TSUKUBA J. MATH.
Vol. 20 No. 2 (1996), 257-262

ON UNICOHERENCE AT SUBCONTINUA*

By

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Abstract. In this paper an Eilenberg-type characterization of unicoherence at subcontinua and a mapping property about this unicoherence are given.

In [5], a localization of the notion of unicoherence, i.e., unicoherence at subcontinua was introduced. Several mapping properties about unicoherence at subcontinua are studied in [1]. This property is related to other properties of unicoherence closely.

The main purpose of this paper is to establish an Eilenberg-type characterization of unicoherence at continua and to show that local homeomorphism preserves unicoherence at subcontinua for locally connected continua. The latter partially answers a question raised by J.J. Charatonik in [1].

1. Preliminary

A continuum is a compact connected metric space. A continuum is unicoherent if the intersection of every two subcontinua having union X is connected; a continuum X is hereditarily unicoherent if every subcontinuum of X is unicoherent. Let Y be a subcontinuum of X; X is unicoherent at Y, denoted $U_n(Y)$, if for each pair of proper subcontinua A and B of X such that $X=A\cup B$ the set $A\cap B\cap Y$ is connected.

Let S^1 denote the unit circle. The mapping $f\in S^{1^X}$ is said to be inessiential ($f\sim 1$) if there exists a mapping $\phi\in R^X$ such that $f(x)=e^{i\phi(x)}$ for every $x\in X$.

The mapping $f\in S^{1^X}$ is said to be inessential on the subspace Y of X ($f\sim 1$ on Y), if there exists a mapping $\phi\in R^Y$ such that $f(x)=e^{i\phi(x)}$ for every $x\in Y$.

S. Eilenberg introduced the property (b) for studying unicoherence. A continuum X is said to have property (b) if for each mapping $f\in S^{1^X}$, there is

*The Project is supported by National Science Foundation of China

AMS (MOS) Sub. Class 54F20, 54F55.
Key words and phrases: Unicoherence at a subcontinuum, property (b), property (b) on a subcontinuum, local homeomorphism, monotone, light.
Received June 10, 1994

$f \sim 1$ ([3] p.63).

We say that a continuum X has property (b) on a subcontinuum Y of X if for each mapping $f \in S^{1X}$, there is $f \sim 1$ on Y.

It is clear to have

LEMMA 1. *Suppose that a continuum X has property (b) on a subcontinuum Y of X and Z is a subcontinuum of Y. Then X has property (b) on Z and Y also has property (b) on Z.*

PROPOSITION 2 ([3]). *Any continuum which has property (b) is unicoherent.*

PROPOSITION 3 ([3]). *Let continuum X be locally connected. The following conditions are equivalent:*

(1) *X is unicoherent;*

(2) *X has property (b).*

PROPOSITION 4 ([5] corollary 1.5). *Let $Y_1, Y_2, \cdots, Y_n$ be a finite collection of subcontinua of a continuum X such that X is $U_n(Y_i)$ for every i, and suppose that for each $i>1$*

$$Y_i \cap \cup\{Y_j : j < i\} \neq \phi.$$

Then X is $U_n(\cup_{i=1}^{n} Y_i)$.

PROPOSITION 5 ([5] Theorem 1.6). *Let Y be a subcontinuum of a continuum X. If X is $U_n(Y)$ and A and B are proper subcontinua of X such that $X = A \cup B$, then the sets $A \cap Y$ and $B \cap Y$ are connected.*

PROPOSITION 6. *If Y is a subcontinuum of a continuum X and Z is a subcontinuum of Y. Suppose that X is $U_n(Y)$ and Y is $U_n(Z)$. Then X is $U_n(Z)$.*

PROOF. Assuming that the conclusion is false, then there is a pair of proper subcontinua A and B of X such that $X = A \cup B$ and $A \cap B \cap Z$ is not connected. Suppose $A \cap B \cap Z = H \cup K$ is a separation. Because X is $U_n(Y)$, by proposition 5, $A \cap Y$ and $B \cap Y$ are all connected. One can assume that both of $A \cap Y$ and $B \cap Y$ are a nonempty proper subcontinuum of Y (Otherwise the case is simple). Then $[(A \cap Y) \cap (B \cap Y)] \cap Z = (A \cap B \cap Y) \cap Z = (A \cap B \cap Z) \cap Y = (H \cup K) \cap Y = H \cup K$. Since $H \cup K \subset Z \subset Y$, this contradicts to that Y is $U_n(Z)$.

PROPOSITION 7 (Corollary 7 of [1]). *Monotone mappings preserve uni-*

coherence at subcontinua.

2. An Eilenberg-type characterization of unicoherence at subcontinua

In this section we give a characterization of unicoherence at subcontinua which is similar with the characterization of unicoherence given by S. Eilenberg.

THEOREM 8. *Suppose that Y is a subcontinuum of a continuum X and for each pair of proper subcontinua A and B of X has property (b) on $A \cap B \cap Y$. Then X is unicoherent at Y.*

PROOF. Suppose X does not be unicoherent at Y. Then there are subcontinua A and B of X such that $X = A \cup B$ and $A \cap B \cap Y$ is not connected. Write $A \cap B \cap Y$ as a disjoint union of nonempty closed subsets C and D. One can assume that $A \cap Y \neq \phi \neq B \cap Y$. Otherwise $A \cap B \cap Y$ must be connected. Define a function $\phi : X \to R$ by

$$\phi(x) = \pi \frac{d(x,C)}{d(x,C)+d(x,D)},$$

for each $x \in X$, and mapping $f : X \to S^1$ by

$$f(x) = \begin{cases} e^{i\phi(x)}, & \text{if } x \in A, \\ e^{-i\phi(x)}, & \text{if } x \in B. \end{cases}$$

Thus the mapping f is well defined and continuous. By hypothesis of property (b) on $A \cap B \cap Y$, one have that $f \sim 1$ on $A \cap B \cap Y$. Then there is a $\xi \in R^{A \cap B \cap Y}$ such that $f(x) = e^{i\xi(x)}$ for each $x \in A \cap B \cap Y$. According to Proposition 5 both of $A \cap Y$ and $B \cap Y$ are connected. There exist integers m and n such that

$$\phi(x) = \xi(x) + 2m\pi, \quad \text{if } x \in A \cap Y$$

and

$$-\phi(x) = \xi(x) + 2n\pi, \quad \text{if } x \in B \cap Y.$$

However, if $x \in C \subset A \cap B \cap Y$, then $\xi(x) = -2m\pi = -2n\pi$ and hence $m=n$. On the other hand, if $x \in D \subset A \cap B \cap Y$, then $\xi(x) = \pi - 2m\pi = -\pi - 2n\pi$ and get $\pi = -\pi$. This contradiction establishes the Theorem.

THEOREM 9. *Let X be a locally connected continuum and Y is its subcontinuum. The following conditions are equivalent:*

(1) *X is unicoherent at Y;*

(2) *X has property* (b) *on* Y.

PROOF. Theorem 7 has established (2) $\Rightarrow$ (1). We prove (1) $\Rightarrow$ (2). For every $f: X \to S^1$ one can let $f = f_2 \circ f_1$ by the monotone-light factorization of f, where $f_1 X \to X'$ is monotone mapping and $f_2 : X' \to S^1$ is light mapping. By Proposition 7, X' is unicoherent at $Y' = f_1(Y)$. Since f_2 is light, X' must be at most 1-dimensional. Hence Y' must contain no any simple closed curve since X' is $U_n(Y)$. Y' is a locally connected continuum that contains no simple closed curve, i.e., it is a dendrite. Thus Y' is unicoherent and locally connected continuum. By Eilenberg's characterization, Y' has property (b). It is not difficult to see that X' has property (b) on Y'. This means that there is a $\psi \in S^{1^{Y'}}$ such that $f_2(x') = e^{i\psi(x')}$, for each $x' \in Y'$. Let $\phi = \psi \circ f_1 \in S^{1^Y}$. Then $f(x) = f_2 f_1(x) = e^{i\phi(x)}$, for each $x \in Y$. This is $f \sim 1$ on Y.

3. Unicoherence at continua under local homeomorphism

It is known that a surjective mapping on a continuum is a local homeomorphism if and only if it is open and n-to-1 for some fixed $n \geq 1$. It is proved that open finite-to-one mapping do not preserve unicoherence at subcontinua, even if the domain space is a linear graph in [1]. In the paper J.J. Charatonik raised a question: Do local homeomorphism perserve unicoherence at continua?

THEOREM 10. *Suppose* X *be a locally connected continuum,* Y *is a subcontinuum of* X *and* X *is* $U_n(Y)$, $f: X \to X'$ *is a local homeomorphism. Then* X' *is* $U_n(Y')$, *here* $Y' = f(Y)$.

PROOF. Whole proof consists of three steps.

CLAIM 1. Y can be covered by finite subcontinua $Y_1, Y_2, \cdots, Y_m$ such that X is $U_n(Y_i)$ and $f|_{Y_i}$ is a homeomorphis, for $i = 1, \cdots, m$.

Since X is locally connected, for any $x \in Y$ there is a connected open neighborhood V_X of x in Y. Moreover one can assume that $f|_{\bar{V}_X}$ a homeomorphism because f is a local homeomorphism.

The local connectedness of X and its unicoherence at Y imply property (b) on Y by Theorem 9, i.e., for each $f: X \to S^1 f|_Y \sim 1$. Thus, by Proposition 1, $f|_{\bar{V}_X} \sim 1$ and this means that X is $U_n(\bar{V}_X)$. By compactness of Y, finite subcontinua $Y_1, \cdots, Y_m$ as required above can be found. Denote $Y_i' = f(Y_i)$.

CLAIM 2. X' is $U_n(Y_i')$.

For any pair of proper subcontinua A' and B' of X' such that $A' \cup B' = X'$ we'll show that $A' \cap B' \cap Y_i'$ is connected. Local homeomorphism between continua is exactly a n-to-one open continuous mapping ([1]) and it is a confluent mapping ([4]). Thus one can get disjoint unions of subcontinua of X

$$f^{-1}(A') = A_1 \cup \cdots \cup A_k \quad \text{and} \quad f^{-1}(B') = B_1 \cup \cdots \cup B_s,$$

here $k, s \le n$ and each of A_j and B_j is mapped onto A' and B' by f respectively. Since $f|_{Y_i}$ is a homeomorphism, it is not difficult to see that only one of A_j and B_j intersects Y_i respectively. Assume that A_1 and B_1 are they. One can consider two subcontinua

$$A_1 \cup \cup\{B_j : B_j \cap A_1 \ne \phi\} \quad \text{and} \quad B_1 \cup \cup\{A_j : A_j \cap B_1 \ne \phi\}.$$

Similarly, consider the rest of A_j and B_j which meet $A_1 \cup \cup\{B_j : B_j \cap A_1 \ne \phi\}$ or $B_1 \cup \cup\{A_j : A_j \cap B_1 \ne \phi\}$ and continue to form unions. Because numbers of A_j and B_j are finite, so finally get two continua

$$A = A_1 \cup \cup\{B_j : j \in J' \subset \{2,\cdots,s\}\} \cup \cup\{A_j : j \in J \subset \{2,\cdots,k\}\}$$

and

$$B = B_1 \cup \cup\{A_j : j \in \{2,\cdots,k\} \setminus J\} \cup \cup\{B_j : j \in \{2,\cdots,s\} \setminus J'\}.$$

Then $A \cup B = X$ and $A_1 \cap B_1 \cap Y_i = A \cap B \cap Y_i$ is connected by unicoherence at Y_i. Therefore $A' \cap B' \cap Y_i' = f(A_1 \cap B_1 \cap Y_i)$ is also connected.

CLAIM 3. It is from Claim 2 that X' is $U_n(Y')$.

$Y' = f(Y) = \cup_{i=1}^m Y_i'$. The finite collection $\{Y_1,\cdots,Y_m\}$ can be selected such that for each $i > 1, Y_i' \cap \cup\{Y_j'; j < i\} \ne \phi$. By Corollary 1.5 of [5], the final conclusion yields.

A unicoherent continuum X is strong unicoherent if for every pair of proper subcontinua A and B such that $X = A \cup B$ both A and B are unicoherent. By Theorem 10 and Theorem 2.1 of [5] we have.

COROLLARY 11. *An image of a locally connected strongly unicoherent continuum under a local homeomorphism is locally connected strongly unicoherent.*

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