn an almost countable productive property in the class of  $C_\pi$ -spaces when  $p \in \omega^*$  (resp.,  $p \in \omega^*$  is semiselective, selective)? (see Theorem 4.11).

## 7. The property $\gamma_p$ and the real line.

In this section we will see that if  $p \in \omega^*$  is such that  $\mathbf{R}$  does not have  $\gamma_p$ , then property  $\gamma_p$  is similar to property  $\gamma$ , and they coincide in the class of Čech-complete spaces. The proof of the next two theorems can be achieved using Theorem 3.18 in [GT<sub>1</sub>] and following completely analogous proofs to those given for Corollary to Lemma 1 and Theorem 5 in [GN], respectively.

**THEOREM 7.1.** *Let  $p \in \omega^*$  such that  $\mathbf{R}$  does not satisfy  $\gamma_p$ . Then, every space with  $\gamma_p$  is zero-dimensional.*

**THEOREM 7.2.** *Let  $p \in \omega^*$  such that  $\mathbf{R}$  does not satisfy  $\gamma_p$ . Then, a space  $X$  has  $\gamma_p$  if and only if  $X$  has  $\mathcal{E}$  and every continuous image of  $X$  in  $\mathbf{R}$  has  $\gamma_p$ .*

Every Čech-complete non-scattered space  $X$  contains a compact subspace

which can be continuously mapped onto  $[0,1]$ . So, in virtue of Theorems 4.5 and 4.7 we obtain the following interesting result.

**THEOREM 7.3.** Let  $p \in \omega^*$  such that  $\mathbf{R}$  does not satisfy  $\gamma_p$ , and let  $X$  be a countable union of Lindelöf Čech-complete subspaces. Then,  $X$  has  $\gamma$  if and only if  $X$  has  $\gamma_p$ .

**PROBLEM 7.4.** Let  $p \in \omega^*$  such that  $\mathbf{R}$  does not satisfy  $\gamma_p$ . Is Theorem 7.3 true for  $k$ -spaces or Baire spaces?

**PROBLEM 7.5.** Is it consistent with ZFC that  $\gamma_p \Rightarrow \gamma$  for every  $p \in \omega^*$  for which  $\mathbf{R}$  does not satisfy  $\gamma_p$ ?

**PROBLEM 7.6.** Let  $X$  be a subset of  $\mathbf{R}$  satisfying  $\gamma_p$  where  $p \in \omega^*$  is such that  $\mathbf{R}$  does not have  $\gamma_p$ . Is  $X$  of strong measure zero?

### 8. Compact spaces and properties $\gamma$ and $\gamma_p$ .

After all we have already analyzed it is natural to ask under what conditions,  $X$  has a compactification  $\kappa X$  such that  $C_\pi(\kappa X)$  is Fréchet-Urysohn. Because of Gerlitz-Nagy's Theorem (see Theorem 4.5), this question may be posed as: Under what conditions on  $X$ , does  $X$  have a scattered compactification? In this Section we are going to give some results in this direction.

The first follows from Theorem 4.5 and the fact that every locally scattered space is scattered.

**THEOREM 8.1.** Let  $X$  be a locally compact (resp., locally compact and Lindelöf) space. Then, the following conditions are equivalent.

- (a)  $X$  has locally  $\gamma$  (resp., satisfies  $\gamma$ ).
- (b) The one-point compactification  $\alpha(X)$  of  $X$  has  $\gamma$ .
- (c) There is a compactification  $\kappa(X)$  of  $X$  with  $\gamma$ .

**PROOF.** We give here the proof of (a)  $\Rightarrow$  (b). For each  $x \in X$  there exist  $V_x$  and  $W_x$  elements of  $\mathcal{N}(x)$  such that  $V_x$  is compact and  $W_x$  satisfies  $\gamma$ . Consider a neighbourhood  $U_x$  of  $x$  such that  $x \in U_x \subset \text{cl} U_x \subset W_x$ . Thus,  $\text{cl} U_x$  has  $\gamma$  and  $V_x \cap \text{cl} U_x$  is a compact set with  $\gamma$ , which implies that  $V_x \cap \text{cl} U_x \in \mathcal{N}(x)$  is scattered. So  $X$  is scattered, since every locally scattered space is scattered. Now, it is not difficult to prove that  $\alpha(X)$  is also scattered. The conclusion in (b) now follows from Theorem 4 in [GN] (see Theorem 4.5).

We cannot dropped the Lindelöf condition in the last Theorem; in fact, the compact space of ordinals  $[0, \omega_1]$  has  $\gamma$  but  $[0, \omega_1)$  does not have this property.

When we consider  $\gamma_M$ -like properties we also have a result concerning compactifications. First, we need a Lemma (In the sequel the symbol  $\vartheta$  will denote one of the properties belonging to  $\{S\gamma_M : \emptyset \neq M \subset \omega^*\} \cup \{W\gamma_M : \emptyset \neq M \subset \omega^*\}$ ).

LEMMA 8.2. *Let  $Y$  be a space with  $\mathcal{A}\mathcal{E}$  where  $\mathcal{A}$  is a network on  $Y$  having an element  $A_0$  with the following property: For every countable collection  $U_1, U_2, \dots$  of neighborhoods of  $A_0$ , the subspace  $Z = Y \setminus \bigcap_{i < \omega} U_i$  has  $\mathcal{A}_Z \vartheta$ . Then  $Y$  satisfies  $\mathcal{A}\vartheta$ .*

PROOF. Suppose that  $\vartheta = S\gamma_M$  where  $\emptyset \neq M \subset \omega^*$ . Let  $\mathcal{G}$  be an open  $\mathcal{A}$ -cover of  $Y$ . Since  $Y$  has  $\mathcal{A}\mathcal{E}$  we can suppose, without loss of generality, that  $\mathcal{G}$  is countable. Let  $\mathcal{G}' = \{G \in \mathcal{G} : A_0 \subset G\}$ .  $\mathcal{G}' = \{G_1, \dots, G_n \dots\}$  is again an open  $\mathcal{A}$ -cover of  $Y$ . By assumption  $Z = Y \setminus \bigcap_{i < \omega} G_i$  has  $\mathcal{A}_Z \vartheta$ , so there is a sequence  $(j_n)_{n < \omega}$  such that  $Z \subset \mathcal{A}_Z - \lim_p G_{j_n}$  for every  $p \in M$ . It happens that  $Y = \mathcal{A} - \lim_p G_{j_n}$  for every  $p \in M$ . ■

THEOREM 8.3. *Let  $X$  be a metrizable locally compact (resp., metrizable locally compact and Lindelöf) space and let  $\emptyset \neq M \subset \omega^*$ . Then, the following are equivalent.*

- (a)  $X$  has locally  $\vartheta$  (resp., satisfies  $\vartheta$ ).
- (b) The one-point compactification  $\alpha(X)$  of  $X$  has  $\vartheta$ .
- (c) There is a compactification  $\kappa(X)$  of  $X$  with  $\vartheta$ .

PROOF. (b)  $\Rightarrow$  (c) is trivial.

(c)  $\Rightarrow$  (a): If  $\kappa(X)$  is scattered, then  $X$  is scattered, so  $X$  has locally  $\gamma$  (Theorem 4.5). Besides, if  $X$  is Lindelöf, then  $X$  is a  $\sigma$ -compact space, so  $X$  again has  $\gamma$  (Corollary 4.7). If  $\kappa(X)$  is a non scattered space, then we obtain the desired conclusion using Theorem 6.1.

(a)  $\Rightarrow$  (b): First case: For some point  $x \in X$ ,  $x$  has a compact non countable neighborhood having  $\vartheta$ . Then every  $\sigma$ -compact metrizable space has  $\vartheta$  (Theorem 6.1). Let  $V_1, \dots, V_n, \dots$  be open neighbourhoods of  $x_0 \in \alpha(X) \setminus X$ . So  $X \setminus \bigcap_{n < \omega} V_n$  is a  $\sigma$ -compact metrizable space that must satisfy  $\vartheta$ . It remains to apply Lemma 8.2.

Second case: For each  $x \in X$  the compact neighborhoods  $V_x$  of  $x$  having  $\vartheta$  are countable. Since every countable compact space is scattered (in fact, in this case  $V_x$  is homeomorphic to a countable ordinal space [dGB]),  $X$  is a scattered

space. Therefore,  $\alpha(X)$  is a scattered space; and we conclude that  $\alpha(X)$  has  $\gamma$  (Theorem 4.5). ■

PROBLEM 8.4. (1) Does every Čech-complete Lindelöf scattered space have a scattered compactification?

(2) Under what conditions on  $X$ , is  $\beta X$  scattered?

Using the facts that every second countable space can be embedded into  $[0,1]^{\aleph_0}$  and every compact non-scattered space contains a closed subset that can be continuously mapped onto  $[0,1]$ , and using Theorem 3.18 in [GT<sub>1</sub>] we obtain:

THEOREM 8.5. *Let  $\emptyset \neq M \subset \omega^*$ . The following conditions are equivalent.*

- (a)  $\mathcal{R}$  has  $\vartheta$ .
- (b) Every second countable space has a compactification with  $\vartheta$ .
- (c) There is a second countable non-scattered space which has a compactification satisfying  $\vartheta$ .

The natural generalization of Theorem 3.18 in [GT<sub>1</sub>] is also true. The symbol  $\pi\chi(x, X)$  denotes the  $\pi$ -character of  $x$  in the space  $X$  ([J]).

THEOREM 8.6. *Let  $\emptyset \neq M \subset \omega^*$  and let  $\mu$  be an uncountable cardinal. The following are equivalent.*

- (a) Every compact space of weight  $\mu$  has  $\vartheta$ .
- (b) Every zero-dimensional compact space of weight  $\mu$  satisfies  $\vartheta$ .
- (c)  $2^\mu$  has  $\vartheta$ .
- (d)  $[0,1]^\mu$  has  $\vartheta$ .
- (e) There is a compact space  $X$  having  $\vartheta$  such that  $\pi\chi(x, X) \geq \mu$  for every  $x \in X$ .

PROOF. (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c) are obvious. Since every compact space of weight  $\mu$  is a continuous image of a closed subset of  $2^\mu$  ([En, Theorem 3.2.2]), (c)  $\Rightarrow$  (a) and (c)  $\Rightarrow$  (d) holds. Besides,  $\pi\chi(x, [0,1]^\mu) \geq \mu$  for every  $x \in [0,1]^\mu$ , so we obtain (d)  $\Rightarrow$  (e). Finally, (e)  $\Rightarrow$  (c) is a consequence of [J, Theorem 3.18]. ■

Observe that Theorems 6.1, 8.3, 8.5 and 8.6 are trivial if we put  $\mathcal{H}\vartheta$  instead of  $\vartheta$  in each condition.

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