TSUKUBA J. MATH.
Vol. 20 No. 1 (1996), 137-149

GEOMETRICAL THEORY OF GRAVITATIONAL AND ELECTROMAGNETIC FIELDS IN HIGHER ORDER LAGRANGE SPACES

By

Radu MIRON and Gheorghe ATANASIU

Abstract. In this paper we shall give an introduction to the geometry of higher order Lagrange spaces. The gravitational field, Einstein equations, as well as the electromagnetic fields and generalized Maxwell equations, are pointed out, too.

Introduction.

Recently, we studied the higher order Lagrange spaces $L^{(k)n}=(M,L)$, [10–14], founded on the notions of k-osculator bundle (Osc^kM,π,M), regular Lagrangian $\mathrm{L}:\mathrm{Osc}^kM\to R$, Euler-Lagrange equations $\overset{\circ}{\mathrm{E}}_i(L)=0$ and the geometrical model $(\mathrm{Osc}^kM,G,\boldsymbol{F})$, where G is the Sasaki lift of the fundamental tensor field g_{ij} of the space $L^{(k)n}$ and $\boldsymbol{F}$ is the natural $F(3,1)$ structure on Osc^kM.

But, g_{ij} can be considered as the gravitational potentials. Therefore, the Einstein equations of G with respect to the canonical metrical connection of $L^{(k)n}$ give us the Einstein equations of the higher order Lagrange space $L^{(k)n}$. The law of conservation is established, too.

We define the electromagnetic potentials as being the covariant components of the Liouville d-vector fields and we obtain the electromagnetic tensors given by (6.2). The generalized Maxwell equations are established, too.

1. Preliminaries. The k-osculator bundle.

In this section, we need the results established in the previous papers [10–14].

Let M be a real n-dimensional C^∞-manifold and (Osc^kM,π,M) its k-osculator bundle, where k is a natural number. The canonical local coordinates on the total space $E=\mathrm{Osc}^kM$ are denoted by $(x^i,y^{(1)i},\cdots,y^{(k)i})$. A coordinate transformation $(x^i,y^{(1)i},\cdots,y^{(k)i})\to(x^i,y^{(1)i},\cdots,\tilde{y}^{(k)i})$ on E is given by

Received April 11, 1994. Revised February 20, 1995

$$\tilde{x}^i = \tilde{x}^i(x^1,\dots,x^n),\ \operatorname{rank}\|\frac{\partial \tilde{x}^i}{\partial x^j}\| = n,$$

$$\tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j},$$

$$2\tilde{y}^{(2)i} = \frac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\frac{\partial \tilde{y}^{(1)i}}{\partial x^{(1)j}} y^{(2)j},$$

(1.1)

$$k\tilde{y}^{(k)i} = \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + 2\frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(1)j}} y^{(2)j} + \dots$$
$$\dots + k\frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j}.$$

If N is a nonlinear connection on E and J is the k-tangent structure [10], then $N_0 = N, N_1 = J(N_0),\cdots,N_{k-1} = J(N_{k-2})$ are k distributions geometrically defined on E, everyone of local dimension n. Let us consider the distribution V_k on E locally generated by the vector fields $\left\{\frac{\partial}{\partial y^{(k)i}}\right\}$. Consequently, the tangent space to E at a point $u \in E$ is given by the direct sum of the vector spaces:

(1.2) $$T_u(E) = N_o(u) \oplus N_1(u) \oplus \cdots \oplus N_{k-1}(u) \oplus V_k(u), \forall u \in E.$$

An adapted basis to the direct decomposition (1.2) is given by

(1.3) $$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \cdots, \frac{\delta}{\delta y^{(k-1)i}}, \frac{\partial}{\partial y^{(k)i}}\right\}, \qquad (i = \overline{1-n}),$$

(1.4) $$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^r_{(1)i}\frac{\partial}{\partial y^{(1)r}} - \cdots - N^r_{(k)i}\frac{\partial}{\partial y^{(k)r}}$$

where

(1.4) $$\begin{cases} \dfrac{\delta}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - N^r_{(1)i}\dfrac{\partial}{\partial y^{(2)r}} - \cdots N^r_{(k-1)i}\dfrac{\partial}{\partial y^{(k)r}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\delta}{\delta y^{(k-1)i}} = \dfrac{\partial}{\partial y^{(k-1)i}} - N^r_{(1)i}\dfrac{\partial}{\partial y^{(k)r}}. \end{cases}$$

The systems of functions $N^i_{(1)j},\cdots,N^i_{(k)j}$ are called the coefficients of the nonlinear connection N.

The dual basis of the basis (1.3) can be given in the form

(1.5) $$\{dx^i, \delta y^{(1)i},\cdots,\delta y^{(k)i}\}, \qquad (i = \overline{1-n}),$$

where

$$(1.6)\qquad \begin{cases} \delta y^{(1)i} = dy^{(1)i} + M^i_{(1)r}dx^r \\ \delta y^{(2)i} = dy^{(2)i} + M^i_{(1)r}dy^{(1)r} + M^i_{(2)r}dx^r, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \delta y^{(k)i} = dy^{(k)i} + M^i_{(1)r}dy^{(k-1)r} + \cdots + M^i_{(k)r}dx^r \end{cases}$$

The systems of functions $M^i_{(1)j},\dots,M^i_{(k)j}$ are called the *dual* coefficients of the nonlinear connection N. There are some relations between the coefficients and the dual coefficients of N, [10].

Let $\Gamma^{(1)},\dots,\Gamma^{(k)}$ be the Liouville vector fields on E. These vector fields are linearly independent on E and have the property $\Gamma^{(k-1)} = J(\Gamma^{(k)}),\dots,$ $\Gamma^{(1)} = J(\Gamma^{(2)})$.

In the adapted basis (1.3), the vectorial field $\Gamma^{(k)}$ can be put in the form

$$(1.7)\qquad \Gamma^{(k)} = z^{(1)i}\frac{\delta}{\delta y^{(1)i}} + 2z^{(2)i}\frac{\delta}{\delta y^{(2)i}} + \dots + kz^{(k)i}\frac{\partial}{\partial y^{(k)i}}.$$

The coefficients $z^{(1)i},\cdots,z^{(k)i}$ from (1.7) are given by

$$(1.7)'\qquad \begin{cases} z^{(1)i} = y^{(1)i}, \\ 2z^{(2)i} = 2y^{(2)i} + M^i_{(1)r}y^{(1)r}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ kz^{(k)i} = ky^{(k)i} + (k-1)M^i_{(1)r}y^{(k-1)r} + \cdots + M^i_{(k-1)r}y^{(1)r}. \end{cases}$$

With respect to (1.1) we have $\tilde{z}^{(\alpha)i} = \frac{\partial \tilde{x}^i}{\partial x^j}z^{(\alpha)j}, (\alpha = 1,\cdots,k)$. Consequently, $z^{(1)i},\cdots,z^{(k)i}$ are d-vector fields. They will be called *the Liouville d-vector fields*.

A k-spray on E is a vector field $S \in X(E)$ which has the property $JS = \overset{(k)}{\Gamma}$. It can be written in the form

$$(1.8)\qquad S = y^{(1)i}\frac{\partial}{\partial x^i} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} - \\ -(k+1)G^i(x, y^{(1)},\cdots,y^{(k)})\frac{\partial}{\partial y^{(k)i}}.$$

With respect to (1.1) its coefficients G^i transform as follows:

$$(1.8)'\qquad (k+1)\tilde{G}^i = (k+1)\frac{\partial \tilde{x}^i}{\partial x^j}G^j - (y^{(1)j}\frac{\partial \tilde{y}^{(k)i}}{\partial x^j} + \\ + 2y^{(2)j}\frac{\partial \tilde{y}^{(k)i}}{\partial y^{(1)j}} + \cdots + ky^{(k)j}\frac{\partial \tilde{y}^{(k)i}}{\partial y^{(k-1)j}}).$$

A k-spray S, with the coefficients G^i, is equivalent to the k-paths of the equations

$$\frac{1}{(k+1)!}\frac{d^{k+1}x^i}{dt^{k+1}} + G^i\left(x, \frac{dx}{dt}, \cdots, \frac{1}{k!}\frac{d^k x}{dt^k}\right) = 0. \tag{1.9}$$

Clearly, (1.9) give us the integral curves of the vector field S.

We repeat an important result, proved in the paper [10]:

THEOREM 1.1. If S is a k-spray, having the coefficients G^i, then the systems of functions $M^i_{(1)j}, \cdots, M^i_{(k)j}$ from the following equalities:

$$\begin{aligned} M^i_{(1)j} &= \frac{\partial G^i}{\partial y^{(k)j}}, \\ M^i_{(2)j} &= \frac{1}{2}(SM^i_{(1)j} + M^i_{(1)r}M^r_{(1)j}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M^i_{(k)j} &= \frac{1}{k}(SM^i_{(k-1)j} + M^i_{(1)r}M^r_{(k-1)j}), \end{aligned} \tag{1.10}$$

are the dual coefficients of a nonlinear connection determined only by the k-spray S.

2. Higher order Lagrange spaces.

We consider the manifold:

$$\tilde{E} = \{(x, y^{(1)}, \cdots, y^{(k)}) \in Osc^k M \mid rank \parallel y^{(1)i} \parallel = 1\}$$

and we give the following:

DEFINITION 2.1. A differentiable Lagrangian of order k on a C^∞-manifold M is a function $L: E \to R$, differentiable on $\tilde{E}$ and continuous in the points of E where $y^{(1)i}$ are nuls.

It follows that

$$g_{ij}(x, y^{(1)}, \ldots, y^{(k)}) = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k)j}} \tag{2.1}$$

is a symmetric d-tensor field of type (o, 2) on $\tilde{E}$.

We say that the differentiable Lagrangian L is a regular if

$$\operatorname{rank} \parallel g_{ij}(x, y^{(1)}, \ldots, y^{(k)}) \parallel = n, \text{ on } \tilde{E}. \tag{2.2}$$

DEFINITION 2.2. We call a Lagrange space of order k a pair $L^{(k)n} = (M, L)$, where L is a regular Lagrangian of order k and the d-tensor field g_{ij} from (2.1) has a constant signature on $\tilde{E}$.

In the case $k = 1$ this definition reduces to that of the Lagrange space $L^n = (M, L)$, [15].

The function L of the space $L^{(k)n}$ is called the fundamental function and the d-tensor field g_{ij} from (2.1) the fundamental (or metric) tensor field of $L^{(k)n}$.

We denote by $g^{ij}(x, y^{(1)}, \cdots, y^{(k)})$, the contravariant tensor field of the fundamental tensor $g_{ij}(x, y^{(1)}, \cdots, y^{(k)})$, i.e. $g_{ir}g^{rj} = \delta_i^j$.

EXAMPLE. Let $R^n = (M, \gamma_{ij}(x))$ be a Riemannian space and $\mathrm{Prol}^k R^n$ its prolongation of order k, given in our paper [11]. We consider the Liouville d-vector field $z^{(k)i}$ constructed by means of the canonical nonlinear connection of the space $\mathrm{Prol}^k R^n$. Then

$$L(x, y^{(1)}, \cdots, y^{(k)}) = \gamma_{ij}(x) z^{(k)i} z^{(k)j} \tag{2.3}$$

is a regular Lagrangian of order k on E, having $g_{ij} = \gamma_{ij}$ as the fundamental tensor field. Thus $L^{(k)n} = (M, L)$ with the Lagrangian (2.3) is a Lagrange space of order k.

Therefore we have:

THEOREM 2.1. If the base manifold M is paracompact, then there exist Lagrange spaces of order $k, L^{(k)n} = (M, L)$.

3. Variational problem for the Lagrangians of order k.

Let $L: E \to R$ be a differentiable Lagrangian of order k and $c: t \in [0,1] \to (x^i(t)) \in M$ a smooth parametrized curve, such that $Imc \subset U, U$ being the domain of a local chart of the differentiable manifold M.

The extension c^* to E of the curve c is given by the mapping:

$$c^* : t \in [0,1] \to (x^i(t), \frac{dx^i}{dt}(t), \cdots, \frac{1}{k!}\frac{d^k x^i}{dt^k}(t)) \in \pi^{-1}(U). \tag{3.1}$$

The integral of action of the Lagrangian L along the curve c is given by:

$$I(c) = \int_0^1 L(x, \frac{dx}{dt}, \cdots, \frac{1}{k!}\frac{d^k x}{dt^k})dt. \tag{3.2}$$

The variational problem regarding the integral of action $I(c)$ leads to the Euler-Lagrange equations $\overset{0}{E}_i(L) = 0$, [12, 13]:

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial L}{\partial y^{(1)i}}\right)+\cdots+(-1)^k\frac{d^k}{dt^k}\left(\frac{\partial L}{\partial y^{(k)i}}\right)=0,$$

(3.3)

$$y^{(1)i}=\frac{dx^i}{dt},\cdots,y^{(k)i}=\frac{1}{k!}\frac{d^kx^i}{dt^k}.$$

The curves c which verify the equations (3.3) are called *the extremal curves* of the integral action $I(c)$.

Now we remark that

$$\overset{k-1}{E}_i(L)=(-1)^{k-1}\frac{1}{(k-1)!}\left\{\frac{\partial L}{\partial y^{(k-1)i}}-\frac{d}{dt}\frac{\partial L}{\partial y^{(k)i}}\right\},\tag{3.4}$$

along the curve c^*, (3.1), is a d-covector field on E, which depends only on the Lagrangian L.

Then we can prove:

THEOREM 3.1 In the Lagrange space of order $k, L^{(k)n}=(M,L)$, the differential equations

$$g^{ij}\overset{k-1}{E}_j(L)=0\tag{3.5}$$

are of the form (1.9), where

$$(k+1)G^i=\frac{1}{2}g^{ij}\left\{\Gamma\left(\frac{\partial L}{\partial y^{(k)j}}\right)-\frac{\partial L}{\partial y^{(k-1)j}}\right\},\tag{3.6}$$

and

$$\Gamma=y^{(1)i}\frac{\partial}{\partial x^i}+2y^{(2)i}\frac{\partial}{\partial y^{(1)i}}+\cdots+ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}.\tag{3.7}$$

They determine k paths, which depend only on the fundamental function L.

Consequently, Theorem 3.1 shows us that there is a k-spray S with the coefficients G^i in (3.6) and (3.7). It will be called *canonical* for the space $L^{(k)n}$.

Now, applying Theorem 1.1, we get:

THEOREM 3.2. For every Lagrange space of order $k, L^{(k)n}=(M,L)$, there exist nonlinear connections determined only by the fundamental function L. One of them has the dual coefficients (1.10), where S is the canonical k-spray and G^i are its coefficients (3.6), (3.7).

The nonlinear connection N from the last theorem is called canonical for the space $L^{(k)n}$.

From now, we shall consider, for $L^{(k)n}$, only the canonical nonlinear

connection N.

4. The geometrical model $\mathbf{H^{(k+1)n} = (Osc^kM, G, F)}$.

For the study of the most important geometrical properties of the Lagrange space of order $k, L^{(k)n}$, we will introduce the so-called geometrical model. In this respect, let us consider the adapted basis (1.3) and its dual basis (1.5) constructed with the canonical nonlinear connection N.

Hence, the Sasaki N-lift, [12], of the fundamental tensor field g_{ij} of the space $L^{(k)n}$ is

$$G = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^{(1)j} \otimes \delta y^{(1)j} + \cdots + g_{ij}\delta y^{(k)i} \otimes \delta y^{(k)j}. \tag{4.1}$$

We can formulate:

THEOREM 4.1. The space $(\tilde{E}, G)$ is a pseudo-Riemannian one and it is determined only by the fundamental function L of the Lagrange space of order $k, L^{(k)n}$.

Now, let $\boldsymbol{F}$ be the $F(3,1)$ structure induced by the canonical nonlinear connection N:

$$\boldsymbol{F}(\frac{\delta}{\delta x^i}) = -\frac{\partial}{\partial y^{(k)i}}, \quad \boldsymbol{F}(\frac{\partial}{\partial y^{(k)i}}) = \frac{\delta}{\delta x^i}, \quad \boldsymbol{F}(\frac{\delta}{\delta y^{(\alpha)i}}) = 0, (\alpha = 1, \ldots, k-1). \tag{4.2}$$

We can prove, without difficulties:

THEOREM 4.2. The structure $\boldsymbol{F}$ has the properties:

1°. $\boldsymbol{F}$ is globally defined on $\tilde{E}$.

2°. Im $\boldsymbol{F} = N_o \oplus V_k$, $Ker\ \boldsymbol{F} = N_1 \oplus \cdots \oplus N_{k-1}$.

3°. rank $\boldsymbol{F}$ = 2n.

4°. $\boldsymbol{F}^3 + \boldsymbol{F} = 0$, (i.e. $\boldsymbol{F}$ is an $F(3,1)$-structure).

5°. $\boldsymbol{F}$ depends only on the Lagrangian L.

6°. The pair $(G, \boldsymbol{F})$ is a metrical $F(3,1)$ structure.

Consequently, the pair $(\mathrm{Osc}^k M, (G, \boldsymbol{F}))$ is a metrical $F(3,1)$ space determined only by the fundamental function L of $L^{(k)n}$ and is denoted by $H^{(k+1)n} = (\mathrm{Osc}^k M, G, \boldsymbol{F})$.. It will be called *the geometrical model* of the Lagrange space of order k. Therefore, the geometry of $L^{(k)n}$ is the geometry of the geometrical model $H^{(k+1)n}$. It can be studied by means of the methods used in the study of the total space of the k-osculator bundle, [11, 13].

For instance, the N-linear connection D on $\tilde{E}$ with the properties $D_X G = D_X \boldsymbol{F} = D_X J = 0, \forall X \in X(\tilde{E})$, is characterized by the conditions:

a) The coefficients of D are given by $D\Gamma(N)=(L^i_{jm},C^i_{(1)jm},\ldots,C^i_{(k)jm})$, only.

b) The h- and v_α-covariant derivatives of the metric tensor g_{ij} of the space $L^{(k)n}$, with respect to D, satisfy

$$g_{ij|m}=0, g_{ij}\overset{(\alpha)}{|}_m=0, \quad (\alpha=1,\ldots,k), \tag{4.3}$$

where

$$g_{ij|m}=\frac{\delta g_{ij}}{\delta x^m}-g_{sj}L^s_{im}-g_{is}L^s_{mj}, g_{ij}\ {}_m=\frac{\delta g_{ij}}{\delta y^{(\alpha)m}}-g_{sj}C^s_{(\alpha)im}-g_{is}C^s_{(\alpha)mj}.$$

We have

THEOREM 4.3. The following properties hold:

1°. If N is the canonical nonlinear connection of the Lagrange space of order k, $L^{(k)n}=(M, L)$, then there exist metrical N-linear connections D on Osc^kM which depend only on the fundamental function L.

2°. There exists only one metrical N-linear connection D in $L^{(k)n}$ whose h-torsion $T^m{}_{ij}$ and v_α-torsion $S^m_{(\alpha)ij}(\alpha=1,\cdots,k)$ vanish. The coefficients $C\Gamma(N)=(L^i_{jm},C^i_{(1)jm},\cdots,C^i_{(k)jm})$ of D are given by the generalized Christoffel symbols

$$\begin{aligned} L^m_{ij}&=\frac{1}{2}g^{ms}\left(\frac{\delta g_{is}}{\delta x^j}+\frac{\delta g_{sj}}{\delta x^i}-\frac{\delta g_{ij}}{\delta x^s}\right),\\ C^m_{(\alpha)ij}&=\frac{1}{2}g^{ms}\left(\frac{\delta g_{is}}{\delta y^{(\alpha)j}}+\frac{\delta g_{sj}}{\delta y^{(\alpha)i}}-\frac{\delta g_{ij}}{\delta y^{(\alpha)s}}\right),(\alpha=1,\ldots,k). \end{aligned} \tag{4.4}$$

3°. The N-linear connection with the coefficients (4.4) depends only on the fundamental function L of the space $L^{(k)n}$.

The proof of this important theorem was given in the paper [12]. $C\Gamma(N)$. is *the canonical metrical connection* of the considered Lagrange space.

Now, we remark that the whole geometrical theory of the Lagrange space of order $k, L^{(k)n}=(M,L)$, can be based on the canonical metrical connection $C\Gamma(N)$.

5. The gravitational field.

Let us consider the canonical metrical connection $C\Gamma(N)$, with the coefficients $(L^i_{jm}C^i_{(1)jm},\cdots,C^i_{(k)jm})$ given in the formula (4.4). The fundamental tensor field $g_{ij}(x,y^{(1)},\cdots,y^{(k)})$ of the Lagrange space of order $k, L^{(k)n}=(M,L)$, is compatible to $C\Gamma(N)$. The conditions of compatibility are in (4.3).

Let $H^{(k+1)n}=(E,G,F)$ be the geometrical model of the space $L^{(k)n}$.

Now, in a fixed local coordinates on E we consider every component of g_{ij} as a gravitational potential. Hence we can take the equations of the gravitational field as follows:

DEFINITION 5.1. The Einstein equations of the geometrical model $H^{(k+1)n}=(E,G,\boldsymbol{F})$, endowed with the canonical connection D, are the Einstein equations of the Lagrange space of order k, $L^{(k)n}$.

Let $\boldsymbol{R}$ be the curvature tensor of the canonical connection D, Ric $\boldsymbol{R}$ its Ricci tensor and R the scalar curvature of $\boldsymbol{R}$.Then the Einstein equation of the space $H^{(k+1)n}$ are expressed by

$$\text{(5.1)}\qquad Ric\boldsymbol{R}-\frac{1}{2}RG=\kappa\mathcal{T},$$

where κ is a constant and $\mathcal{T}$ is the energy-momentum tensor field.
With respect to the direct decomposition (1.2), determined by the canonical nonlinear connection N, the curvature tensor has the following essential components:

$$\begin{aligned}
&\boldsymbol{R}(X^H,Y^H)Z^H=[D_X^H,D_Y^H]Z^H-D^H_{[X^H,Y^H]}Z^H-\sum_{\varphi=1}^{k}D^{\upsilon_\varphi}_{[X^H,Y^H]}Z^H,\\
&\boldsymbol{R}(X^H,Y^{\upsilon_\alpha})Z^H=[D_X^H,D_Y^{\upsilon_\alpha}]Z^H-D^H_{[X^H,Y^{\upsilon_\alpha}]}Z^H-\sum_{\varphi=1}^{k}D^{\upsilon_\varphi}_{[X^H,Y^{\upsilon_\alpha}]}Z^H,\\
&\boldsymbol{R}(X^{\upsilon_\beta},Y^{\upsilon_\alpha})Z^H=[D_X^{\upsilon_\beta},D_Y^{\upsilon_\alpha}]Z^H-\sum_{\varphi=1}^{k}D^{\upsilon_\varphi}_{[X^{\upsilon_\beta},Y^{\upsilon_\alpha}]}Z^H,\alpha\le\beta(\alpha,\beta=1,\cdots,k).
\end{aligned}\qquad\text{(5.2)}$$

If we take $X^H=\dfrac{\delta}{\delta x^i}, X^{\upsilon_\beta}=\dfrac{\delta}{\delta y(\beta)i}$, $(\beta=1,\cdots,k)$ and we denote the components of Ric $\boldsymbol{R}$ in this adapted basis:

$$\begin{aligned}
&R_{ij}=R_i^{\ s}{}_{js}, P^1_{(\alpha)ij}=P_{(\alpha)i}^{\ \ s}{}_{js}, P_{(\alpha)ij}=P_{(\alpha)i}^{\ \ s}{}_{sj},\\
&P^1_{(\alpha\beta)ij}=P_{(\alpha\beta)i}^{\ \ s}{}_{js}, P^2_{(\alpha\beta)ij}=P_{(\alpha\beta)i}^{\ \ s}{}_{sj}, S_{(\alpha)ij}=S_{(\alpha)i}^{\ \ s}{}_{js}
\end{aligned}\qquad\text{(5.3)}$$

and the scalar curvature R:

$$\text{(5.3)}\qquad R=g^{ij}(R_{ij}+S_{(1)ij}+\cdots+S_{(k)ij}),$$

then we obtain from (5.1) in the adapted basis:

THEOREM 5.1. The Einstein equations of the Lagrange space of order k $L^{(k)n}$, corresponding to the canonical metrical connection $C\Gamma(N)$ are given by

$$
\begin{aligned}
&R_{ij} - \frac{1}{2} R g_{ij} = x T_{ij}, P^1_{(\alpha)ij} = x T^1_{(\alpha)ij}, P^2_{(\alpha)ij} = -x T^2_{(\alpha)ij}, \\
&S_{(\alpha)ij} - \frac{1}{2} R g_{ij} = x T_{(\alpha)ij}, P^1_{(\alpha\beta)ij} = x T^1_{(\alpha\beta)ij}, P^2_{(\alpha\beta)ij} = -x T^2_{(\alpha\beta)ij},
\end{aligned} \tag{5.4}
$$

where $T_{ij}, \cdots, T_{(\alpha,\beta)ij}$ are d-tensor fields. They are the components of the energy-momentum tensor field in the adapted basis.

Also, we can prove:

THEOREM 5.2. The law of conservation, with respect to $C\Gamma(N)$ in $L^{(k)n}$, is given by

$$
\begin{aligned}
&(R^i_j - \frac{1}{2} R \delta^i_j)_{|i} + \sum_{\varphi=1}^{k} P^i_{(\varphi)j} \,|\, {}^{(\varphi)}_{i} = 0, \\
&(S^i_{(\beta)j} - \frac{1}{2} R \delta^i_j)|^{(\beta)}_i - P^{2i}_{(\beta)j|i} - \sum_{\varphi=1}^{\beta-1} P^{2i}_{(\alpha\beta)j}|^{(\varphi)}_i + \sum_{\varphi=\beta+1}^{k} P^i_{(\beta\varphi)j}|^{\varphi}_i = 0,
\end{aligned} \tag{5.5}
$$

where $R^i_j = g^{is} R_{sj},$, etc.

6. Electromagnetic fields

Let $z^{(1)i}, \cdots, z^{(k)i}$ be the Liouville d-vector fields (1.7)' constructed by means of the canonical nonlinear connection N. Then the d-covector fields

$$
z^{(1)}{}_i = g_{ij} z^{(1)j}, \cdots, z^{(k)}{}_i = g_{ij} z^{(k)j}, \tag{6.1}
$$

depend only on the fundamental function L of the Lagrange space of order k.

Therefore, in the preferential local coordinates the covariant d-vector fields $z^{(1)i}, \cdots, z^{(k)i}$ will be called electromagnetic potentials. The d-tensors

$$
F^{(\alpha)}{}_{ji} = \frac{1}{2}\left(\frac{\delta z^{(\alpha)}{}_i}{\delta x^j} - \frac{\delta z^{(\alpha)}{}_j}{\delta x^i}\right), f^{(\alpha\beta)}{}_{ji} = \frac{1}{2}\left(\frac{\delta z^{(\alpha)}{}_i}{\delta y^{(\beta)j}} - \frac{\delta z^{(\alpha)}{}_j}{\delta y^{(\beta)i}}\right), \tag{6.2}
$$

will be called the electromagnetic tensor fields of the space $L^{(k)n}$.

Obviously, it is necessary to prove that $F^{(\alpha)}{}_{ji}$ and $f^{(\alpha\beta)}{}_{ji}$ are d-tensor fields on $\tilde{E}$. In this respect we shall consider the deflection tensors of the canonical metrical connection $C\Gamma(N)$. These are:

$$
D^{(\alpha)i}{}_j = z^{(\alpha)i}{}_{|j}, d^{(\alpha\beta)i}{}_j = z^{(\alpha)i}|^{(\beta)}_j, \tag{6.3}
$$

where „$|$" and „$|^{(\beta)}$" are, respectively, the h - and v_β – covariant derivatives with respect to $C\Gamma(N)$. The covariant deflections tensors are given by

$$
D^{(\alpha)}{}_{ij} = g_{is} D^{(\alpha)s}{}_j, d^{(\alpha\beta)}{}_{ij} = g_{is} d^{(\alpha\beta)s}{}_j. \tag{6.3}'
$$

Then, we have:

PROPOSITION 6.1. The electromagnetic tensor fields (6.2) have the following expressions:

$$F^{(\alpha)}{}_{ji} = \frac{1}{2}(D^{(\alpha)}{}_{ij} - D^{(\alpha)}{}_{ji}), f^{(\alpha\beta)}{}_{ji} = \frac{1}{2}(d^{(\alpha\beta)}{}_{ij} - d^{(\alpha\beta)}{}_{ji}). \tag{6.4}$$

We shall see that the electromagnetic tensor fields $F^{(\alpha)}{}_{ij}$ and $f^{(\alpha\beta)}{}_{ij}$ satisfy some laws of conservations – called the *generalized Maxwell equations*. Indeed, the covariant deflection tensors $D^{(\alpha)}{}_{ij}, d^{(\alpha\beta)}{}_{ij}$ satisfy the Ricci identities with respect to $C\Gamma(N)$. Using these identities it follows:

THEOREM 6.1. The electromagnetic tensor fields $F^{(\alpha)}{}_{ij}, f^{(\alpha\beta)}{}_{ij}$ of the space $L^{(k)n}$ satisfy the following generalized Maxwell equations

$$2(F^{(\alpha)}{}_{pi|q} + F^{(\alpha)}{}_{iq|p} + F^{(\alpha)}{}_{qp|i}) = \mathcal{O}_{(i,p,q)}\{z^{(\alpha)r}R_{ripq} - \sum_{\varphi=1}^{k} d^{(\alpha\varphi)}{}_{ir}R^{(\varphi)r}_{(o)pq}\},$$

$$2(F^{(\alpha)}{}_{pi}|^{(\beta)}_{q} + F^{(\alpha)}{}_{iq}|^{(\beta)}_{p} + F^{(\alpha)}{}_{qp}|^{(\beta)}_{i} + f^{(\alpha\beta)}{}_{pi|q} + f^{(\alpha\beta)}{}_{iq|p} + f^{(\alpha\beta)}{}_{qp|i}) =$$

$$= \mathcal{O}_{(i,p,q)}\{z^{(\alpha)r}(P_{(\beta)ripq} - P_{(\beta)riqp}) - \sum_{\varphi=1}^{k} d^{(\alpha\varphi)}{}_{ir}(P^{(\varphi)r}_{(\beta)\ pq} - P^{(\varphi)r}_{(\beta)\ qp})\},$$

$$2(f^{(\alpha\beta)}{}_{pi}|^{(\gamma)}_{q} + f^{(\alpha\beta)}{}_{iq}|^{(\gamma)}_{p} + f^{(\alpha\beta)}{}_{qp}|^{(\gamma)}_{i}) = \mathcal{O}_{(i,p,q)}\{z^{(\alpha)r}P_{(\beta\gamma)ripq} -$$

$$- (d^{(\alpha\beta)}{}_{ir}C^{r}_{(\gamma)pq} - d^{(\alpha\gamma)}{}_{ir}C^{r}_{(\beta)pq}) -$$

$$- \sum_{\varphi=1}^{k} d^{(\alpha\varphi)}{}_{ir}P^{(\varphi)r}_{(\beta\gamma)pq} \qquad (\gamma \neq \beta),$$

$$2(f^{(\alpha\beta)}{}_{pi}|^{(\beta)}_{q} + f^{(\alpha\beta)}{}_{iq}|^{(\beta)}_{p} + f^{(\alpha\beta)}{}_{qp}|^{(\beta)}_{i}) = \mathcal{O}_{(i,p,q)}\{z^{(\alpha)r}S_{(\alpha)ripq} -$$

$$- \sum_{\varphi=1}^{k} d^{(\alpha\beta)}{}_{ir}R^{(\varphi-\beta)r}_{(\beta)\ pq}\}.$$

Using the Bianchi identities of $C\Gamma(N)$ we can prove:

THEOREM 6.2. If the canonical metrical connection D is torsion less, then the electromagnetic tensors $F^{(\alpha)}{}_{ji}, f^{(\alpha\beta)}{}_{ji}$ verify the following generalized Maxwell equations:

$$F^{(\alpha)}{}_{ip}|_{q} + F^{(\alpha)}{}_{pq|i} + F^{(\alpha)}{}_{qi|p} = 0,$$

$$F^{(\alpha)}{}_{ip}|^{(\beta)}_{q} + F^{(\alpha)}{}_{pq}|^{(\beta)}_{i} + F^{(\alpha)}{}_{qi}|^{(\beta)}_{p} + f^{(\alpha\beta)}{}_{ip|q} + f^{(\alpha\beta)}{}_{pq|i} + f^{(\alpha\beta)}{}_{qi|p} = 0,$$

$$f^{(\alpha\beta)}{}_{ip}|^{(\gamma)}_{q} + f^{(\alpha\beta)}{}_{pq}|^{(\gamma)}_{i} + f^{(\alpha\beta)}{}_{qi}|^{(\gamma)}_{p} = 0, (\gamma \neq \beta),$$

$$f^{(\alpha\beta)}{}_{ip}|^{(\beta)}_{q} + f^{(\alpha\beta)}{}_{pq}|^{(\beta)}_{i} + f^{(\alpha\beta)}{}_{qi}|^{(\beta)}_{p} = 0.$$

A good example is given by the Lagrange space of order k, $L^{(k)n}$, with the fundamental function L from (2.3). Of course it has $\gamma_{ij}(x)$ as gravitational potentials. Its Einstein equations are those classical and the electromagnetic tensors vanish.

In the particular case k = 1 all the previous theory reduces to that given for the Lagrange space $L^n(M, L)$. It can be find in the book [15].

ACKNOWLEDGMENT. The first author would like to thank Prof. Dr. Takahashi T., the head of the Department of Mathematics of Tsukuba University, for his kind invitation to give a lecture, in Sept. 1993, on the problems contained in the present paper.

References

[1] G. S. Asanov, Finsler Geometry, Relativity and Gauge Theories, D. Reidel, Dordrecht, 1985.
[2] G. S. Asanov, Jet Extension of Finslerian Gauge Approach, Fortschr. Phys. **38** (1990) 8, 571 610.
[3] Gh. Atanasiu, Metrical Kawaguchi Connections on the Total Space of a vector bundle and some Special Subordinated Cases, Tensor, N.S. **46** (1987), 248–264.
[4] Gh. Atanasiu, Structures Finsler presque horsymplectiques, An. Stiint-Univ. "Al. I. Cuza" Iasi s I a Mat. **30** (1984) 4, 15–18.
[5] R. G. Beil, Finsler Gauge Transformations and General Relativity, Int. J. Theor. Phys. **31** (1992), 1025–1044.
[6] M. Crampin, Alternative Lagrangians in particle Dynamics, Diff. Geom. and its Appl. Proc. of the Conference, Aug. 24–30 (1986), Brno-Czechoslovakia.
[7] M. Crampin, W. Sarlet and F. Cantrijn, Higher order equations and higher order Lagrangian Mechanics, Math. Proc. Camb. Phil. Soc. **99** (1986), 565–587.
[8] M. de Léon and P. Rodrigues, Generalized classical mechanics and fields theory, North-Holland, 1985.
[9] P. Libermann and M. Ch. Marle, Symplectic Geometry and Analytical Mechanics, Reidel. Publ. Comp., 1987.
[10] R. Miron and Gh. Atanasiu, Differential Geometry of the k-osculator bundle, Rev. Roumaine Math. Pures Appl. **41**(1996) 3-4, 205-236.
[11] R. Miron and Gh. Atanasiu, Prolongations of the Riemannian, Finslerian and Lagrangian structures, Rev. Roumaine Math. Pures Appl. **41**(1996) 3-4, 237-249.
[12] R. Miron and Gh. Atanasiu, Higher order Lagrange spaces, Rev. Roumaine Math. Pures Appl. **41**(1996) 3-4, 251-262.
[13] R. Miron et Gh. Atanasiu, Compendium sur les espaces Lagrange d'ordre supérieur, Sem. Mecanica. Univ. Timisoara, Romania (1994), 32 pg.
[14] R. Miron and Gh. Atanasiu, Lagrange spaces of second order. Mathl. Comput. Modelling, Pergamon, Vol. 20, No. 4/5, (1994), 41–56.
[15] R. Miron and M. Anastasiei, The Geometry of Lagrange spaces. Theory and Applications, Kluwer Acad. Publ. FTPH **59** (1994).
[16] R. Miron, A Lagrangian theory of relativity I, II, An. Stiint. Univ. "Al. I. Cuza" Iasi s I a Mat. **32** (1986), 37–62 and **33** (1987), 137–149.

[17] R. Miron and M. Tatoiu – Radivoiovici, Extended Lagrangian Theory of electromagnetism, Rep. Math. Phys. **21** (1988), 193–229.

[18] R. Miron, Techiques of Finsler Geometry in the Theory of vector bundles, Acta Sci. Math. **49** (1985), 119–129.

[19] R. Miron, Vector bundles Finsler Geometry, Proc. of the Nat. Seminar on Finsler spaces, Brasov, (1982), 147–188.

[20] D. J. Saunders, The Geometry of Jet Bundles, Cambridge Univ. Press, 1989.

[21] K. Yano and S. Ishihara, Tangent and Cotangent Bundles. Differential Geometry, Dekker, New-York, 1973.

University "Al. I. Cuza" Iasi
Faculty of Mathematics
6600 Iasi, Romania

University "Transilvania" Brasov
Faculty of Sciences
2200 Brasov, Romania