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ON HOMOGENEITY OF HYPERSPACE OF RATIONALS

By

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Abstract. We show, assuming analytic determinacy, that the hyperspace consisting of compact sets of rational numbers is topologically homogeneous.

Introduction

For a metric space X with metric d, we consider the set $\mathscr{K}(X)$ of all non-empty compact subsets of X. We metrize $\mathscr{K}(X)$ by the *Hausdorff metric* as follows:

$$d_H(A,B)=\max\left\{\sup_{x\in B}d(x,A),\sup_{x\in A}d(x,B)\right\}.$$

The metric space $(\mathscr{K}(X),d_H)$ thus obtained is called the *hyperspace of compact sets* of X. Relationship between metric and topological properties of X and $\mathscr{K}(X)$ has been studied. Here we study homogeneity of hyperspaces. A topological space X is *homogeneous* if each point of X can be carried to any other by a homeomorphism of X onto itself.

The hyperspaces $\mathscr{K}(X)$ are known to be homogeneous for many spaces X. For example, it is known that if X is a Peano continuum, then $\mathscr{K}(X)$ is homeomorphic to the Hilbert cube $[0,1]^{\omega}$ and hence homogeneous. For $X=2^{\omega}$ (the Cantor space) or ω^{ω} (the Baire space), $\mathscr{K}(X)$ is homeomorphic to X itself and hence homogeneous. For $X=\boldsymbol{Q}$, the space of rational numbers, the situation was less trivial, since no simple topological characterization of $\mathscr{K}(\boldsymbol{Q})$ has been known. In fact, $\mathscr{K}(\boldsymbol{Q})$ is homogeneous under certain set theoretical assumption, as our main theorem shows. Under the same assumption, $\mathscr{K}(\boldsymbol{Q})$ is characterized as the unique subspace of 2^{ω} which is meager in 2^{ω} and everywhere property co-analitic (for the terminology, see Section 1).

THEOREM. *Assume analytic determinacy. The hyperspace $\mathscr{K}(\boldsymbol{Q})$ of compact sets of rational number, with the Hausdorff metric, is homogeneous.*

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Analytic determinacy is the statement that every two person infinite game on ω is determined if its payoff set is analytic as a subset of ω^{ω}. This assumption is strictly weaker than the existence of a measurable cardinal. The reader may consult [1] for details.

§1. Proof of The Theorem

For the sake of technical convenience, we regard $\boldsymbol{Q}$ as a countable dense subset of the Cantor space 2^{ω}, not of the real line. This may cause no confusion because every countable dense subset of perfect Polish space is in fact homeomorphic to the space of rational numbers. The inclusion $\boldsymbol{Q} \subset 2^{\omega}$ induces another topological inclusion $\mathscr{K}(\boldsymbol{Q}) \subset \mathscr{K}(2^{\omega})$ in an obvious way.

We need the notions of analytic and co-analytic sets. For general information about analytic and co-analytic sets, we refer the reader to [2]. A subset of Polish space X is *analytic* if it is the projection of a Borel subset of $X \times Y$ where Y is Polish. The class of analytic sets is closed under countable unions, countable intersections, continuous images and continuous preimages, while it is *not* closed under complements. The complement of an analytic set is called *co-analytic*. A subset E of a Polish space X is *everywhere properly co-analytic* if $E \cap U$ is co-analytic but not analytic for every basic neighborhood U in X.

We will use a result of J. Steel [3] in descriptive set theory. This is the only place we should mention analytic determinacy.

LEMMA 1.1. (Steel) *Assume analytic determinacy. Suppose $A, B \subset 2^{\omega}$ are everywhere properly co-analytic and meager. Then there is a homeomorphism $h: 2^{\omega} \approx 2^{\omega}$ such that $h[A] = B$.*

LEMMA 1.2. *The space $\mathscr{K}(\boldsymbol{Q})$ is meager as a subset of $\mathscr{K}(2^{\omega})$.*

PROOF. Let $\mathscr{P}$ be the set of non-empty compact subsets of 2^{ω} without isolated points. Then $\mathscr{P}$ is a dense G_{δ} subset of $\mathscr{K}(2^{\omega})$. To see that $\mathscr{P}$ is G_{δ}, note

$$K \in \mathscr{P} \Leftrightarrow (\forall N)[K \cap N \neq \emptyset \Rightarrow K \cap N \text{ is infinite}]$$

(where N runs over basic clopen subsets of 2^{ω}). For each N, we have

$$\{K : K \cap N \text{ is infinite}\} = \mathscr{K}(2^{\omega}) \setminus \bigcup_{n<\omega} \{K : K \cap N \text{ has at most } n \text{ points}\}.$$

The sets under countable union on the right hand side are closed. So the condition

" $K\cap N$ is infinite" determines a G_δ set of $K\in\mathscr{K}(2^\omega)$ for each fixed basic clopen set $N\subset 2^\omega$. It now follows immediately that $\mathscr{P}$ is G_δ, since there are only countably many basic clopen sets in 2^ω.

Since every countable compact set has an isolated point, $\mathscr{P}$ is disjoint from $\mathscr{K}(\boldsymbol{Q})$. Being disjoint from a dense G_δ set, $\mathscr{K}(\boldsymbol{Q})$ is meager. ■

LEMMA 1.3. *The space $\mathscr{K}(\boldsymbol{Q})$ is everywhere properly co-analytic as a subset of $\mathscr{K}(2^\omega)$.*

The proof of this lemma is given in Section 2. Here we prove the main theorem taking Lemma 1.3 for granted.

PROOF OF THE MAIN THEOREM: We show that every two points H and K in $\mathscr{K}(\boldsymbol{Q})$ have arbitrarily small homeomorphic clopen neighborhoods. Let U and V be any neighborhoods in $\mathscr{K}(\boldsymbol{Q})$ of H and K respectively. There are clopen subsets U' and V' of $\mathscr{K}(2^\omega)$ such that $H\in U'\cap\mathscr{K}(\boldsymbol{Q})\subset U$ and $K\in V'\cap\mathscr{K}(\boldsymbol{Q})\subset V$. As compact zero-dimensional metric spaces without isolated points, all non-empty clopen subsets of $\mathscr{K}(2^\omega)$ are homeomorphic to the Cantor space 2^ω. Let $h:U'\approx 2^\omega$ and $k:V'\approx 2^\omega$. By Lemmas 1.2 and 1.3, $h[U'\cap\mathscr{K}(\boldsymbol{Q})]$ and $k[V'\cap\mathscr{K}(\boldsymbol{Q})]$ are both everywhere properly co-analytic meager subsets of 2^ω. By Lemma 1.1, there is a homeomorphism of 2^ω onto itself which maps $h[U'\cap\mathscr{K}(\boldsymbol{Q})]$ onto $k[V'\cap\mathscr{K}(\boldsymbol{Q})]$. So the neighborhoods $U'\cap\mathscr{K}(\boldsymbol{Q})$ and $V'\cap\mathscr{K}(\boldsymbol{Q})$ are homeomorphic.

Thus we have proved that H and K have arbitrarily small homeomorphic clopen neighborhoods. Then Bernstein type back-and-forth construction yields a homeomorphism of $\mathscr{K}(\boldsymbol{Q})$ onto itself which maps H to K. ■

§2. The Cantor-Bendixson Number

For each $K\in\mathscr{K}(2^\omega)$, let ∂K be the set of all accumulation points of K. Since K is compact, ∂K is also compact, though it may be empty. By transfinite induction on ξ, define $\partial^\xi K$ as follows: $\partial^0 K=K$, $\partial^{\xi+1}K=\partial(\partial^\xi K)$, and $\partial^\lambda K=\bigcap_{\xi<\lambda}\partial^\xi K$ when λ is a limit ordinal. If K is countable, there is a countable ordinal ξ such that $\partial^\lambda K=\emptyset$. The smallest such ξ must be a successor ordinal because each $\partial^\eta K$ is compact. The *Cantor-Bendixson number*, denoted by $|K|_{CB}$, of countable compact subset K of 2^ω is the unique countable ordinal ξ such that $\partial^\xi K\neq\emptyset$ and $\partial^{\xi+1}K=\emptyset$.

We need the following "coding procedure" of countable well-ordering relations. Let $\{r_i : i < \omega\}$ be a fixed one-to-one enumeration of $\boldsymbol{Q}$. For each $\alpha \in 2^\omega$ let

$$Z(\alpha) = \{r_i : \alpha(i) = 0\},$$

and then define WO to be the set of $\alpha \in 2^\omega$ such that $Z(\alpha)$ is well-ordered by usual linear-ordering of $\boldsymbol{Q}$. For each $\alpha \in \mathrm{WO}$, let $|\alpha|_{\mathrm{WO}}$ be the order type of $(Z(\alpha), <)$.

LEMMA 2.1. (Folklore) *The set* WO *is co-analytic set which is not analytic.*

LEMMA 2.2. *The Cantor-Bendixson number is unbounded on each non-empty clopen subset of* $\mathscr{K}(\boldsymbol{Q})$.

PROOF. We show that each non-empty clopen subset of $\mathscr{K}(\boldsymbol{Q})$ contains an element whose Cantor-Bendixson number is ξ, where ξ is an arbitrary countable non-zero ordinal.

Suppose that a clopen subset E of $\mathscr{K}(\boldsymbol{Q})$ is given. Without loss of generality, we may assume E is of the form $\langle N_0, N_1, \cdots, N_k \rangle \cap \mathscr{K}(\boldsymbol{Q})$ where $N_0, N_1, \cdots, N_k$ are basic clopen set in 2^ω and $\langle N_0, N_1, \cdots, N_k \rangle$ is the Vietoris neighborhood:

$$\langle N_0, N_1, \cdots, N_k \rangle = \left\{ K \in \mathscr{K}(2^\omega) : K \subset \bigcup_{i \le k} N_i \ \& \ (\forall i \le k)[K \cap N_i \ne \emptyset] \right\}.$$

Now, since $N_0 \cap \boldsymbol{Q}$ is homeomorphic to $\boldsymbol{Q}$, there is a subset K_0 of $N_0 \cap \boldsymbol{Q}$ homeomorphic to the ordinal space $\omega^\xi + 1$ whose Cantor-Bendixson number is ξ. Pick $x_i \in N_i \cap \boldsymbol{Q}$ for $i = 1, \ldots, k$. Then

$$K = K_0 \cup \{x_1, \ldots, x_k\}$$

is a compact set belonging to $\langle N_0, N_1, \ldots, N_k \rangle \cap \mathscr{K}(\boldsymbol{Q})$ and its Cantor-Bendixson number is exactly ξ. ∎

Deffne a relation $S(F, \alpha)$ as the conjunction of the following clauses:

(1) F is a function on ω into $\mathscr{K}(2^\omega)$;

(2) r_0 is the smallest element of $Z(\alpha)$;

(3) if both r_i and r_j is in $Z(\alpha)$ and if $r_i < r_j$, then $F(j) \subset \partial F(i)$.

Then S is identified with a subset of $(\mathscr{K}(2^\omega))^\omega \times 2^\omega$. In fact:

LEMMA 2.3. *The relation S on $(\mathscr{K}(2^\omega))^\omega \times 2^\omega$ is Borel.*

PROOF. The only non-trivial part is the computation of the relation $K_0 \subset \partial K_1$ for $K_0, K_1 \in \mathscr{K}(2^\omega)$. This relation is in fact Borel as a subset of $\mathscr{K}(2^\omega) \times \mathscr{K}(2^\omega)$

because

$$K_0 \subset \partial K_1 \Leftrightarrow (\forall N)[K_0 \cap N \neq \emptyset \Rightarrow [K_1 \cap N \text{ is infinite}]$$

where N runs over basic clopen neighborhoods in 2^ω. ■

LEMMA 2.4. *The following relation R is co-analytic:*

$$R(K,\alpha) \Leftrightarrow K \in \mathscr{K}(\boldsymbol{Q}) \,\&\, [\alpha \notin \mathrm{WO} \vee |K|_{\mathrm{CB}} \leq |\alpha|_{\mathrm{WO}}].$$

PROOF. We shall show the equivalence

$$R(K,\alpha) \Leftrightarrow K \in \mathscr{K}(\boldsymbol{Q}) \,\&\, R_0(K,\alpha)$$

where R_0 is defined, using S in Lemma 2.3, as follows:

$R_0(K,\alpha) \Leftrightarrow (\forall F)(\forall \beta)[S(F,\beta) \,\&\, F(0)=K \Rightarrow (Z(\alpha),<)$ cannot be embedded into any initial segment of $(Z(\beta),<)]$.

Indeed, a pair (F,β) in the relation S represents, provided that $\beta \in \mathrm{WO}$, a sequence of non-empty compact sets $\langle K_\xi : \xi < |\beta|_{\mathrm{WO}} \rangle$ such that $K_\eta \subset \partial K_\xi$ for $\xi < \eta < |\beta|_{\mathrm{WO}}$. Then by transfinite induction we have $K_\xi \subset \partial^\xi K_0$ for every $\xi < |\beta|_{\mathrm{WO}}$. Thus the Cantor-Bendixson number $|K|_{\mathrm{CB}}$ is the maximum possible length of such sequences starting with $K_0 = K$. This means if $\beta \in \mathrm{WO}$ and if $S(F,\beta)$ holds for some F, then $|\beta|_{\mathrm{WO}} \leq |K|_{\mathrm{CB}}$ should be the case. The relation R_0 expresses the situation that as far as $\alpha \in \mathrm{WO}, |\alpha|_{\mathrm{WO}}$ is not less than any of such $|\beta|_{\mathrm{WO}}$. These observation proves the equivalence as required. Using this equivalence, one can show that R is co-analytic by simple computation of relations. ■

LEMMA 2.5. *If a set $\mathscr{A} \subset \mathscr{K}(\boldsymbol{Q})$ is analytic as a subset of $\mathscr{K}(2^\omega)$ then*

$$\sup\{|K|_{\mathrm{CB}} : K \in \mathscr{A}\} < \omega_1.$$

PROOF. By contradiction. Suppose there is an analytic subset $\mathscr{A}$ of $\mathscr{K}(\boldsymbol{Q})$ on which the Cantor-Bendixson number is unbounded: $\sup\{|K|_{\mathrm{CB}} : K \in \mathscr{A}\} = \omega_1$. Then the equivalence

$$\begin{aligned} \alpha \in \mathrm{WO} &\Leftrightarrow (\exists K)[K \in \mathscr{A} \,\&\, (\alpha \in \mathrm{WO} \,\&\, |\alpha|_{\mathrm{WO}} \leq |K|_{\mathrm{CB}}] \\ &\Leftrightarrow (\exists K)[K \in \mathscr{A} \,\&\, \neg R(K,\alpha)] \end{aligned}$$

shows that WO would be an analytic set. But in fact WO is not an analytic set as shown in Lemma 2.1. This contradiction proves Lemma 2.5. ■

PROOF OF LEMMA 1.3: Since $\mathscr{K}(\boldsymbol{Q})$ is co-analytic as subspace of $\mathscr{K}(2^\omega), U \cap \mathscr{K}(\boldsymbol{Q})$ is also co-analytic for each basic clopen subset U of $\mathscr{K}(2^\omega)$.

By Lemmas 2.2 and 2.5, we know that $U\cap\mathscr{K}(\boldsymbol{Q})$ is not analytic. Hence $\mathscr{K}(\boldsymbol{Q})$ is everywhere properly co-analytic. ∎

In our proof of the main theorem we needed the assumption of analytic determinacy only to obtain an autohomeomorphism of $\mathscr{K}(\boldsymbol{Q})$. Thus, the following problem arises:

PROBLEM: Is $\mathscr{K}(\boldsymbol{Q})$ homogeneous in ZFC?

References

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