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UNIMODULARITY OF FINITE DIMENSIONAL HOPF ALGEBRAS

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Introduction.

D. Radford [4] proved that a finite dimensional Hopf algebra A is a symmetric algebra if and only if

(1) A is unimodular, and

(2) the square of the antipode s^2 is an inner algebra-automorphism.

It is well-known that the 4-dimensional Hopf algebra of Sweedler shows that Condition (2) does not necessarily imply Condition (1).

We present in this paper an example which shows that the converse does not hold either.

The construction.

Let n be a positive integer and m_i an integer $\geqq 2$ for $1 \leqq i \leqq n$. Let k be a field that contains a primitive m_ith root of unity η_i and let $\omega \in k$ be an element satisfying $\omega^{m_i} = 1$. We may assume that ω is a primitive mth root of unity for some positive integer m. We note that m divides m_i.

As a general case we shall construct a Hopf algebra B over k, which is generated as an algebra by g_i, x_i subject to the relations;

$$g_j g_i = g_i g_j, \quad g_k^{m_k} = 1, \quad x_k^{m_k} = 0, \quad x_j g_i = \omega g_i x_j,$$
$$x_k g_k = \eta_k g_k x_k, \quad x_i g_j = \omega^{-1} g_j x_i, \quad x_j x_i = \omega x_i x_j, \quad \text{for } 1 \leqq k \leqq n, \quad 1 \leqq i \lneqq j \leqq n.$$

First, let $F = k[G_1, \cdots, G_n, G_1^{-1}, \cdots, G_n^{-1}, X_1, \cdots, X_n]$ be the free algebra on $3n$ noncommuting indetarminates. We form the so-called free Hopf algebra

$$F = k[G_1, \cdots, G_n, G_1^{-1}, \cdots, G_n^{-1}, X_1, \cdots, X_n]/(G_i G_i^{-1} - 1, G_i^{-1} G_i - 1).$$

The coalgebra structure maps $F \to F \otimes F$ and $F \to k$ are the algebra-homomorphisms determined by

$$G_i \mapsto G_i \otimes G_i, \quad X_i \mapsto X_i \otimes G_i + 1 \otimes X_i,$$

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and

$$G_i \mapsto 1, \quad X_i \mapsto 0$$

respectively. The antipode is the anti-algebra-homomorphism determined by

$$G_i \mapsto G_i^{-1}, \quad X_i \mapsto -X_i G_i^{-1}.$$

Second, let L be the ideal of F generated by the following six subsets of F:

(1) $\{G_j G_i - G_i G_j | 1 \leqq i \lneqq j \leqq n\}$

(2) $\{G_k^{m_k} - 1 | 1 \leqq k \leqq n\}$

(3) $\{X_k^{m_k} | 1 \leqq k \leqq n\}$

(4) $\{X_j G_i - \omega G_i X_j, X_i G_j - \omega^{-1} G_j X_i | 1 \leqq i \lneqq j \leqq n\}$

(5) $\{X_k G_k - \eta_k G_k X_k | 1 \leqq k \leqq n\}$

(6) $\{X_j X_i - \omega X_i X_j | 1 \leqq i \lneqq j \leqq n\}$.

Let B indicate F/L. Then denoting cosets by the small case letters, we have the multiplicative relations stated above.
Finally, we show the following proposition.

PROPOSITION 1. *(a) The ideal L is a Hopf ideal of F, so B is the Hopf algebra.*

(b) B has as a basis the set $\{g_1^{p_1} \cdots g_n^{p_n} \cdot x_1^{q_1} \cdots x_n^{q_n} | 0 \leqq p_i, q_i \leqq m_i - 1\}$. Hence $dim B = \Pi m_i^2$.

(c) The antipode s of B is given by

$$s(g_i) = g_i^{-1}, \quad s(x_i) = -x_i g_i^{-1} = -\eta_i^{-1} g_i^{-1} x_i.$$

Hence $s^2(x_i) = \eta_i^{-1} x_i$ and the order of s is $(L.C.M.\ of\ \{m_i\}) \times 2$.

PROOF. (a) We note that in a Hopf algebra an ideal generated by skew primitives is a Hopf ideal.

One can see that the elements of the above sets (1), (2) and (5) are all skew primitives. Thus the ideal I, J and K generated by (1), (2) and (5), respectively, are Hopf ideals.

Since the elements of the set (4) are skew primitives modulo I, the ideal I' generated by these, namely I and (4), is a Hopf ideal; and since the elements of (6) are skew primitives modulo I', similarly the ideal I'' generated by these is also a Hopf ideal.

On the other hand, the elements of (3) are skew primitives modulo K (c.f. [2, Lem. 2.5.], [6]). Thus the ideal K' generated by the two sets K and (3) is a Hopf ideal.

Therefore the sum $I''+J+K'$ is a Hopf ideal, and is just L.

(b) For $1\leqq i\leqq n$ let $k[H_i], k[Y_i]$ be the polynomial algebras on 1 indeterminate H_i, Y_i, respectively, over k. Set $T_i=k[H_i]/(H_i^{m_i}-1), U_i=k[Y_i]/(Y_i^{m_i})$, and $M=T_1\otimes\cdots\otimes T_n\otimes U_1\otimes\cdots\otimes U_n$. Denoting cosets by the small case letters, we see that T_i has a basis $\{h_i^j|0\leqq j\leqq m_i-1\}$ and U_i has a basis $\{y_i^j|0\leqq j\leqq m_i-1\}$. Therefore M has the basis $\{h_1^{p_1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_n^{q_n}|0\leqq p_i,q_i\leqq m_i-1\}$.

We define a left B-module structure on M.

First of all, we define an algebra-homomorphism from the free Hopf algebra F to $End_k(M)$, the algebra of endomorphisms of M, as follows: for $M\ni m=h_1^{p_1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_n^{q_n}\quad(0\leqq p_i,q_i\leqq m_i-1)$,

$$G_i\cdot m=h_1^{p_1}\otimes\cdots\otimes h_i^{p_i+1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_n^{q_n},$$

$$X_i\cdot m=\omega^{p_1+\cdots+p_{i-1}}\eta_i^{p_i}\omega^{-(p_{i+1}+\cdots+p_n)}\omega^{q_1+\cdots+q_{i-1}}h_1^{p_1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_i^{q_i+1}\otimes\cdots\otimes y_n^{q_n},$$

but $p_0=q_0=0$. It is well-defined.

Next, we must show that the algebra-homomorphism annihilates the ideal L. For that purpose, it suffices to check on the six sets generating L.

On the sets (1), (2) and (3), it is clear.

On the set (4) for $1\leqq i\lneqq j\leqq n$,

$$\begin{aligned}X_jG_i\cdot m&=X_j\cdot h_1^{p_1}\otimes\cdots\otimes h_i^{p_i+1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_n^{q_n}\\&=\omega^{p_1+\cdots+p_{j-1}+1}\eta_j^{p_j}\omega^{-(p_{j+1}+\cdots+p_n)}\omega^{q_1+\cdots+q_{j-1}}\\&\qquad h_1^{p_1}\otimes\cdots\otimes h_i^{p_i+1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_j^{q_j+1}\otimes\cdots\otimes y_n^{q_n},\\G_iX_j\cdot m&=G_i\cdot(\omega^{p_1+\cdots+p_{j-1}}\eta_j^{p_j}\omega^{-(p_{j+1}+\cdots+p_n)}\omega^{q_1+\cdots+q_{j-1}}\\&\qquad h_1^{p_1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_j^{q_j+1}\otimes\cdots\otimes y_n^{q_n})\\&=\omega^{p_1+\cdots+p_{j-1}}\eta_j^{p_j}\omega^{-(p_{j+1}+\cdots+p_n)}\omega^{q_1+\cdots+q_{j-1}}\\&\qquad h_1^{p_1}\otimes\cdots\otimes h_i^{p_i+1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_j^{q_j+1}\otimes\cdots\otimes y_n^{q_n}.\end{aligned}$$

Thus

$$(X_jG_i-\omega G_iX_j)\cdot M=(0).$$

Similarly,

$$\begin{aligned}X_iG_j\cdot m&=X_i\cdot h_1^{p_1}\otimes\cdots\otimes h_j^{p_j+1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_n^{q_n}\\&=\omega^{p_1+\cdots+p_{i-1}}\eta_i^{p_i}\omega^{-(p_{i+1}+\cdots+p_n+1)}\omega^{q_1+\cdots+q_{i-1}}\end{aligned}$$

$$h_1^{p_1}\otimes\cdots\otimes h_j^{p_j+1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_i^{q_i+1}\otimes\cdots\otimes y_n^{q_n},$$

$$G_jX_i\cdot m=G_j\cdot(\omega^{p_1+\cdots+p_{i-1}}\eta_i^{p_i}\omega^{-(p_{i+1}+\cdots+p_n)}\omega^{q_1+\cdots+q_{i-1}}$$
$$h_1^{p_1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_i^{q_i+1}\otimes\cdots\otimes y_n^{q_n})$$
$$=\omega^{p_1+\cdots+p_{i-1}}\eta_i^{p_i}\omega^{-(p_{i+1}+\cdots+p_n)}\omega^{q_1+\cdots+q_{i-1}}$$
$$h_1^{p_1}\otimes\cdots\otimes h_j^{p_j+1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_i^{q_i+1}\otimes\cdots\otimes y_n^{q_n}.$$

Thus

$$(X_iG_j-\omega^{-1}G_jX_i)\cdot M=(0).$$

Therefore the algebra-homomorphism annihilates the set (4).

In a similar fashion, one can check that it takes zero on the sets (5), (6).

Hence the map annihilates the ideal L, so M can be seen as a left B-module.

Define a linear map $\phi: M\to B$ by $h_1^{p_1}\otimes\cdots\otimes h_n^{p_n}\otimes y_1^{q_1}\otimes\cdots\otimes y_n^{q_n}\mapsto g_1^{p_1}\cdots g_n^{p_n}\cdot x_1^{q_1}\cdots x_n^{q_n}$. It is easy to see that ϕ is surjective. Define a B-module map $\psi: B\to M$ by $a\mapsto a\cdot 1\otimes\cdots\otimes 1$. Then $\psi\circ\phi$ is the identity map, hence ϕ is bijective.

Therefore the set of the statement is a basis of B.

(c) It is clear.

This completes the proof of the proposition.

In this way we can form the Hopf algebra B, which is a generalization of [3, 4.4].

Now we shall notice the unimodularity. A Hopf algebra is called unimodular if the left integral space and the right one are the same (c.f. [3]).

LEMMA 2. *(a) Set* $\Lambda=(1+g_1+\cdots+g_1^{m_1-1})\cdots(1+g_n+\cdots+g_n^{m_n-1})\cdot x_1^{m_1-1}\cdots x_n^{m_n-1}$. *Then* Λ *is a non-zero left integral.*

(b) B is unimodular if and only if $\eta_i=\omega^{-n+2i-1}$ *for* $1\leqq i\leqq n$.

Let B be unimodular, then we have

(c) n is even,

(d) $m=m_i$ *for* $1\leqq i\leqq n$, *i.e.* η_i *is a primitive mth root of unity,*

(d') $(n-2i+1,m)=1$, *i.e.* $n-2i+1$ *and m are relatively prime, for* $1\leqq i\leqq n$,

(d'') $(p,m)=1$ *for each odd prime* $p\leqq n-1$,

(e) $s^2(x_i)=\eta_i^{-1}x_i=\omega^{n-2i+1}x_i$.

PROOF. (a) Clear.

(b) It follows that $\Lambda g_i=\omega^{-n+2i-1}\eta_i^{-1}\Lambda$ and $\Lambda x_i=0$. Indeed, $\Lambda g_i=$

$\omega^{-(n-i)}\eta_i^{-1}\omega^{i-1}\Lambda = \omega^{-n+2i-1}\eta_i^{-1}\Lambda$. Hence B is unimodular if and only if $\eta_i = \omega^{-n+2i-1}$ for $1 \leqq i \leqq n$.

(c) Suppose that n is odd. Then by (b) $\eta_{\frac{n+1}{2}} = \omega^0 = 1$, a contradiction.

(d) Recall that m divides m_i for all i. Since $\eta_i^m = 1$ holds by (b) and η_i is a primitive m_ith root of unity, m_i divides m. Hence $m = m_i$.

(d') This follows from (b) and (d).

(d") This follows from (c) and (d').

(e) By (b) and Proposition 1.(c).

This completes the proof of the lemma.

We note that (d), (d') and (d") are equivalent under the conditions $\eta_i = \omega^{-n+2i-1}$ and (c), and notice that Lemma 2 shows us the way of direct construction of unimodular Hopf algebra B, that is;

PROPOSITION 3. *Suppose that integers n and $m \geqq 2$ satisfy the conditions* (c), (d") *in Lemma 2. For a primitive mth root of unity ω, set $\eta_i = \omega^{-n+2i-1}$ (Note that this is a primitive mth root of unity). Then the Hopf algebra B is unimodular.*

Let $B_{n,m}$ indicate B in case of Proposition 3.

The relation between the unimodularity and s^2

We only consider the unimodular Hopf algebra $B_{n,m}$ and show that Condition (1) does not necessarily imply Condition (2).

LEMMA 4. *Suppose in addition that $n \geqq 4$ and m is divided by 4. Let $\mu = \frac{m}{4}$ and $\nu = \frac{n}{2}$, set $z = x_{\nu-1}^{\mu} + x_{\nu}^{\mu} \in B_{n,m}$ (Note that neither $x_{\nu-1}^{\mu}$ nor x_{ν}^{μ} is zero, so $z \neq 0$). Then there is an element $\zeta_i \in k$ such that $zg_i = \zeta_i g_i z$ for $1 \leqq i \leqq n$.*

PROOF.

It is easy to see that the statement is true for $1 \leqq i \leqq \nu - 2$ or $\nu + 1 \leqq i \leqq n$.

i) $i = \nu - 1$:

$$\begin{aligned} x_{\nu-1}^{\mu} g_{\nu-1} &= \eta_{\nu-1}^{\mu} g_{\nu-1} x_{\nu-1}^{\mu} \\ &= \omega^{-3\mu} g_{\nu-1} x_{\nu-1}^{\mu} \quad (\text{since } \eta_{\nu-1} = \omega^{-3}) \\ &= \omega^{\mu} g_{\nu-1} x_{\nu-1}^{\mu}, \\ x_{\nu}^{\mu} g_{\nu-1} &= \omega^{\mu} g_{\nu-1} x_{\nu}^{\mu}. \end{aligned}$$

$$x_\nu^\mu g_{\nu-1} = \omega^\mu g_{\nu-1} x_\nu^\mu .$$

Thus it follows that $zg_{\nu-1} = \omega^\mu g_{\nu-1} z$, so set $\zeta_{\nu-1} = \omega^\mu$.

ii) $i = \nu$:

$$\begin{aligned} x_{\nu-1}^\mu g_\nu &= \omega^{-\mu} g_\nu x_{\nu-1}^\mu, \\ x_\nu^\mu g_\nu &= \eta_\nu^\mu g_\nu x_\nu^\mu \\ &= \omega^{-\mu} g_\nu x_\nu^\mu . \quad (\text{since } \eta_\nu = \omega^{-1}). \end{aligned}$$

Thus it follows that $zg_\nu = \omega^{-\mu} g_\nu z$, so set $\zeta_\nu = \omega^{-\mu}$.

This completes the proof.

THEOREM 5. *Let n, m be as in Proposition 3. If $n \geqq 4$ and m is divided by 4, the square of antipode s^2 of $B_{n,m}$ is not inner.*

PROOF. Let z, μ and ν be as in Lemma 4. It follows that

$$s^2(z) = \omega^{3\mu} x_{\nu-1}^\mu + \omega^\mu x_\nu^\mu .$$

Now suppose that $s^2(z) = azb$ for some $a, b \in B_{n,m}$.

Write

$$a = \sum \alpha_{p_1 \cdots p_n q_1 \cdots q_n} g_1^{p_1} \cdots g_n^{p_n} \cdot x_1^{q_1} \cdots x_n^{q_n},$$

$$\text{where} \quad \alpha_{p_1 \cdots p_n q_1 \cdots q_n} \in k, 0 \leqq p_i, q_i \leqq m-1,$$

$$b = \sum \beta_{p_1' \cdots p_n' q_1' \cdots q_n'} g_1^{p_1'} \cdots g_n^{p_n'} \cdot x_1^{q_1'} \cdots x_n^{q_n'},$$

$$\text{where} \quad \beta_{p_1' \cdots p_n' q_1' \cdots q_n'} \in k, 0 \leqq p_n', q_n' \leqq m-1,$$

and divide a into $a_0 + a_1$, where a_0 is the part of $q_i = 0$ and a_1 is the rest, b into $b_0 + b_1$, where b_0 is the part of $q_i' = 0$ and b_1 is the rest.

Then

$$\begin{aligned} azb &= (a_0 + a_1) z (b_0 + b_1) \\ &= a_0 z b_0 + a_0 z b_1 + a_1 z b_0 + a_1 z b_1 \\ &= a_0 z b_0 + w, \quad \textit{where} \quad w = a_0 z b_1 + a_1 z b_0 + a_1 z b_1, \\ &= c_0 z + w, \\ &\qquad \textit{where } c_0 = \sum \gamma_{r_1 \cdots r_n} g_1^{r_1} \cdots g_n^{r_n}, \quad \gamma_{r_1 \cdots r_n} \in k, 0 \leqq r_i \leqq m-1, \\ &\qquad \textit{by Lemma } 4, \\ &= c_0 x_{\nu-1}^\mu + c_0 x_\nu^\mu + w. \end{aligned}$$

So $s^2(z) = azb$ implies

$$\omega^{3\mu} x_{\nu-1}^{\mu} + \omega^{\mu} x_{\nu}^{\mu} = c_0 x_{\nu-1}^{\mu} + c_0 x_{\nu}^{\mu} + w.$$

Thus $w = 0$ since w is a sum of terms that contain at least $\mu + 1$ x's as factors, and $c_0 = \gamma_{0\cdots 0} \cdot 1$ since $\gamma_{r_1 \cdots r_n} = 0$ if some $r_i \neq 0$. Set $\gamma = \gamma_{0\cdots 0}$. Then

$$\omega^{3\mu} x_{\nu-1}^{\mu} + \omega^{\mu} x_{\nu}^{\mu} = \gamma x_{\nu-1}^{\mu} + \gamma x_{\nu}^{\mu}.$$

Again comparing the coefficients, we have $\omega^{3\mu} = \gamma = \omega^{\mu}$, so $\omega^{(3\mu-\mu)} = \omega^{2\mu} = \omega^{\frac{m}{2}} = 1$, a contradiction since ω is a primitive mth root of unity.

Therefore for any $a, b \in B_{n,m}$, $s^2(z) \neq azb$. This implies that s^2 cannot be inner.

This completes the proof.

REMARKS. *For another* $B_{n,m}$, *we have the following:*

(1) *For any m, the square of antipode* s^2 *of* $B_{2,m}$ ([3, 4.4]) *is inner. More precisely,*

$$s^2(?) = (g_1^p g_2^q)^{-1} \cdot ? \cdot (g_1^p g_2^q),$$

where p, q are integers such that $p + q \equiv -1$ *modulo m.*

(2) *Suppose that* $n \geqq 4$ *and either that m is odd or that m is even but is not divided by* 4. *Then the square of antipode of* $B_{n,m}$ *is inner as follows. Fix an integer l such that* $m = 2l + 1 (\frac{m}{2} = 2l + 1$, *resp.). Set* $g = g_1^l \cdots g_{\frac{n}{2}-1}^l \cdot g_{\frac{n}{2}}^p \cdot g_{\frac{n}{2}+1}^q \cdot g_{\frac{n}{2}+2}^l \cdots g_n^l$, *where p and q are integers such that* $p + q \equiv -1$ *modulo m. Then*

$$s^2(?) = g^{-1} \cdot ? \cdot g.$$

(3) *In general if both the dimension of a Hopf algebra and the order of the square of antipode* s^2 *are odd, then* s^2 is inner [1, Prop. 1].

Therefore we have the following;

THEOREM 6. *There exists a finite dimensional unimodular Hopf algebra such that the square of antipode is not inner.*

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