ON A REFINEMENT OF ANTI-SOUSLIN TREE PROPERTY

To the memory of the late Dr Simauti Takakazu,
the late Dr Maehara Shoji and the late Dr Hirose Ken

By

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§0. Introduction.

The field of uncountable trees contains so many unusual parts that has provided various examples and counter-examples in the fields around set theory and general topology (see Todorcevic [5]). A Souslin tree and a special Aronszajn tree are famous examples among them. The former is characterized mainly by the property that it has no uncountable antichain, and the latter by the one that it is a countable union of antichains. As seen here, the antichain properties often play a main role in describing tree characters. Anti-Souslin tree with which we shall concern is also in such a case. It is defined as a tree in which every uncountable set contains an uncountable antichain (Baumgartner [1], See Remark 1). In the above, countability and uncountability are the only scales for the size of infinite antichains. More refined scales of meaningful sense appears in Devlin and Shelah [3] and Shelah [4], where they introduce the notions of "stationary" and "club" for the subsets of a tree, and prove that e.g. for an ω_1-tree:

(*) it is collectionwise hausdorff under interval topology, if it has no stationary antichain,

(**) the existence of stationary antichains does not imply the existence of club antichains.

Thus it is expected that there would be some significant differences between those notions that are obtained from the definition of anti-Souslin property by replacing one or both occurrences of the word "uncountable" by "stationary" or "club". The present paper investigates the implicational relationships between these new notions. Consequently they are reduced to four different notions. We also try to

clarify their relationship to Q-embeddability and R-embeddability which are closely related with anti-Souslin property. In the end, one question remains open.

§1. Basic notions and summary of the results.

By an ω_1-tree we mean a well-founded tree $T = (T, <)$ such that

(i) T has a unique minimal element,

(ii) for every ordinal $\alpha < \omega_1$, the set $T_\alpha = \{x \mid ht(x) = \alpha\}$ is countable and non-empty, where $ht(x)$ means the height of x (in T),

(iii) T has no element of height ω_1,

(iv) For every distinct two nodes s, t on a limit level, the sets $\{x \in T \mid x < s\}$ and $\{x \in T \mid x < t\}$ are distinct.

If A is a set of ordinals, the set $\{x \in T \mid ht(x) \in A\}$ is denoted by $T \mid A$. For $s, t \in T, (s, t]$ denotes the interval $\{x \in T \mid s < x \le t\}$. We define also $(s, t), [s, t), [s, t]$ in parallel. A set of incomparable elements of T is called an anti-chain. A subset S of an ω_1-tree is called stationary (resp. club) if $ht'' S$ is stationary (resp. club) in ω_1. An ω_1-tree T is called Q- (resp. R-) embeddable if there exists a function $e : T \to Q$(resp. R) such that whenever $x < y$ in T then $e(x) < e(y)$ in Q(resp. R).

Now we introduce nine notions on ω_1-trees. Let each of the letters X and Y stand for one of the letters C, S and U, which stand for "club", "stationary" and "uncountable" respectively. Then an XY-tree is defined to be an ω_1-tree in which every X set contains a Y antichain. Hence, in particular, a UU-tree is an ω_1-tree in which every uncountable set contains an uncountable antichain; this is none other than an anti-Souslin tree (see Introduction). We write XY for the class of all XY-trees and QE (resp. RE) for the class of all Q- (R-) embeddable trees. We often write $A \to B$ instead of $A \subseteq B$ and $A \not\to B$ instead of $A \not\subseteq B$ for classes A and B. Trivial are:

(i) $SC = UC = US = \emptyset$,

(ii) $CC \to CS \to CU$,

(iii) $SS \to CS$, and $SS \to SU$,

(iv) $UU \to SU \to CU$.

As mentioned above, the notion UU is known under the name anti-Souslin, and the following are famous:

THEOREM 1. (Baumgartner [1]) (i) $QE \to RE \to UU$,

(ii) $(\lozenge)QE \not\leftarrow RE \not\leftarrow UU$.

We also use Jensen's principle $\lozenge$ or $\lozenge^*$ to determine those facts that are independent of the standard set-theoretical axioms. The principles are explained in Sections 3 and 4.

We note preliminarily the following facts, whose proofs are in the next section:

THEOREM 2. (i) $QE \to SS$,,

(ii) $CU = SU = UU$,

(iii) $CC = \emptyset$.

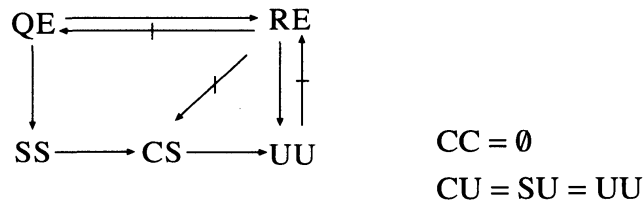It is known that

THEOREM 3. (Shelah [4]) *If* $\diamondsuit^*$, *there is an* **R**-*embeddable* ω_1-*tree that has no stationary antichain.*

Hence:

THEOREM 4. $(\diamondsuit^*)RE \nrightarrow CS$.
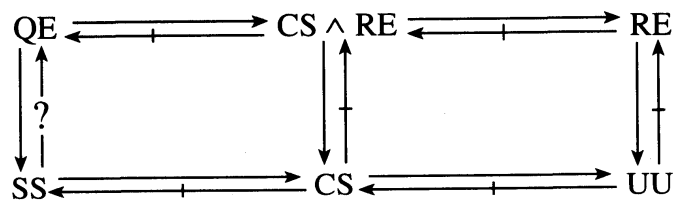
The facts described so far are summarized as follows:

$$QE \rightleftarrows RE$$

$$SS \longrightarrow CS \longrightarrow UU \qquad CC = \emptyset$$
$$CU = SU = UU$$

In Sections 3 and 4, toward the completion of this diagram, we show the following (see Remark 2):

THEOREM 5. (i) $(\diamondsuit) CS \nrightarrow RE$,

(ii) $(\diamondsuit^*)RE \wedge CS \nrightarrow SS$.

But the next question yet remains open:

QUESTION. *Does* SS *imply* RE *(or more strongly* QE*)?*

Thus we have finally:

$$QE \rightleftarrows CS \wedge RE \rightleftarrows RE$$

$$SS \rightleftarrows CS \rightleftarrows UU$$

§2. On CC and CU.

In this section, we prove Theorem 2. Every statement of the theorem can be proved within ZFC alone,

2.1. $QE \to SS$.

To show that a QE-tree T is in SS, suppose S is a stationary subset of T. The

property QE obviously implies that S is a countable union of anti-chains. Besides S is stationary, so, at least one of the anti-chains is stationary. q.e.d.

2.2. CU = SU = UU.

It suffices to show that $\mathrm{CU} \to \mathrm{UU}$. Suppose that T is in CU and U is an uncountable subset of T. For each ordinal $\alpha \in \omega_1$, take such a node t_α of T_α that has an extension in U. The set $\{t_\alpha \mid \alpha < \omega_1\}$ is club, so by the assumption, it contains an uncountable antichain, say A. For each member of A, pick out one of its extensions from U. Then the set of thus selected nodes in U is obviously an uncountable antichain. q.e.d.

2.3. CC = ∅.

We show that every ω_1-tree T has a club subset that contains no club antichain. Fix a stationary set $E \subseteq \omega_1$ such that $\omega_1 \setminus E$ is also stationary, and take a partition $\{F_\xi \mid \xi < \omega_1\}$ of $\omega_1 \setminus E$ such that each $F_\xi, \xi < \omega_1$, is stationary. Let $\langle \alpha_\xi \mid \xi < \omega_1 \rangle$ enumerate E increasingly. First, for every $\alpha \in E$, take a node $t(\alpha) \in T_\alpha$ arbitrarily. Next, for each $\beta \in \omega_1 \setminus E$, taking a unique ξ such that $\beta \in E_\xi$, take such a node $t(\beta) \in T_\beta$ that is comparable with $t(\alpha_\xi)$. Then the set $S = \{t(\beta) \mid \beta < \omega_1\}$ is trivially club. But it contains no club anti-chain. For, if C is a club subset of S, then since the set E is stationary, $t(\alpha_\xi) \in C$ for more ξ, and similarly for some β in $F_\xi, t(\beta) \in C$, besides the two nodes are comparable, thus C is not an anti-chain, q.e.d.

§3. CS dose not imply R-embeddability.

The aim is to construct a CS tree T which is not R-embeddable. Our construction requires the hypothesis $\Diamond$ that asserts the existence of a sequence

$$\langle S_\alpha \mid \alpha < \omega_1 \cap \mathrm{Lim} \rangle$$

such that for every subset X of ω_1, there are stationarily many $\alpha \in \omega_1$ satisfying $X \cap \alpha = S_\alpha$. Such a sequence has the property that for any partition $\langle P_\xi \mid \xi < \omega_1 \rangle$ of ω_1 and for any subset X of ω_1 satisfying $\forall \xi (X \cap P_\xi$ is countable), there are stationarily many α such that $X \cap \bigcup \{P_\xi \mid \xi < \alpha\} = X \cap \alpha = S_\alpha$. Hence from $\Diamond$ we obtain a sequence

$$\langle \Diamond_\lambda \mid \lambda \in \omega_1 \rangle$$

such that:

(1) $\Diamond_\lambda$ is a countable subset of $\bigcup \{^\alpha \omega \times R \mid \alpha < \lambda\}$, and

(2) whenever a subset X of $\bigcup \{^\alpha \omega \times R \mid \alpha < \omega_1\}$ satisfies that $(^\alpha \omega \times R) \cap X$ is countable for all $\alpha < \omega_1$, then there are stationarily many $\lambda < \omega_1$ satisfying

$\cup\{^{\alpha}\omega\times R\mid\alpha<\lambda\}\cap X=\Diamond_{\lambda}$. Fix it. We construct a tree T and a function $e:T\to Q^{+}\cup\{0\}$ such that

(T1) for $\alpha<\omega_{1},T_{\alpha}\subset{}^{\alpha}\omega$,

(T2) $f<_{T}g$ iff $f\subset g$ (function extension), for $f,g\in T$,

(T3) for every $s\in T$ and every positive $q\in Q$, there exist $t\in T$ such that $s<t$ and $e(t)<e(s)+q$ and e is strictly increasing on the interval $[s,t]$,

(T4) $\forall s,t\in T(s<t\wedge e''(s,t]\subseteq Q^{+}\to e(s)<e(t))$,

(T5) $\forall s\in T\forall\beta>ht(s)\forall q\in Q^{+}\exists t\in T_{\beta}(s<t\wedge e''(s,t]\subseteq Q^{+}\wedge e(t)\le e(s)+q)$,

(T6) $\forall t\in T(e(t)>0\to\exists s<t(e''(s,t]\subseteq Q^{+}))$,

(T7) $e(0)=0$ (note that $0\in T_{0}$).

Fix a surjection q from ω to Q^{+}. The construction is by induction on the levels.

(i) $T_{0}=\{0\},e(0)=0$,

(ii) $T_{\alpha+1}=\{x^{\frown}\langle n\rangle\mid x\in T_{\alpha},n\in\omega\},e(x^{\frown}\langle n\rangle)=e(x)+q(n)$.

(iii) Now let λ be a limit. To define T_{λ}, we first fix an increasing sequence $\langle\lambda_{n}\mid n<\omega\rangle$ unbounded in λ and associate $t_{x,q}\in{}^{\lambda}\omega$ with every pair of $x\in T\mid\lambda$ and $q\in Q^{+}$ as follows: first taking a sequence $\{x_{n}\mid n\in\omega\}$ increasing in $T\mid\lambda$ such that $x_{0}=x$ and x_{k+1} satisfies that $(1)x_{k}<x_{k+1},(2)ht(x_{k+1})>\lambda_{k},(3)e(x_{k+1})<e(x)+q$, and $(4)e''(x_{k},x_{k+1}]\subseteq Q^{+}$, then put $t_{x,g}=\cup\{x_{k}\mid k\in\omega\}$. The definition of T_{λ} is devided into two cases.

CASE 1. $\Diamond_{\lambda}$ is an embedding: $(T\mid\lambda)\to R$ (namely, $\Diamond_{\lambda}(x)<\Diamond_{\lambda}(y)$ for every x,y in $T\mid\lambda$ with $x<y$). Define a sequence $\langle y_{n}\mid n\in\omega\rangle$ increasing in $T\mid\lambda$ so that $y_{0}=0$ and for every $k\in\omega$, $(1)y_{k}<y_{k}+1$, $(2)\lambda_{k}<ht(y_{k}+1)$, and $(3)\exists y>y_{k}$ $(\Diamond_{\lambda}(y)\ge q(k))\to\Diamond_{\lambda}(y_{k}+1)\ge q(k)$. Put $s=\cup\{y_{k}\mid k\in\omega\}$. Put:

$$T_{\lambda}=\{s\}\cup\{t_{x,q}\mid x\in(T\mid\lambda),q\in Q^{+}\},$$

and $e(s)=0,e(t_{x,q})=e(x)+q$.

CASE 2. Otherwise. Define:

$$T_{\lambda}=\{t_{x,q}\mid x\in(T\mid\lambda),q\in Q^{+}\}\quad\text{and}\quad e(t_{x,q})=e(x)+q.$$

T is thus defined.

CLAIM 1. T is not in RE.

For, suppose that $d:T\to R$ were an embedding. Then, since the set

$$C=\{\lambda\in\omega\mid\forall x\in(T\mid\lambda)\forall q\in Q^{+}(\exists y\in T(x<y\wedge d(y)\ge q\to\exists\text{ such }y\in T\mid\lambda)\}$$

is club, there is a limit $\lambda\in C$ such that

$$d\cap(T\mid\lambda)\times R=\Diamond_{\lambda}.$$

Let s be such as in the definition of T_λ. Arbitrarily take $t \in T_{\lambda+1}$ and $k \in \omega$ so that $s < t$ and $d(s) < q(k) < d(t)$. Recall $y(k)$ in the definition of s. Since $\lambda \in C$, there is $y \in T \mid \lambda$ such that $y > y(k)$ and $q(k) \leq d(y)(= \Diamond_\lambda(y))$, hence $q(k) \leq d(y(k+1))$, so $d(y(k+1)) \not< d(s)$ despite $y(k+1) < s$, which contradicts embedding property of d.

CLAIM 2. $T \in CS$

PROOF. Recall the definition of T_λ for limit λ. We observe:

(*) For stationarily many $\lambda, \forall x \in T_\lambda(e(x) > 0)$.

from the properties (T1) and (T2), we have:

(**) whenever $e(x) > 0$, there is $y < x$ such that $e(y) = 0 \wedge e''(y, x] \subseteq Q^+$.

To show $T \in CS$, let X be any club subset of T. By the fact (*) we have a stationary set $Y \subset X$ such that $\forall y \in Y(e(y) > 0)$, and by (**), with each $y \in Y$, we can associate $n(y) \in T$ such that

$$n(y) < y \wedge e(n(y)) = 0 \wedge e''(n(y), y] \subseteq Q^+.$$

Since the function $n()$ is regressive, by the pressing down lemma (and using the countability of each level T_λ), we have a stationary subset $Y_0 \subseteq Y$ such that $n''Y_0 = \{z\}$ for some $z \in T$. Then there is a $q \in Q$ such that $Y_1 = Y_0 \cap e^{-1}\{q\}$ is a stationary subset of T. Since e is increasing on $[z, w]$ for every $w \in Y_1, Y_1$ is an antichain. q.e.d.

§4. CS does not imply SS.

The purpose is to prove Theorem 5. In fact, assuming $\Diamond^*$, we show that for arbitrarily given stationary set $E \subseteq \omega_1$, there is an ω_1-tree T with an R-embedding $e: T \to R$ such that (1) $e''(T \mid E) \subseteq Q$ and (2) $T \mid (\omega_1 \setminus E)$ contains no stationary antichain.

So, if $\omega_1 \setminus E$ is also stationary, this tree is in $CS \setminus SS$, since the condition (1) obviously implies CS, and the condition (2) implies $\neg$ SS. Basic techniques and ideas come from Devlin and Shelah [3] and Shelah [4]. To simplify our argument, we use the hypothesis $\Diamond^*$ in the following form:

LEMMA 1. $(\Diamond^*)$ There is a sequence $\langle \Diamond_\lambda^* \mid \lambda \in \omega_1 \rangle$ such that

(1) $\Diamond_\lambda^*$ is a countable family of countable subsets of $\bigcup\{^\alpha\omega \mid \alpha < \lambda\}$,

(2)for all $X \subset \bigcup\{^\alpha\omega \mid \alpha < \omega_1\}$, if $\forall \alpha < \omega_1 \mid X \cap^\alpha \omega \mid \leq \aleph_0$, then there are club-many $\lambda < \omega_1$ such that the set $X \cap \bigcup\{^\alpha\omega \mid \alpha < \lambda\}$ belongs to $\Diamond_\lambda^*$.

Fix such a sequence. By induction on the levels, we construct a desired tree T and R-embedding e so that $T_\alpha \subset^\alpha\omega$, where $f <_T g$ is defined by $f \subset g$.

$T_0 = \{\emptyset\}$ and $e(\emptyset) = 0$.

$T_{\beta+1} = \{x \frown \langle n \rangle \mid x \in T_\beta, n \in \omega\}$ and $e(x \frown \langle n \rangle) = e(x) + 1/n$.

Toward defining T_λ for a limit ordinal λ, fix a sequence $\langle \lambda_n \mid n < \omega \rangle$ increasing unbounded in λ. We are assuming inductively the following:

$(*) \forall x \in (T \mid \lambda) \forall q \in Q^+ \forall \alpha \in [ht(x), \lambda) \exists y \in T_\alpha \; e(y) \le e(x) + q$.

The definition is divided into two cases.

CASE 1. $\lambda \in E$, where E is a given stationary set. Associate $t_{x,q} \in {}^\lambda \omega$ with each pair of $x \in T \mid \lambda$ and $q \in Q^+$ satisfying $q > e(x)$ in the following manner: First take a sequence $\langle x_n \mid n \in \omega \rangle \subseteq T \mid \lambda$ so that $x_0 = x$ and for each $n > 0, ht(x_n) \ge \lambda_n \wedge x_{n-1} < x_n \wedge e(x_n) < q$, and then put $t_{x,q} = \cup\{x_n \mid n \in \omega\}$.

Now put $T_\lambda = \{t_{x,q} \mid x \in (T \mid \lambda), q \in Q, q > e(x)\}$ and $e(t_{x,q}) = q$.

CASE 2. Otherwise. Let $\{D_n^* \mid n \in \omega\}$ enumerate $\Diamond_\lambda^*$. For every pair of $x \in T \mid \lambda$ and $q \in Q$ with $q > e(x)$, define $y_n \in T \mid \lambda$ and $q_n \in Q$ by induction on $n \in \omega$ so that (1) $y_0 = x, q_0 = q$, (2) for $n > 0, y_n > y_{n-1}, ht(y_n) \ge \lambda_n, e(y_n) < q_{n-1}$,

$\exists z \in T \mid \lambda (z > y_{n-1} \wedge e(z) < q_{n-1} \wedge z \in D_{n-1}^* \cap (T \mid \lambda)) \rightarrow \exists \bar{z} \le y_n (z \in D_{n-1}^* \cap T \mid \lambda)$,

$q_n = (e(y_n) + q_{n-1})/2$,

and put $s_{x,q} = \cup\{y_n \mid n \in \omega\} \in {}^\lambda \omega$. Then put $T = \{s_{x,q} \mid x \in (T \mid \lambda), q \in Q^+, q > e(x)\}$,

and $e(s_{x,q}) = \sup\{e(y_n) \mid n < \omega\}$.

T_λ is thus defined. It remains to show that T is as desired. But the check is simple. We only show that $T \mid (\omega_1 \setminus E)$ has no stationary antichain. Let $X \subseteq T(\mid \omega_1 \setminus E)$ be an antichain. Put $C = \{\lambda \in \omega_1 \mid \forall x \in (T \mid \lambda) \forall q \in Q \exists y > x(e(y) < q$ $\wedge y \in X) \rightarrow \exists y \in (T \mid \lambda)(e(y) < q \wedge y \in X)\}$. Then C is club, so there are club-many $\lambda \in C$ such that $X \cap (T \mid \lambda) \in \Diamond_\lambda^*$. But for all such λ, we have $X \cap T_\lambda = \emptyset$. (Because, if $u \in X \cap T_\lambda$, then since $\lambda \notin E, u = s_{x,q}$ for some $x \in (T \mid \lambda)$ and $q \in Q$ with $e(x) < q$. Take n so that $X \cap (T \mid \lambda) = D_n^*$. Since $u > y_n \wedge e(u) < q_n \wedge u \in X$, it follows from $\lambda \in C$ that $\exists \bar{z} \in T \mid \lambda(z > y_n \wedge e(z) < q_n \wedge z \in D_n^* \cap T \mid \lambda)$. Hence by the definition of y_{n+1}, there is $z \in X$ such that $z \le y_{n+1}$. But this, together with $y_{n+1} < u \in X$, contradicts antichain property of X.) q.e.d.

REMARKS. Remark 1. The notion anti-Souslin was originally called by the name non-Souslin in Baumgartner [1]. The use of the present name is based on the suggestion by Baumgartner [2].

Remark 2. The tree T constructed in Lemma 3.9 of Shelah [4] has (among others) the property $T \notin CS \setminus RE$, which asserts Theorem 5 (i). Its construction

however is what uses a hypothesis stronger than $\lozenge$.

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