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ON ISOMORPHISMS OF A BRAUER CHARACTER RING ONTO ANOTHER

Dedicated to Professor Hiroyuki Tachikawa

By

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1. Introduction

Throughout this paper G, Z and Q denote a finite group, the ring of rational integers and the rational field respectively. Moreover we write $\overline{Z}$ to denote the ring of all algebraic integers in the complex numbers and $\overline{Q}$ to denote the algebraic closure of Q in the field of complex numbers. For a finite set S, we denote by $|S|$ the number of elements in S.

Let $Irr(G)=\{\chi_1,\cdots,\chi_h\}$ be the complete set of absolutely irreducible complex characters of G. Then we can view $\chi_1,\cdots,\chi_h$ as functions from G into the complex numbers. We write $\overline{Z}R(G)$ to denote the $\overline{Z}$-algebra spanned by $\chi_1,\cdots,\chi_h$. For two finite groups G and H, let λ be a $\overline{Z}$-algebra isomorphism of $\overline{Z}R(G)$ onto $\overline{Z}R(H)$. Then we can write

$$\lambda(\chi_i)=\sum_{j=1}^{h}a_{ij}\chi'_j,\quad (i=1,\cdots,h)$$

where $a_{ij}\in\overline{Z}$ and $Irr(H)=\{\chi'_1,\cdots,\chi'_h\}$. In this case we write A to denote the $h\times h$ matrix with (i,j)-entry equal to a_{ij} and say that A is afforded by λ with respect to $Irr(G)$ and $Irr(H)$.

As is well known, concerning the isomorphism λ, we have the following two results, which seem to be most important. (For example see Theorem 1.3 (ii) and Lemma 3.1 in [5])

(i) $|c_G(c_i)|=|c_H(c'_{i'})|$, $(i=1,\cdots,h)$ *where* $\{c_1,\cdots,c_h\}$ *and* $\{c'_{1'},\cdots,c'_{h'}\}$ *are complete sets of representatives of the conjugate classes in* G *and* H *respectively and* $c_i\xrightarrow{\lambda}c'_{i'}$ $(i=1,\cdots,h)$. (Concerning a symbol "$c_i\xrightarrow{\lambda}c'_{i'}$", see the definition in [5] and also the definition in section 2 in this paper)

(ii) *A is unitary where A is the matrix afforded by λ with respect to $Irr(G)$ and $Irr(H)$.*

In this paper our main objective is to give a necessary and sufficient condition

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under which the above statements (i) and (ii) hold, concerning an isomorphism λ of a Brauer character ring onto another, and to state a generalization of theorems of Saksonov and Weidman about character tables of finite groups. (See Theorem 2, Corollary 2.1 in [3] and Theorem 3 in [4])
From now on, when we consider homomorphisms from an algebra to another, unless otherwise specified, we shall only deal with algebra homomorphisms.

2. Preliminaries

We fix a rational prime number p and use the following notation with respect to a finite group G.

G_o: the set of all p-regular elements of G

$Cl(G_o)=\{\mathfrak{C}_1=\{1\},\cdots,\mathfrak{C}_r\}$: the complete set of p-regular conjugate classes in G

$\{c_1=1,\cdots,c_r\}$: a complete set of representatives of $\mathfrak{C}_1,\cdots,\mathfrak{C}_r$ respectively

$IBr(G)=\{\varphi_1=1,\cdots,\varphi_r\}$: the complete set of irreducible Brauer characters of G, which can be viewed as functions from G_o into the complex numbers

For any subring R of the field of complex numbers such that $1\in R$, we write $RBR(G)$ to denote the ring of linear combinations of $\varphi_1,\cdots,\varphi_r$ over R. That is, $RBR(G)$ is the R-algebra spanned by $\varphi_1,\cdots,\varphi_r$. In particular we use the notation $BR(G)$ instead of $ZBR(G)$ and say that $BR(G)$ is the Brauer character ring of G. Moreover we add the following notation.

$G(\overline{Q}/Q)$: the Galois group of $\overline{Q}$ over Q

If $A=(a_{ij})$ is a matrix over $\overline{Q}$, then for $\sigma\in G(\overline{Q}/Q)$ we write A^σ to denote the matrix (a_{ij}^σ). We use the common notation X^* for the conjugate transpose of a matrix X.

Now we define characteristic class functions on G_0.

DEFINITION 2.1. We define class functions f_i *on* G_o $(i=1,\cdots,r)$ as follows

$$f_i(c_i)=1,\quad f_i(c_j)=0\quad(i\neq j).$$

In this case we say that these class functions are the characteristic class functions on G_0 and that f_i corresponds to $\mathfrak{C}_i$ or $\mathfrak{C}_i$ corresponds to f_i $(i=1,\cdots,r)$.

Now we prove an easy lemma concerning characteristic class functions on G_0.

LEMMA 2.2. Let $\{f_1,\cdots,f_r\}$ be the complete set of characteristic class functions on G_0. Then we have

$$f_i \in \overline{Q}BR(G), \quad (i=1,\cdots,r).$$

PROOF. Let $\hat{f}_i$ be a characteristic class function of G such that $\hat{f}_i|_{G_o} = f_i$ where $\hat{f}_i|_{G_o}$ indicates the restriction of f_i to G_o. Then each $\hat{f}_i$ is written as a $\overline{Q}$-linear combination of $\chi_1,\cdots,\chi_h$. That is,

$$\hat{f}_i = \sum_{j=1}^{h} (|\mathfrak{C}_i|/|G|)\overline{\chi_j(c_i)}\chi_j, \quad (i=1,\cdots,r) \tag{2.1}$$

For each absolutely irreducible complex character χ_i of G, $\chi_i|_{G_o}$ is written as a Z-linear combination of $\varphi_1,\cdots,\varphi_r$. That is,

$$\chi_i|_{G_o} = \sum_{j=1}^{r} d_{ij}\varphi_j, \quad (i=1,\cdots,h) \tag{2.2}$$

where (d_{ij}) is the decomposition matrix of G.

By virtue of the formulas (2.1) and (2.2), we can conclude that $f_i \in \overline{Q}BR(G)$, $(i=1,\cdots,r)$ as required. Q.E.D.

We are given two finite groups G and H. For G and H we assume that there exists an isomorphism λ of $\overline{Z}BR(G)$ onto $\overline{Z}BR(H)$. Then it follows that the rank of $BR(G)$= the rank of $BR(H)$ and $|Cl(G_o)| = |Cl(H_o)|$. We also can extend λ to an isomorphism $\hat{\lambda}$ of $\overline{Q}BR(G)$ onto $\overline{Q}BR(H)$ by linearity. By Lemma 2.2 we have $f_i \in \overline{Q}BR(G)$. Here we use the following additional notation.

$Cl(H_o) = \{\mathfrak{C}'_1 = \{1\},\cdots,\mathfrak{C}'_r\}$

$\{c'_1 = 1',\cdots,c'_r\}$: a complete set of representatives of $\mathfrak{C}'_1,\cdots,\mathfrak{C}'_r$ respectively

$\{f'_1,\cdots,f'_r\}$: the complete set of characteristic class functions on H_o where f'_i corresponds to $\mathfrak{C}'_i$, $(i=1,\cdots,r)$.

$IBr(H) = \{\varphi'_1 = 1,\cdots,\varphi'_r\}$.

We now show a lemma which is actually the key step in the proof of Lemma 2.4.

LEMMA 2.3. In the above situation, $\hat{\lambda}(f_i)$ is a characteristic class function on H_o, $(i=1,\cdots,r)$.

PROOF. Since $\overline{Q}BR(G)f_i = \overline{Q}f_i \cong \overline{Q}$, $\overline{Q}BR(G)f_i$ is a minimal ideal of $\overline{Q}BR(G)$ and so f_i is a (central) primitive idempotent, $(i=1,\cdots,r)$. Since $\hat{\lambda}(f_i) \in \overline{Q}BR(H)$, we can write

$$\hat{\lambda}(f_i) = \sum_{j=1}^{r} a_j f'_j, \quad a_j \in \overline{Q} \tag{2.3}$$

Since $f_i^2 = f_i$ *and* $f'_i f'_j = 0$ $(i \neq j)$, by the formula (2.3) we have

$$\hat{\lambda}(f_i) = \sum_{j=1}^{r} a_j^2 f'_j.$$

Thus $a_j^2 = a_j$, $(j = 1\cdots,r)$. Hence $a_j = 0$ or $a_j = 1$, $(j = 1\cdots,r)$. It follows that $\hat{\lambda}(f_i) = f'_j$ for some $j \in \{1,\cdots,r\}$, because f_i is a primitive idempotent, hence the result. Q.E.D.

Now we define a bijection from $Cl(G_o)$ to $Cl(H_o)$ through the isomorphism λ as follows. For a p-regular conjugate class $\mathfrak{C}_i$ of G, $\mathfrak{C}_i$ corresponds to a characteristic class function f_i on G_o. Since by Lemma 2.3 $\hat{\lambda}(f_i)$ is also a characteristic class function $f'_{i''}$ on H_o, $\hat{\lambda}(f_i) = f'_{i''}$ corresponds to a p-regular conjugate class $\mathfrak{C}'_{i''}$ of H. Here we assign $\mathfrak{C}'_{i''}$ to $\mathfrak{C}_i$ $(i = 1,\cdots,r)$. Thus we get a one-to-one correspondence between $Cl(G_o)$ and $Cl(H_o)$:

$$c_i \in \mathfrak{C}_i \rightarrow f_i \rightarrow \hat{\lambda}(f_i) = f'_{i''} \rightarrow \mathfrak{C}'_{i''} \ni c'_{i''}$$

where $i \rightarrow i''$ $(i = 1\cdots,r)$ is a permutation. In this case we write $\mathfrak{C}_i \xrightarrow{\lambda} \mathfrak{C}'_{i''}$ or $c_i \xrightarrow{\lambda} c'_{i''}$ $(i = 1\cdots,r)$.

Keeping the above notation, we give the following lemma concerning the Brauer character table of G. This lemma plays a fundamental role in the proof of Theorem 3.1. The proof is the same as that of Theorem 2.2 in [5] and so we omit its proof.

LEMMA 2.4. $(\varphi_i(c_j)) = (\lambda(\varphi_i)(c'_{j''}))$ *($r \times r$ matrices)* where $c_j \xrightarrow{\lambda} c'_{j''}$, $(j = 1,\cdots,r)$.

3. Main theorems

Let G and H be two finite groups with Cartan matrices C and C' respectively. Let λ be an isomorphism of $\overline{Z}BR(G)$ onto $\overline{Z}BR(H)$ and $A = (a_{ij})$ be the matrix afforded by λ with respect to $IBr(G) = \{\varphi_1,\cdots,\varphi_r\}$ and $IBr(H) = \{\varphi'_1,\cdots,\varphi'_r\}$. We set $Cl(G_o) = \{\mathfrak{C}_1,\cdots,\mathfrak{C}_r\}$ and $Cl(H_o) = \{\mathfrak{C}'_1,\cdots,\mathfrak{C}'_r\}$ and assume that $c_i \in \mathfrak{C}_i$, $c'_i \in \mathfrak{C}'_i$ and $c_i \xrightarrow{\lambda} c'_{i''}$ where $i \rightarrow i''$ $(i = 1,\cdots,r)$ is a permutation. We write $\boldsymbol{m}$ to denote the vector with i-th entry equal to $|C_G(c_i)|$ and $\boldsymbol{m}'$ to denote the vector with i-th entry equal to $|C_H(c'_{i''})|$, $(i = 1,\cdots,r)$. Then we have the following two theorems.

THEOREM 3.1. *With the above notation,* $\boldsymbol{m} = \boldsymbol{m}'$ *iff* $A^*CA = C'$. *This necessarily happens if* $CA = AC'$, *in which case A is clearly unitary.*

PROOF. To prove this theorem, we introduce some simplifying notation: Write P to denote the $r \times r$ matrix with (i,j)-entry equal to $\varphi_i(c_j)$ and similarly write P' for the matrix with (i,j)-entry equal to $\varphi'_i(c'_{j''})$.

Since $\lambda(\varphi_i) = \sum_{k=1}^{r} a_{ik}\varphi'_k$ where $A = (a_{ij})$, by Lemma 2.4 we have

$$\varphi_i(c_j) = \lambda(\varphi_i)(c'_{j''}) = \sum_{k=1}^{r} a_{ik}\varphi'_k(c'_{j''}).$$

This implies that $P = AP'$. Also, if B is the diagonal matrix with (i,i)-entry equal to $|C_G(c_i)|$, it follows that $P^*CP = B$ by Theorem 60.5 in [2]. Similarly $(P')^*C'P' = B'$, where B' is the diagonal matrix with (i,i)-entry equal to $|C_H(c'_{i''})|$. Here we note that $B = B'$ *iff* $\boldsymbol{m} = \boldsymbol{m}'$. Since $P^* = (P')^*A^*$, we have the two equations

$$(P')^*A^*CAP' = B \quad \text{and} \quad (P')^*C'P' = B'.$$

It is now obvious that $B = B'$ *iff* $A^*CA = C'$.

Now suppose $CA = AC'$. Then we show that A is unitary. If we write $J = A^*A$, then we have $(P')^*JC'P' = B$. Thus $(B')^{-1}B = (P')^{-1}(C')^{-1}JC'P'$. This is a diagonal matrix with rational entries and this shows that J has rational eigenvalues. But J has algebraic integer entries, and so must have integer eigenvalues. Thus $(B')^{-1}B$ is a diagonal matrix with positive integer diagonal entries. Also, A is invertible over $\overline{Z}$ and thus A^* is too. It follows that $det(J) = det((B')^{-1}B) = 1$ and so $(B')^{-1}B$ is the identity matrix I. It follows that $J = A^*A = I$ and so A is unitary, as required. Q.E.D.

THEOREM 3.2. *If $CA = AC'$, then we have*

(i) $\lambda(\varphi_i) = \varepsilon_i\varphi'_{i'}$ *where the ε_i are roots of 1 and $i \to i'$ $(i = 1,\cdots,r)$ is a permutation.*

(ii) *The Brauer character tables of G and H are the same.*

PROOF. (i) Now we pay attention to the fact that if $\alpha \in \overline{Z}$ and $|\alpha^\sigma| \le 1$ (an absolute value) for all $\sigma \in G(\overline{Q}/Q)$, then $\alpha = 0$ or α is a root of 1.

If we use the same notation as in the proof of Theorem 3.1, then we have $A = P(P')^{-1}$ and so A has entries that lie in a field with an abelian Galois group. Thus $(A^*)^\sigma = (A^\sigma)^*$ for all $\sigma \in G(\overline{Q}/Q)$. Since A is unitary by Theorem 3.1, A^σ is automatically unitary for all $\sigma \in G(\overline{Q}/Q)$. Hence we have the equation with respect to the i-th row of A^σ.

$$\sum_{j=1}^{r} a_{ij}^\sigma \overline{a_{ij}^\sigma} = \sum_{j=1}^{r} \left|a_{ij}^\sigma\right|^2 = 1, \quad (i = 1,\cdots,r)$$

Hence we have $\left|a_{ij}^\sigma\right| \le 1$ for all $\sigma \in G(\overline{Q}/Q)$. This implies that $a_{ij} = 0$ or a_{ij} is a root of 1 because of the above attention. Thus it follows that for each $i \in \{1,\cdots,r\}$, there exists $i' \in \{1,\cdots,r\}$ such that $a_{ii'}$ is a root of 1 and $a_{ij} = 0$ $(j \ne i')$. Hence $\lambda(\varphi_i) = \varepsilon_i\varphi'_{i'}$ where $\varepsilon_i = a_{ii'}$ is a root of 1 and $i \to i'$ $(i = 1,\cdots,r)$ is a permutation.

(ii) We state a one-to-one correspondence μ between $IBr(G)$ and $IBr(H)$

through the isomorphism λ as follows. By (i) of this theorem, we have $\lambda(\varphi_i)=\varepsilon_i\varphi'_{i'}$ $(i=1,\cdots,r)$ where the ε_i are roots of 1. Here we assign $\varphi'_{i'}$ to $\varphi_i : \mu(\varphi_i)=\varphi'_{i'}$ $(i=1,\cdots,r)$. Then μ can be extended to an isomorphism of $BR(G)$ onto $BR(H)$ by linearity. (See the proof of Lemma 3.2 in [5]) By Lemma 2.4 we have $(\varphi_i(c_j))=(\varphi'_{i'}(c'_{j''}))$ ($r\times r$ matrices) where $c_j \xrightarrow{\mu} c'_{j''}$ $(j=1,\cdots,r)$. That is, G and H have the same Brauer character table. Thus the result follows. Q.E.D.

REMARK. If the condition $\boldsymbol{m}=\boldsymbol{m}'$ in Theorem 3.1 holds, then we can easily prove $|G|=|H|$. But we can give examples such that for two finite groups G, H with $|G|\neq|H|$, a matrix A is unitary where A is afforded by an isomorphism of $BR(G)$ onto $BR(H)$. Actually, such an example is given by taking G and H to be any two p-groups of different orders. Another example can be found in [1]. (p = 2, G = the symmetric group S_4 on 4 symbols and H = the dihedral group D_6 of order 12. See the examples of section 91 in [1])

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