

ALMOST KÄHLER STRUCTURES ON THE RIEMANNIAN PRODUCT OF A 3-DIMENSIONAL HYPERBOLIC SPACE AND A REAL LINE

By

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1. Introduction.

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the Kähler form is closed (or equivalently $\mathbb{C}_{x,y,z} \nabla_x J Y, Z = 0$ for $X, Y, Z \in \mathfrak{X}(M)$, where \mathbb{C} and $\mathfrak{X}(M)$ denotes the cyclic sum and the Lie algebra of all differentiable vector fields on M respectively). A Kähler manifold, which is defined by $\nabla J = 0$, is necessarily an almost Kähler manifold. A non-Kähler almost Kähler manifold is called a strictly almost Kähler manifold. It is well-known that an almost Kähler manifold with integrable almost complex structure is a Kähler manifold. Concerning the integrability of almost Kähler manifolds, the following conjecture by S. I. Goldberg is known ([1]):

CONJECTURE. *A compact almost Kähler Einstein manifold is a Kähler manifold.*

The second author has proved that the above conjecture is true for the case where the scalar curvature is nonnegative ([4]). However, the above conjecture is still open in the case where the scalar curvature is negative. Recently, the authors proved that a $2n (\geq 4)$ -dimensional hyperbolic space H^{2n} cannot admit (compatible) almost Kähler structure ([3]).

In the present paper, we consider about (compatible) almost Kähler structures on the Riemannian product $H^3 \times R$ of a 3-dimensional hyperbolic space H^3 and a real line R . We construct an example of strictly almost Kähler structure (J, g) on the Riemannian product $H^3 \times R$ and determine the automorphism group of the almost Kähler manifold $(H^3 \times R, J, g)$. To our knowledge, this is the first example of strictly almost Kähler symmetric space. Moreover, we prove that the Riemannian product $H^3 \times R$ provided with a (compatible) almost Kähler structure (J, g) cannot be a universal (almost Hermitian) covering of any compact almost Kähler manifold (Theorem 2 in

section 3).

2. Preliminaries.

Let H^3 be a 3-dimensional hyperbolic space of constant sectional curvature -1 . Then, the Riemannian product $H^3 \times \mathbf{R}$ can be regarded as a Riemannian manifold (\mathbf{R}_+^4, g) equipped with the Riemannian metric g defined by

$$g = \frac{1}{x_1^2} \sum_{i=1}^3 dx_i \otimes dx_i + dx_4 \otimes dx_4,$$

where $\mathbf{R}_+^4 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_1 > 0\}$.

We put $X_i = x_1(\partial/\partial x_i)$, $i = 1, 2, 3$, and $X_4 = \partial/\partial x_4$. Then $\{X_1, X_2, X_3, X_4\}$ forms a global orthonormal frame field on $H^3 \times \mathbf{R}$. Direct calculation implies

$$(2.1) \quad [X_1, X_i] = -[X_i, X_1] = X_i$$

for $i = 2, 3$, and are otherwise zero. We set

$$\nabla_{X_i} X_j = \sum_{k=1}^4 \Gamma_{ijk} X_k,$$

for $1 \leq i, j \leq 4$, where ∇ denotes the Levi-Civita connection on $H^3 \times \mathbf{R}$. Then, by (2.1), we have

$$(2.2) \quad \Gamma_{ii1} = -\Gamma_{i1i} = 1$$

for $i = 2, 3$, and are otherwise zero.

Let (J, g) be an almost Hermitian structure on $H^3 \times \mathbf{R}$. We put

$$(2.3) \quad JX_i = \sum_{j=1}^4 J_{ij} X_j,$$

for $1 \leq i \leq 4$. Then we may easily observe that the 4×4 matrix (J_{ij}) is a skew-symmetric orthogonal matrix, i.e. the equalities

$$J_{ij} = -J_{ji}, \quad \sum_{k=1}^4 J_{ik} J_{jk} = \delta_{ij}$$

holds for $1 \leq i, j \leq 4$, and, furthermore, that the matrix (J_{ij}) is of the form

$$(I) \quad \begin{pmatrix} 0 & J_{12} & J_{13} & J_{14} \\ -J_{12} & 0 & J_{14} & -J_{13} \\ -J_{13} & -J_{14} & 0 & J_{12} \\ -J_{14} & J_{13} & -J_{12} & 0 \end{pmatrix}$$

or

$$(II) \begin{pmatrix} 0 & J_{12} & J_{13} & J_{14} \\ -J_{12} & 0 & -J_{14} & J_{13} \\ -J_{13} & J_{14} & 0 & -J_{12} \\ -J_{14} & -J_{13} & J_{12} & 0 \end{pmatrix}$$

with $J_{12}^2 + J_{13}^2 + J_{14}^2 = 1$.

3. An example of strictly almost Kähler structure on $H^3 \times R$.

The aim of this section is to construct an example of a strictly almost Kähler structure on the Riemannian product $(H^3 \times R, g)$ and to show Theorem 2.

We assume that (J, g) is an almost Kähler structure on the Riemannian product $(H^3 \times R, g)$. Then, the almost Kähler condition $\mathfrak{S}_{i,j,k}g((\nabla_{X_i} J)X_j, X_k) = 0$ and (2.2) yields the following system of first order partial differential equations:

$$(3.1) \quad \begin{cases} X_1 J_{23} - X_2 J_{13} + X_3 J_{12} - 2J_{23} = 0, \\ X_1 J_{24} - X_2 J_{14} + X_4 J_{12} - J_{24} = 0, \\ X_1 J_{34} - X_3 J_{14} + X_4 J_{13} - J_{34} = 0, \\ X_2 J_{34} - X_3 J_{24} + X_4 J_{23} = 0. \end{cases}$$

We may regard the triple (J_{12}, J_{13}, J_{14}) as a unit vector in the 3-dimensional Euclidean space R^3 . First of all, we may observe that the unit vector (J_{12}, J_{13}, J_{14}) has the following property.

PROPOSITION 1. *The vector (J_{12}, J_{13}, J_{14}) varies with the variable x_4 on an open subdt of $H^3 \times R$.*

PROOF. We assume that the vector (J_{12}, J_{13}, J_{14}) is independent on the variable x_4 . Then, the system of partial differential equations (3.1) reduces to the following:

$$(3.2) \quad \begin{cases} X_1 J_{23} - X_2 J_{13} + X_3 J_{12} - 2J_{23} = 0, \\ X_1 J_{24} - X_2 J_{14} - J_{24} = 0, \\ X_1 J_{34} - X_3 J_{14} - J_{34} = 0, \\ X_2 J_{34} - X_3 J_{24} = 0. \end{cases}$$

Now, we suppose that the matrix (J_{ij}) is of the form (I). Then, by (2.1), (2.2) and (3.2), we have

$$(3.3) \quad \begin{cases} \Delta J_{12} - 2X_1 J_{12} + 3J_{12} = 0, \\ \Delta J_{13} - 2X_1 J_{13} + 3J_{13} = 0, \\ \Delta J_{14} - 2X_1 J_{14} + 4J_{14} = 0. \end{cases}$$

From (3.3), we have

$$J_{12}\Delta J_{12} + J_{13}\Delta J_{13} + J_{14}\Delta J_{14} + 3 + J_{14}^2 = 0,$$

and hence

$$\sum_{i=1}^4 \{(X_i J_{12})^2 + (X_i J_{13})^2 + (X_i J_{14})^2\} = 3 + J_{14}^2,$$

since $J_{12}^2 + J_{13}^2 + J_{14}^2 = 1$, and we conclude that

$$(3.4) \quad \sum_{i,j,k=1}^4 (X_i J_{jk})^2 = 4(3 + J_{14}^2).$$

Next, from the equality above, we have

$$\sum_{i,j,k=1}^4 (X_i J_{jk}) X_l X_i J_{jk} = 4J_{14} X_l J_{14},$$

for each X_l . Thus, by a direct calculation, we obtain

$$(3.5) \quad \begin{aligned} & \sum_{l,i,j,k=1}^4 (X_l X_i J_{jk})^2 \\ &= 4 \sum_l (X_l J_{14})^2 + 4J_{14} \sum_l X_l X_l J_{14} - \sum_{l,i,j,k} (X_l J_{jk}) X_l X_l X_i J_{jk} \\ &= 4 \sum_l (X_l J_{14})^2 + 4J_{14} (\Delta J_{14} + 2X_1 J_{14}) \\ &\quad - \left\{ \sum_{l,i,j,k} (X_l J_{jk}) X_l X_l X_i J_{jk} - 2 \sum_{l,j,k} (X_l J_{jk}) X_l X_l J_{jk} \right. \\ &\quad \left. + 2 \sum_{i,j,k} (X_i J_{jk}) X_l X_i J_{jk} - \sum_{i \geq 2} \sum_{j,k} (X_i J_{jk})^2 \right\} \\ &= 4 \sum_l (X_l J_{14})^2 + 16J_{14} (X_1 J_{14} - J_{14}) - \{32J_{14} X_1 J_{14} \\ &\quad - 32(3 + J_{14}^2) - 4 \sum_i (X_i J_{14})^2 - 12 \sum_i (X_1 J_{1i})^2\} \\ &= 8 \sum_i (X_i J_{14})^2 + 12 \sum_i (X_1 J_{1i})^2 - 16J_{14} X_1 J_{14} + 16J_{14}^2 + 96. \end{aligned}$$

From (3.4) and (3.5), we find that $\sum_{i,j,k} (X_i J_{jk})^2$ and $\sum_{i,j,k,l} (X_l X_i J_{jk})^2$ are both bounded. By applying the similar argument in [3] along x_1 -curve, we can deduce a contradiction. More precisely, let γ_1 be any integral curve of X_1 . Then, we

obtain

$$(3.6) \quad \lim_{x_1 \rightarrow \infty} X_1 J_{ij} = 0 \quad (1 \leq i, j \leq 4).$$

along the geodesic γ_1 (see [3]). We denote by $\bar{\varphi}_a (a = 2, 3)$ isometries of H^3 such that $(\bar{\varphi}_a)_* X_1$ is orthogonal to X_1 and, $(\bar{\varphi}_2)_* X_1$ and $(\bar{\varphi}_3)_* X_1$ are orthogonal to each other along γ_1 . Let $\varphi_a(x_1, x_2, x_3, x_4) = (\bar{\varphi}_a(x_1, x_2, x_3), x_4) (a = 2, 3)$ be the naturally induced isometries of $H^3 \times R$, and we define almost complex structures $J_{(a)} (a = 2, 3)$ on $H^3 \times R$ by $J_{(a)} = (\varphi_a)_*^{-1} \circ J \circ (\varphi_a)_*$. Because J is independent on x_4 , so are $J_{(a)}$. We may easily check that $(J_{(a)}, g)$ are almost Kähler structures on $H^3 \times R$. Thus, by similar argument as above, we obtain

$$(3.7) \quad \lim_{x_1 \rightarrow \infty} X_1 J_{(a)ij} = 0 \quad (1 \leq i, j \leq 4, a = 2, 3).$$

along the geodesic γ_1 . Moreover, by semi-Kähler condition $\sum_{a=1}^4 \nabla_a J_{aj} = 0$ ($j = 1, 2, 3, 4$), we have

$$\begin{aligned} & \sum_{i=1}^4 \{(\nabla_i J_{12})^2 + (\nabla_i J_{13})^2 + (\nabla_i J_{14})^2\} \\ &= \sum_i \{(X_i J_{12})^2 + (X_i J_{13})^2 + (X_i J_{14})^2\} + 1 + J_{14}^2 \\ &+ 2(J_{23} X_2 J_{13} + J_{24} X_2 J_{14} + J_{32} X_3 J_{12} + J_{34} X_3 J_{14}) \\ &= 4 + 2J_{14}^2 - 2(J_{13} X_2 J_{23} + J_{14} X_2 J_{24} + J_{12} X_3 J_{32} + J_{14} X_3 J_{34}) \\ &= 2 - 2(J_{13} \nabla_2 J_{23} + J_{14} \nabla_2 J_{24} + J_{12} \nabla_3 J_{32} + J_{14} \nabla_3 J_{34}) \\ &= 2 + 2(J_{13} \nabla_1 J_{13} + J_{14} (\nabla_1 J_{14} + \nabla_3 J_{34})) + J_{12} \nabla_1 J_{12} + J_{14} (\nabla_1 J_{14} + \nabla_2 J_{24}) \\ &= 2, \end{aligned}$$

and hence, we have

$$(3.8) \quad \sum_{i,j,k=1}^4 (\nabla_i J_{jk})^2 = 8,$$

where $\nabla_i J_{jk} = g((\nabla_{X_i} J)X_j, X_k)$. From (3.6), (3.7) and (3.8), we can derive a contradiction (see [3]).

In the case where (J_{ij}) is of the form (II), we also arrive at a contradiction. Now, we write down an example of strictly almost Kähler structure (J, g) on $H^3 \times R$.

EXAMPLE. We define an almost complex structure J by

$$(J_{ij}) = \begin{pmatrix} 0 & \cos x_4 & \sin x_4 & 0 \\ -\cos x_4 & 0 & 0 & -\sin x_4 \\ -\sin x_4 & 0 & 0 & \cos x_4 \\ 0 & \sin x_4 & -\cos x_4 & 0 \end{pmatrix}$$

with respect to the orthonormal frame field $\{X_i\}_{i=1,2,3,4}$ defined in the preceding section. Then, it is easy to verify that (J, g) is a strictly almost Kähler structure on $H^3 \times R$.

In the rest of this section, we shall prove the following Theorem 2. First of all, we recall an integral formula on a 4-dimensional compact almost Kähler manifold. Let $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ be a 4-dimensional compact almost Kähler manifold. Then, we see that the square of the first Chern class $c_1(\bar{M})$ is given by the following formula (cf. [2], [5]):

$$(3.9) \quad c_1(\bar{M})^2 = \frac{1}{16\pi^2} \int_{\bar{M}} \{ (\bar{\tau}^*)^2 - 2 \| \bar{\rho}^{*sym} \|^2 + 2 \| \bar{\rho}^{*skew} \|^2 - \frac{1}{4} (\bar{\tau}^* + \bar{\tau}) \| \bar{\nabla} \bar{J} \|^2 + (\bar{\rho}, \bar{D}) \} d\bar{M},$$

where $\bar{\rho}^*, \bar{\tau}^*, \bar{\tau}, \bar{\nabla}, d\bar{M}$ denote the Ricci *-tensor, *-scalar curvature, scalar curvature, Levi-Civita connection on \bar{M} and the volume element of \bar{M} respectively, and $\bar{\rho}^{*sym}$ (resp. $\bar{\rho}^{*skew}$) the symmetric (resp. skew-symmetric) part of $\bar{\rho}^*$, and $(\bar{\rho}, \bar{D}) = \sum_{a,b,i,j=1}^4 \bar{\rho}_{ab} (\bar{\nabla}_a \bar{J}_{ij}) \bar{\nabla}_b \bar{J}_{ij}$. Here we put $\bar{\nabla}_a \bar{J}_{ij} = \bar{g}((\bar{\nabla}_{\bar{X}_a} \bar{J}) \bar{X}_i, \bar{X}_j)$ and $\bar{\rho}_{ab} = \bar{\rho}(\bar{X}_a, \bar{X}_b)$ for a local orthonormal frame field $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4\}$ on \bar{M} .

THEOREM 2. *Let (g, J) be a (compatible) almost Kähler structure on the Riemannian product $H^3 \times R$. Then, the almost Kähler manifold $(H^3 \times R, J, g)$ cannot be a universal (almost Hermitian) covering of any compact almost Kähler manifold.*

PROOF. Let (g, J) be a (compatible) almost Kähler structure on the Riemannian product $H^3 \times R$. We assume that there exists a compact almost Kähler manifold $(\bar{M}, \bar{J}, \bar{g})$ whose universal (almost Hermitian) covering is the almost Kähler manifold $(H^3 \times R, J, g)$. We denote by p the covering map from $H^3 \times R$ onto \bar{M} . For any point $\bar{p} \in \bar{M}$, we may choose a local orthonormal frame field $\{\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4\}$ near \bar{p} in such a way that $p_*(X_i) = \bar{X}_i, (i = 1, 2, 3, 4)$. We set $\bar{J}\bar{X}_i = \sum_{j=1}^4 \bar{J}_{ij} \bar{X}_j (i = 1, 2, 3, 4)$. for the proof, without loss of essentially, it is sufficient to consider the case where (J_{ij}) (and hence (\bar{J}_{ij})) is of the form (I). We may easily observe that $\sum_{i,j=1}^4 (\bar{X}_i \bar{J}_{ij})^2$ gives rise to a differentiable function

on \bar{M} . Since \bar{M} is a locally product Riemannian manifold of 3-dimensional hyperbolic space and a real line, it follows that the Euler class $\chi(\bar{M})$ of \bar{M} vanishes. Further, since \bar{M} is conformally flat, it follows that the first Pontrjagin class $p_1(\bar{M})$ of \bar{M} also vanishes. Thus, by the Wu's theorem ([6]), we have $c_1(\bar{M})^2 = 0$. On one hand, by direct calculation, from (3.9), we see that

$$\begin{aligned} c_1(\bar{M})^2 &= \frac{1}{8\pi^2} \int_{\bar{M}} \left(\|\bar{\nabla} \bar{J}\|^2 - \sum_{i,j=1}^4 \sum_{a=1}^3 (\bar{\nabla}_a \bar{J}_{ij})^2 \right) d\bar{M} \\ &= \frac{1}{8\pi^2} \int_{\bar{M}} \sum_{i,j=1}^4 (\bar{\nabla}_a \bar{J}_{ij})^2 d\bar{M}. \end{aligned}$$

Thus, it must follow that $\nabla_4 \bar{J}_{ij} = 0 (1 \leq i, j \leq 4)$ everywhere on \bar{M} , and hence, $\nabla_4 J_{ij} = X_4 J_{ij} = 0 (1 \leq i, j \leq 4)$ everywhere on M . But this contradicts Proposition 1, which completes the proof of theorem.

4. Automorphisms of the example of almost Kähler manifold $(H^3 \times R, J, g)$.

A differentiable transformation φ on an almost hermitian manifold (M, J, g) is called an automorphism if φ is a isometry and pseudo-holomorphic transformation, that is, φ satisfies

$$\varphi^* g = g \text{ and } \varphi_* \circ J = J \circ \varphi_*,$$

where φ_* denotes the differential map of φ and φ^* its dual map. We denote by $\text{Aut}_M(J, g)$ the set of all automorphisms on (M, J, g) . It is obvious that the set $\text{Aut}_M(J, g)$ is a group under the composition of maps, and we call it the automorphism group on (M, J, g) . In this section, we shall determine the automorphism group $\text{Aut}_{H^3 \times R}(J, g)$ of the example of strictly almost Kähler manifold $(H^3 \times R, J, g)$ constructed in the preceding section.

Let $\varphi \in \text{Aut}_{H^3 \times R}(J, g)$. We set $\varphi_*(X_i) = \sum_{j=1}^4 \varphi_{ij} X_j$ for $i = 1, 2, 3, 4$. Then, we see that 4×4 matrix (φ_{ij}) is of the following form

$$(4.1) \quad \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & 0 \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } (\varphi_{ij})_{1 \leq i, j \leq 3} \in SO(3),$$

$$(4.2) \quad \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & 0 \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & 0 \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ with } -(\varphi_{ij})_{1 \leq i, j \leq 3} \in SO(3),$$

since φ is an orientation-preserving isometry. We notice that $\varphi_{ij} (1 \leq i, j \leq 3)$ are independent on the variable x_4 . Since φ satisfies $\varphi_*^{-1} \circ J \circ \varphi_* = J$, we have, in particular,

$$(\varphi_*^{-1} \circ J \circ \varphi_*)X_1 = JX_1.$$

We now suppose that the matrix (φ_{ij}) is of the form (4.1). Then, by a direct calculation, we have

$$\begin{aligned} & (\varphi_*^{-1} \circ J \circ \varphi_*)X_1 \\ &= \{(\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21})J_{12} + (\varphi_{11}\varphi_{23} - \varphi_{13}\varphi_{21})J_{13}\}X_2 \\ &+ \{(\varphi_{11}\varphi_{32} - \varphi_{12}\varphi_{31})J_{12} + (\varphi_{11}\varphi_{33} - \varphi_{13}\varphi_{31})J_{13}\}X_3 \\ &+ (\varphi_{13}J_{12} - \varphi_{12}J_{13})X_4 \\ &= (\varphi_{33} \cos(x_4 + c_4) - \varphi_{32} \sin(x_4 + c_4))X_2 \\ &+ (-\varphi_{23} \cos(x_4 + c_4) + \varphi_{22} \sin(x_4 + c_4))X_3 \\ &+ (\varphi_{13} \cos(x_4 + c_4) - \varphi_{12} \sin(x_4 + c_4))X_4, \end{aligned}$$

for some constant $c_4 \in \mathbf{R}$, and hence

$$(4.3) \quad \begin{pmatrix} \varphi_{33} & -\varphi_{32} & \varphi_{31} \\ -\varphi_{23} & \varphi_{22} & -\varphi_{21} \\ \varphi_{13} & -\varphi_{12} & \varphi_{11} \end{pmatrix} \begin{pmatrix} \cos(x_4 + c_4) \\ \sin(x_4 + c_4) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos x_4 \\ \sin x_4 \\ 0 \end{pmatrix}.$$

Similarly, we see that the condition $(\varphi_*^{-1} \circ J \circ \varphi_*)X_i = JX_i, (i = 2, 3, 4)$ implies the same equality (4.3). From (4.3), it follows that

$$(\varphi_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos c_4 & \sin c_4 & 0 \\ 0 & -\sin c_4 & \cos c_4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, the automorphism $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ satisfies the following system of first order partial differential equations:

$$\frac{\partial \varphi_1}{\partial x_1} = \frac{1}{x_1} \varphi_1, \quad \frac{\partial \varphi_1}{\partial x_2} = \frac{\partial \varphi_1}{\partial x_3} = \frac{\partial \varphi_1}{\partial x_4} = 0,$$

$$\begin{aligned} \frac{\partial \varphi_2}{\partial x_2} &= \frac{1}{x_1} \varphi_1 \cos c_4, & \frac{\partial \varphi_2}{\partial x_3} &= -\frac{1}{x_1} \varphi_1 \sin c_4, & \frac{\partial \varphi_2}{\partial x_1} &= \frac{\partial \varphi_2}{\partial x_4} = 0, \\ \frac{\partial \varphi_3}{\partial x_2} &= \frac{1}{x_1} \varphi_1 \sin c_4, & \frac{\partial \varphi_3}{\partial x_3} &= \frac{1}{x_1} \varphi_1 \cos c_4, & \frac{\partial \varphi_3}{\partial x_1} &= \frac{\partial \varphi_3}{\partial x_4} = 0, \\ \frac{\partial \varphi_4}{\partial x_4} &= 1, & \frac{\partial \varphi_4}{\partial x_1} &= \frac{\partial \varphi_4}{\partial x_2} = \frac{\partial \varphi_4}{\partial x_3} = 0. \end{aligned}$$

Solving this system of differential equations, we find that the automorphism φ can be express as the form

$$\begin{aligned} \varphi(x_1, x_2, x_3, x_4) &= (e^{c_1} x_1, e^{c_1}((\cos c_4)x_2 - (\sin c_4)x_3) + c_2, \\ &e^{c_1}((\sin c_4)x_2 + (\cos c_4)x_3) + c_3, x_4 + c_4) \end{aligned}$$

for $c_i \in \mathbf{R}, i = 1, 2, 3, 4$.

Next, we suppose that the matrix (φ_{ij}) is of the form (4.2). Then, in the same way, we have

$$\begin{pmatrix} -\varphi_{33} & \varphi_{32} & -\varphi_{31} \\ \varphi_{23} & -\varphi_{22} & \varphi_{21} \\ -\varphi_{13} & \varphi_{12} & -\varphi_{11} \end{pmatrix} \begin{pmatrix} \cos(-x_4 + c_4) \\ \sin(-x_4 + c_4) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos x_4 \\ \sin x_4 \\ 0 \end{pmatrix},$$

which implies

$$(\varphi_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos c_4 & \sin c_4 & 0 \\ 0 & \sin c_4 & -\cos c_4 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and hence, we have

$$\begin{aligned} \varphi(x_1, x_2, x_3, x_4) &= (e^{c_1} x_1, e^{c_1}((\cos c_4)x_2 + (\sin c_4)x_3) + c_2, \\ &e^{c_1}((\sin c_4)x_2 - (\cos c_4)x_3) + c_3, -x_4 + c_4) \end{aligned}$$

for $c_i \in \mathbf{R}, i = 1, 2, 3, 4$.

We can summarize the above arguments as follows.

PROPOSITION 3. *The automorphism group $\text{Aut}_{H^3 \times \mathbf{R}}(J, g)$ of the almost Kähler manifold $(H^3 \times \mathbf{R}, J, g)$ is isomorphic to a solvable subgroup of affine transformation group $GL(4, \mathbf{R}) \times \mathbf{R}^4$ (embedded in $GL(5, \mathbf{R})$), which consists of the elements*

$$(4.4) \quad \begin{pmatrix} e^{c_1} & 0 & 0 & 0 & 0 \\ 0 & e^{c_1} \cos c_4 & -e^{c_1} \sin c_4 & 0 & c_2 \\ 0 & e^{c_1} \sin c_4 & e^{c_1} \cos c_4 & 0 & c_3 \\ 0 & 0 & 0 & 1 & c_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$(4.5) \quad \begin{pmatrix} e^{c_1} & 0 & 0 & 0 & 0 \\ 0 & e^{c_1} \cos c_4 & e^{c_1} \sin c_4 & 0 & c_2 \\ 0 & e^{c_1} \sin c_4 & -e^{c_1} \cos c_4 & 0 & c_3 \\ 0 & 0 & 0 & -1 & c_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $c_i \in \mathbf{R}, i=1,2,3,4..$

From Proposition 3, we may easily see that the group $\text{Aut}_{\mathbf{H}^3 \times \mathbf{R}}(J, g)$ acts transitively on $\mathbf{H}^3 \times \mathbf{R}$ and that the identity component $\text{Aut}_{\mathbf{H}^3 \times \mathbf{R}}(J, g)_0$ of $\text{Aut}_{\mathbf{H}^3 \times \mathbf{R}}(J, g)$ is a subgroup consists of the elements of the form (4.4) and acts simply transitively on $\mathbf{H}^3 \times \mathbf{R}$. Taking account of Theorem 2, we see that there does not exist a discrete uniform subgroup of $\text{Aut}_{\mathbf{H}^3 \times \mathbf{R}}(J, g)_0$ (i.e. discrete subgroup Γ of $\text{Aut}_{\mathbf{H}^3 \times \mathbf{R}}(J, g)_0$ such that the orbit space $\Gamma \backslash \mathbf{H}^3 \times \mathbf{R}$ is compact).

REMARK 1. We may easily find that the system of differential equations $\nabla_{X_i} J = 0 (i=1,2,3,4)$ has no solution, and thus the Riemannian product $(\mathbf{H}^3 \times \mathbf{R}, g)$ can not admit a (compatible) Kähler structure.

REMARK 2. Let ψ be an isometry on $(\mathbf{H}^3 \times \mathbf{R}, g)$ and (J, g) be an almost Kähler structure constructed in the preceding section. Then, almost Hermitian structure $(\psi(J), g)$ is also an almost Kähler structure on $(\mathbf{H}^3 \times \mathbf{R}, g)$, where $\psi(J)$ is defined by $\psi_*^{-1} \circ J \circ \psi_*$. The automorphism group $\text{Aut}_{\mathbf{H}^3 \times \mathbf{R}}(\psi(J), g)$ is determined by the automorphism group $\text{Aut}_{\mathbf{H}^3 \times \mathbf{R}}(J, g)$. Indeed, the map $\text{Aut}_{\mathbf{H}^3 \times \mathbf{R}}(J, g) \ni \varphi \mapsto \psi^{-1} \circ \varphi \circ \psi \in \text{Aut}_{\mathbf{H}^3 \times \mathbf{R}}(\psi(J), g)$ is an isomorphism.

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