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Sp(n)-EQUIVARIANT HARMONIC MAPS BETWEEN COMPLEX PROJECTIVE SPACES

Dedicated to Professor Hideki Ozeki on his sixtieth birthday

By

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Introduction

On existence of harmonic maps, Guest [2] constructed equivariant harmonic maps from a flag manifold to a complex Grassmannian manifold, and Ohnita [5] developed a method of studying equivariant maps from a compact homogeneous space to a complex projective space and investigated equivariant harmonic maps from a compact irreducible Hermitian symmetric space to a complex projective space, in detail. In particular, Ohnita classified equivariant harmonic maps relative to a unitary group between complex projective spaces.

In this paper, we study existence and harmonicity of $Sp(n)$-equivariant maps between complex projective spaces, by using the fact the symplectic group $Sp(n)$ acts a $(2n-1)$-dimensional complex projective space CP^{2n-1} transitively. In section 4 we determine all complex irreducible representations of $Sp(n)$, which define $Sp(n)$-equivariant maps from CP^{2n-1} to CP^{m} (Theorem 4.3), with the aid of the restriction rule of representations of $Sp(n)$, due to Koike and Terada [3, 4], Zhelobenko [6]. In section 5 we prove that the associated $Sp(n)$-equivariant maps are harmonic for any $Sp(n)$-invariant Riemannian metric on CP^{2n-1} (Theorem 5.2). In particular, we get $Sp(n)$-equivariant minimal immersions from CP^{2n-1} to CP^{m}, but not $SU(2n)$-equivariant.

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§1. Complex line bundles and harmonic maps into a complex projective space.

In this section, we recall known facts due to Ohnita [5]. Let CP^m be an m-dimensional complex projective space with the Fubini-Study metric. We denote by $\langle , \rangle$ the standard Hermitian inner product on C^{m+1}. Let $\pi : C^{m+1} \setminus \{0\} \to CP^m$ be the canonical projection. Then $C^{m+1} \setminus \{0\}$ is a principal bundle over CP^m with the structure group $C^* = C - \{0\}$. Let $E = (C^{m+1} \setminus \{0\}) \times_{C^*} C$ be the universal bundle over CP^m. The fiber E_x over each $x \in CP^m$ is the complex 1-dimensional subspace of C^{m+1} determined by x. Thus E is a holomorphic subbundle of the trivial bundle $\underline{C}^{m+1} = CP^m \times C^{m+1}$ over CP^m. Let $E^\perp$ be the subbundle of $\underline{C}^{m+1}$ whose fiber at x is the orthogonal complement of E_x in C^{m+1}. The bundles E, E^* and $E^\perp$ have natural Hermitian connected structures. We give $E^* \otimes E^\perp$ the tensor product Hermitian connected structure. Then there exists a natural bundle isomorphism $h : T^{(1,0)}CP^m \to E^* \otimes E^\perp$ preserving connections.

Let $M = G/K$ be an n-dimensional compact homogeneous space with a compact connected Lie group G and $\varphi : M \to CP^m$ a smooth map. Consider the exact sequence of pull-back vector bundles over M:

$$0 \to \varphi^{-1}(E^* \otimes E) \xrightarrow{i} \varphi^{-1}(E^* \otimes \underline{C}^{m+1}) \xrightarrow{j} \varphi^{-1}(E^* \otimes E^\perp) \to 0,$$

where i is the natural inclusion and j is given by the orthogonal projection along E. Pulling back $h : T^{(1,0)}CP^m \to E^* \otimes E^\perp$ by φ, we get a connection-preserving bundle isomorphism $h : \varphi^{-1}(T^{(1,0)}CP^m) \to \varphi^{-1}(E^* \otimes E^\perp)$.

Let (σ, C) be a complex 1-dimensional representation of the structure group K and $L = P \times_\sigma C$ a complex line bundle over M associated with a principal bundle (P, π, M, K). Then the vector space $C^\infty(L)$ of all smooth sections of L can be identified with the vector space $C^\infty(P, C)_K$ of all C-valued smooth functions $\tilde{f}$ on P satisfying the condition $\tilde{f}(uk) = \sigma(k)^{-1}\tilde{f}(u)$ for each $u \in P$ and $k \in K$, by the correspondence $C^\infty(L) \ni f \mapsto \tilde{f} \in C^\infty(P, C)_K, \tilde{f}(u) = u^{-1}(f(\pi(u)))$ for each $u \in P$.

We consider a system $\{\varphi_0, \ldots, \varphi_m\}$ in $C^\infty(L)$ with no common zeros. Let $\{\tilde{\varphi}_0, \ldots, \tilde{\varphi}_m\}$ be the corresponding system in $C^\infty(P, C)_K$. We define a smooth map $\tilde{\varphi} : P \to C^{m+1} \setminus \{0\}$ by $\tilde{\varphi} : \{\tilde{\varphi}_0, \ldots, \tilde{\varphi}_m\}$. Since $\tilde{\varphi}$ satisfies $\tilde{\varphi}(uk) = \sigma(k)^{-1}\tilde{\varphi}(u)$ for each $u \in P$ and $k \in K$, the map $\tilde{\varphi} : P \to C^{m+1} \setminus \{0\}$ becomes a bundle homomorphism from (P, π, M, K) to $(C^{m+1} \setminus \{0\}, \pi, CP^m, C^*)$ with the homomorphism $\sigma^{-1} : K \to C^*$ of the structure groups. Therefore $\tilde{\varphi}$ induces a smooth map $\varphi : M \to CP^m$ and the diagram below is commutative:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\varphi}} & C^{m+1}\setminus\{0\} \\ \pi\downarrow & & \downarrow\pi \\ M & \xrightarrow[\varphi]{} & CP^m. \end{array}$$

Let $H = E^*$ be the hyperplane bundle over CP^m. Conversely, every smooth map $\varphi : M \to CP^m$ is obtained in this manner by considering the pull-back complex line bundle $\varphi^{-1}H$ over M and a system of $m+1$ sections of $\varphi^{-1}H$ given by homogeneous coordinates on CP^m.

We denote by ∇^M the Riemannian connection of M and endow the principal bundle P with a connection Γ. Then in the associated line bundle L, the covariant differentiation ∇^L is induced by Γ. For $X \in C^\infty(TM^C)$, we denote by $X^* \in C^\infty(TP^C)$ the horizontal life of X to P with respect to Γ.

We denote by $\tau^{(1,0)} \in C^\infty(\varphi^{-1}T^{(1,0)}CP^m)$ the $(1,0)$-component of the tension field τ for the map φ. Then we have

$$\begin{aligned} h(\tau^{(1,0)})\tilde{\varphi} &= h(\sum_{i=1}^{n} (\nabla_{e_i}(d\varphi)^{(1,0)})(e_i))\tilde{\varphi} \\ &= j(\sum_{i=1}^{n} (e_i^* e_i^* \tilde{\varphi} - (\nabla^M_{e_i} e_i)^* \tilde{\varphi}) - 2\sum_{i=1}^{n} \frac{\langle d\tilde{\varphi}(e_i^*), \tilde{\varphi}\rangle}{\langle \tilde{\varphi}, \tilde{\varphi}\rangle} d\tilde{\varphi}(e_i^*)) \\ &= j(-(\Delta^L\varphi)^{\sim} - 2\sum_{i=1}^{n} \frac{\langle (\nabla^L_{e_i}\varphi)^{\sim}, \tilde{\varphi}\rangle}{\langle \tilde{\varphi}, \tilde{\varphi}\rangle} (\nabla^L_{e_i}\varphi)^{\sim}), \end{aligned}$$

where $\{e_i\}$ denotes a local orthonormal frame field on M and $\Delta^L = -\sum_{i=1}^{n} (\nabla^L_{e_i}\nabla^L_{e_i} - \nabla^L_{\nabla^M_{e_i} e_i})$.

PROPOSITION 1.1 (Ohnita [5]). *φ is a harmonic map if and only if the system* $\{\varphi_0, \ldots, \varphi_m\}$ satisfies

$$(\nabla^L_{\varphi})^{\sim} + 2\sum_{i=1}^{n} \frac{\langle (\nabla^L_{e_i}\varphi)^{\sim}, \tilde{\varphi}\rangle}{\langle \tilde{\varphi}, \tilde{\varphi}\rangle} (\nabla^L_{e_i}\varphi)^{\sim} = \mu\tilde{\varphi}$$

for some function μ on P.

§2. Construction and harmonicity of equivariant maps.

We are concerned with G-equivariant maps from an n-dimensional compact homogeneous manifold $M = G/K$ with a compact connected semisimple Lie group G to CP^m with the Fubini-Study metric.

Let $Aut(CP^m)$ be the group of all holomorphic isometries of CP^m. $Aut(CP^m)$ is identified with a projective unitary group $PU(m+1)$. A map $\varphi : M \to CP^m$ is called G-equivariant if there exists a Lie group homomorphism $\rho: G \to Aut(CP^m)$ satisfying $\rho(a)\circ\varphi = \varphi\circ\gamma_a$ for each $a \in G$, where γ_a denotes the natural action of G on M.

We endow M with a G-invariant metric g. Let (G, π, M, K) be the standard principal bundle on M and (σ, C) a complex 1-dimensional unitary representation of K. Then the associated complex line bundle $L = G\times_\sigma C$ becomes a G-homogeneous vector bundle with a Hermitian fiber metric $\langle , \rangle$.

Let V be a complex $(m+1)$-dimensional irreducible G-submodule of $C^\infty(L)$. Choose a unitary basis $\{\varphi_0, \ldots, \varphi_m\}$ of V with respect to the L^2-inner product. Let $\{\tilde{\varphi}_0, \ldots, \tilde{\varphi}_m\}$ be the corresponding system in $C^\infty(G, C)_K$. By using this system, we obtain maps $\tilde{\varphi}_V = \{\tilde{\varphi}_0, \ldots, \tilde{\varphi}_m\}: G \to C^{m+1}\setminus\{0\}$ and $\varphi_V = (\varphi_0, \ldots, \varphi_m): M \to CP^m$.

We define a unitary representation $\rho_V: G \to U(m+1)$ by $L_a(\tilde{\varphi}_0, \ldots, \tilde{\varphi}_m) = (\tilde{\varphi}_0, \ldots, \tilde{\varphi}_m)\rho_V(a)$ for $a \in G$, where L_a is the left action of G on $C^\infty(G, C)_K$. Then the map φ_V is G-equivariant with respect to ρ_V. Hence we have

$$\tilde{\varphi}_V(a) = (\rho_V(a))v_0\ ,\ \varphi_V(a\cdot o) = \pi((\rho_V(a))v_0)\ \text{ for each } a \in G,$$

where $o = eK \in M$ and $v_0 = \tilde{\varphi}_V(e) \in C^{m+1}\setminus\{0\}$.

On the other hand, let $\varphi: M \to CP^m$ be a G-equivariant map relative to a Lie group homomorphism $\rho: G \to Aut(CP^m)$. There exists a unitary representation $\tilde{\rho}: \tilde{G} \to SU(m+1)$ of the finite covering group $\tilde{G}$ of G such that the diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{\rho}} & SU(m+1) \\ \pi\downarrow & & \downarrow \\ G & \xrightarrow{\rho} & PU(m+1) \end{array}$$

is commutative. Take $v_0 \in S^{2m+1}$ with $\varphi(o) = Cv_0$. Then we have $\varphi(a\cdot o) = \rho(a)\varphi(o) = \rho(a)\pi(v_0) = \pi(\tilde{\rho}(\tilde{a})v_0)$ for each $\tilde{a} \in \tilde{G}$ with $\pi(\tilde{a}) = a \in G$. In particular, we have $\tilde{\rho}(\tilde{K})Cv_0 \subset Cv_0$. Hence there is a real-valued linear form λ_0 on $\mathfrak{k}$ such that $\tilde{\rho}(X)v_0 = \sqrt{-1}\lambda_0(X)\,v_0$ for each $X \in \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K. Put $W = Cv_0$. Then W is a complex 1-dimensional $\tilde{K}$-submodule of C^{m+1}. Consider the associated homogeneous line bundle $L = \tilde{G}\times_{\sigma^*} W^*$ over $M = \tilde{G}/\tilde{K}$, where (σ^*, W^*) is the dual $\tilde{K}$-module of W. We define a map $\tilde{\varphi} = (\tilde{\varphi}_0, \ldots, \tilde{\varphi}_m):\ \tilde{G} \to (W^*)^{m+1} \approx C^{m+1}$ by $(\tilde{\varphi}_i(a))(w) = \langle\,\tilde{\rho}(a)\,w, \varepsilon_i\rangle\ (i = 0, \ldots, m)$ for each $a \in \tilde{G}$ and $w \in W$, where $\{\varepsilon_0, \ldots, \varepsilon_m\}$ denotes the standard basis of C^{m+1}. Each $\tilde{\varphi}_i$ satisfies $\tilde{\varphi}_i(ak) = \sigma^*(k)^{-1}\tilde{\varphi}_i(a)$ for each $a \in \tilde{G}$ and $k \in \tilde{K}$, therefore we have that $\tilde{\varphi}_i \in C^\infty(\tilde{G}, W^*)_{\tilde{K}}$. Let $\{\varphi_0, \ldots, \varphi_m\}$ be the corresponding system of $\{\tilde{\varphi}_0, \ldots, \tilde{\varphi}_m\}$ on $C^\infty(L)$ and V the $\tilde{G}$-submodule of $C^\infty(L)$ spanned by $\varphi_0, \ldots, \varphi_m$. If $\tilde{\rho}$

is irreducible, then V is an irreducible $\tilde{G}$-module and φ is equivalent to $\varphi_V = (\varphi_0, \ldots, \varphi_m)$.

Now we recall the following.

PROPOSITION 2.1 (Ohnita [5]). *Suppose that a homogeneous space $M = G/K$ with a G-invariant metric g satisfies the condition $[\mathfrak{k}, \mathfrak{m}] = \mathfrak{m}$. Then a G-equivariant map $\varphi : M \to CP^m$ is a harmonic map if and only if $(\sum_{i=1}^n \tilde{\rho}(X_i)^2)v_0 \in \boldsymbol{R}v_0$, where $\{X_1, \ldots, X_n\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to g.*

PROPOSITION 2.2 (Ohnita [5]). *Suppose that $M = G/K$ with the G-invariant Riemannian metric g_G induced by an Ad(G)-invariant inner product of $\mathfrak{g}$ satisfies the condition $[\mathfrak{k}, \mathfrak{m}] = \mathfrak{m}$. Then a G-equivariant map $\varphi = \varphi_V : M \to CP^m$ is a harmonic map.*

§3. Representations of symplectic group.

We consider the case $G = Sp(n)$ $(n \geq 2)$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}^C$ and $\mathfrak{t}^C$ the complexification of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. $\mathfrak{t}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$. Let (,) be an Ad(G)-invariant inner product on $\mathfrak{g}$ defined by -1 times the Killing form of $\mathfrak{g}$. Let $\Sigma(\subset \mathfrak{t})$ be the root system of $\mathfrak{g}^C$ relative to $\mathfrak{t}$. We have a root space decomposition of $\mathfrak{g}^C$:

$$\mathfrak{g}^C = \mathfrak{t}^C + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^C ; (adH)X = \sqrt{-1}(\alpha, H)X \text{ for } H \in \mathfrak{t}\}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a fundamental root system of Σ. Choose a lexicographic order $>$ on Σ such that the set of simple roots with respect to $>$ coincides with Π. Note that the Dynkin diagram corresponding to $\mathfrak{g}^C$ is given by the following:

$$\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} - \cdots - \overset{\alpha_{n-1}}{\circ} \Leftarrow \overset{\alpha_n}{\circ}.$$

Put $\Sigma^+ = \{\alpha \in \Sigma ; \alpha > 0\}$. Let $\{\Lambda_i\}$ be the fundamental weights of $(\mathfrak{g}^C, \mathfrak{t}^C)$ corresponding to Π:

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Λ_i is given by

$$\Lambda_i = \alpha_1 + \alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n).$$

We put $\Pi_0=\{\alpha_2,\ldots,\alpha_n\}$ and $\Sigma_0=\Sigma\cap\{\Pi_0\}_Z$, where $\{\Pi_0\}_Z$ denotes the subgroup of $\mathfrak{t}$ generated by Π_0 over $\boldsymbol{Z}$.

We note that $G=Sp(n)$ acts $\boldsymbol{C}P^{2n-1}$ transitively. The isotropy subgroup K of G at $[1,0,\ldots,0]\in \boldsymbol{C}P^{2n-1}$ is given by

$$K=\left\{\begin{pmatrix} e^{i\theta} & 0\cdots0 & 0 & 0\cdots0 \\ 0 & A & 0 & -\overline{B} \\ \vdots & 0\cdots0 & e^{-i\theta} & 0\cdots0 \\ 0 & B & 0 & \overline{A}\end{pmatrix}\in M_{2n}(\boldsymbol{C})\,;\ \begin{matrix}\theta\in\boldsymbol{R}, \\ \begin{pmatrix} A & -\overline{B} \\ B & \overline{A}\end{pmatrix}\in SU(2n-2)\end{matrix}\right\}.$$

Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to (,). Then the complexifications $\mathfrak{k}^C$ and $\mathfrak{m}^C$ of $\mathfrak{t}$ and $\mathfrak{m}$ are given by

$$\mathfrak{k}^C=\mathfrak{t}^C+\sum_{\alpha\in\Sigma_0}\mathfrak{g}_\alpha,\quad \mathfrak{m}^C=\sum_{\alpha\in\Sigma-\Sigma_0}\mathfrak{g}_\alpha,$$

respectively. Set $\Sigma_{\mathfrak{m}}^+=\Sigma^+-\Sigma_0$ and $\Sigma_{\mathfrak{m}}^-=-\Sigma_{\mathfrak{m}}^+$. We define subspaces $\mathfrak{m}^\pm$ of $\mathfrak{g}^C$ by

$$\mathfrak{m}^\pm=\sum_{\alpha\in\Sigma_{\mathfrak{m}}^\pm}\mathfrak{g}_\alpha.$$

We choose $E_\alpha\in\mathfrak{g}_\alpha$ for $\alpha\in\Sigma$ with the following properties and fix them once and for all:

$$[E_\alpha,E_{-\alpha}]=\sqrt{-1}\alpha,\ (E_\alpha,E_{-\alpha})=1,\ \overline{E}_\alpha=E_{-\alpha}\ \text{ for }\alpha\in\Sigma,$$

where we denote by $X\mapsto\overline{X}$ the complex conjugation of $\mathfrak{g}^C$ with respect to the real form $\mathfrak{g}$. We see that $[\mathfrak{k},\mathfrak{m}]=\mathfrak{m}$. Put $Z_{\mathfrak{k}}=\{k\Lambda_1\,;k\in\boldsymbol{Z}\}$. For $k\Lambda_1\in Z_{\mathfrak{k}}$, we can define a complex 1-dimensional unitary representation $\sigma_{k\Lambda_1}$ of K by $\sigma_{k\Lambda_1}(a)=\exp(\sqrt{-1}(k\Lambda_1,X))$ for each $a\in K$, where $a=\exp X$ and $X\in\mathfrak{k}$. Using this representation $(\sigma_{k\Lambda_1},\boldsymbol{C})$ of K, we construct a homogeneous complex line $[E_\alpha,E_{-\alpha}]=\sqrt{-1}\alpha,\ (E_\alpha,E_{-\alpha})=1,\ \overline{E}_\alpha=E_{-\alpha}$ for $\alpha\in\Sigma$, bundle $L_k=Sp(n)\times_{\sigma_{k\Lambda_1}}\boldsymbol{C}$ over $\boldsymbol{C}P^{2n-1}=Sp(n)/K$. Conversely, for each homogeneous complex line bundle L over $\boldsymbol{C}P^{2n-1}=Sp(n)/K$, there exists an element $k\Lambda_1\in Z_{\mathfrak{k}}$ such that $L=L_k$.

LEMMA 3.1. *Let $\rho:Sp(n)\to GL(V)$ be a complex irreducible representation of $Sp(n)$ with $\xi\in\mathfrak{t}$ as its highest weight and $\langle,\rangle$ an $Sp(n)$-invariant Hermitian inner product of V. Choose a nonzero weight vector $v_\xi\in V$ for the highest weight ξ. Suppose that there exists a nonzero vector $w\in V$ and an element $\lambda\in\mathfrak{k}$ such that $\rho(X)w=\sqrt{-1}(\lambda,X)w$ for each $X\in\mathfrak{t}$. Then we have $\langle w,v_\xi\rangle\neq0$.*

PROOF. We define a complex valued linear function F by $F(X)=\langle\rho(X)v_\xi,w\rangle$

for $X \in \mathfrak{g}^C$. For each $X \in \mathfrak{k}^C$, we have

$$F(X) = \langle \rho(X) v_\xi, w \rangle = -\langle v_\xi, \rho(X) w \rangle = \sqrt{-1}(\lambda, X)\langle w, v_\xi \rangle.$$

For each $Y \in \mathfrak{m}^+$, we have $F(Y) = 0$ because $\rho(Y) v_\xi = 0$.

For each $Z \in \mathfrak{m}^-$, we have

$$F(Z) = \langle \rho(Z) v_\xi, w \rangle = -\langle v_\xi, \rho(Z) w \rangle = 0$$

because $\rho(Z) w$ is a linear combination of non-highest weight vectors. Thus we have $F(\mathfrak{g}^C) \subset \boldsymbol{C}\langle v_\xi, w \rangle$. If $\langle v_\xi, w \rangle = 0$, then we get $F \equiv 0$. But we have $V = \sum_{j=0}^{N} \rho(\mathfrak{g}^C)^j v_\xi$ for a sufficiently large integer N by the irreducibility of ρ, thus we obtain $w = 0$. Hence $\langle v_\xi, w \rangle \neq 0$. q.e.d.

LEMMA 3.2. *Let* $\rho : Sp(n) \to GL(V)$ *be a complex irreducible representation of* $Sp(n)$. *For every* $\lambda \in \mathfrak{k}$, *put*

$$W_\lambda = \{w \in V; \rho(X)w = \sqrt{-1}(\lambda, X)w \text{ for each } X \in \mathfrak{k}\}$$

Then we have $\dim_{\mathbb{C}} W_\lambda = 0$ *or* 1.

PROOF. As in Lemma 3.1, we denote by v_ξ a highest weight vector of ρ and by $\langle , \rangle$ an $Sp(n)$-invariant inner product of V. We define a linear map $f : W_\lambda \to \boldsymbol{C}$ by $f(w) = \langle w, v_\xi \rangle$ for $w \in W_\lambda$. By Lemma 3.1, f is injective. Hence we have $\dim_{\mathbb{C}} W_\lambda = 0$ or 1. q.e.d.

For $k \in \boldsymbol{Z}$, we set $W_k = (\sigma_{k\Lambda_1}, \boldsymbol{C})$. let $D(Sp(n))$ be the set of all dominant integral forms of $\mathfrak{t}$. By Lemma 3.2, we obtain $\dim Hom_k(V_\Lambda, W_k) = 0$ or 1 for each $\Lambda \in D(Sp(n))$, where V_Λ is a representation space of an irreducible representation of $Sp(n)$ with highest weight Λ. We put

$$D(Sp(n), K; k) = \{\Lambda \in D(Sp(n)); \dim Hom_K(V_\Lambda, W_k) = 1\}.$$

For each $\Lambda \in D(Sp(n), K; k)$, we obtain the $Sp(n)$-equivariant map corresponding to Λ. We shall determine the elements of $D(Sp(n), K; k)$ for $k \in \boldsymbol{Z}$.

As is well-known, there is a bijective correspondence between the sets of equivalence classes of irreducible representations of a complex semisimple Lie group and its compact real form by using the unitarian trick of Weyl. So we identify the representations of $Sp(n, \boldsymbol{C})$ and $Sp(n)$.

§4. Construction of Sp(n)-equivariant maps.

We take a Cartan subalgebra $\mathfrak{t}^C$ of $\mathfrak{g}^C = \mathfrak{sp}(n, \boldsymbol{C})(n \geq 2)$ as follows:

$$\mathfrak{t}^C = \left\{ \begin{pmatrix} \varepsilon_1 & & & & & \\ & \ddots & & & & \\ & & \varepsilon_n & & & \\ & & & -\varepsilon_1 & & \\ & & & & \ddots & \\ & & & & & -\varepsilon_n \end{pmatrix} ; \varepsilon_i \in \boldsymbol{C} \right\}.$$

Then the root system Σ of $\mathfrak{g}^C$ is given by

$$\Sigma = \{\pm(\varepsilon_i \pm \varepsilon_j)\}_{1 \leq i < j \leq n} \cup \{\pm 2\varepsilon_i\}_{1 \leq i \leq n}.$$

We take a simple root system Π of Σ as follows:

$$\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = 2\varepsilon_n\}.$$

Then the weight lattice P and the set of dominant integral weights P_+ are given by

$$P = \boldsymbol{Z}\varepsilon_1 + \boldsymbol{Z}\varepsilon_2 + \cdots + \boldsymbol{Z}\varepsilon_n,$$

$$P_+ = \{f_1\varepsilon_1 + f_2\varepsilon_2 + \cdots + f_n\varepsilon_n \in P; f_1 \geq f_2 \geq \cdots \geq f_n \geq 0\}.$$

There is a one-to-one correspondence between the equivalence classes of the irreducible representation of a connected complex semisimple Lie group G and the elements of P_+. We identify each element of P_+ with the irreducible representation corresponding to it.

In general any sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots)$ of nonnegative integers and containing only finitely many nonzero terms is called a *partition*. We consider each element of P_+ as a partition and identify each partition with the Young diagram corresponding to it. For a partition λ, the length of λ is defined to be the number of nonzero terms in λ and is denoted by $\ell(\lambda)$, the size of λ is defined to be the sum of all terms in λ and is denoted by $|\lambda|$, i.e., $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n + \cdots$. If partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n, \ldots)$ satisfy the condition $\lambda_i \geq \mu_i$ for all $i \geq 1$, we say that the Young diagram λ contains the Young diagram μ and denote it by $\lambda \supset \mu$. If $\lambda \supset \mu$, put μ on λ with the same top-left corner and remove μ out of λ. Then the resulting diagram is called a *skew diagram* and is denoted by $\lambda - \mu$. A skew diagram each column of which consists of either zero or one square is called a *horizontal strip*.

We recall the following.

THEOREM 4.1 (Koike and Terada [4], Zhelobenko [6]). *Let* λ *be a partition of length at most* n *and* $\lambda_{Sp(n,C)}$ *the irreducible character of* $Sp(n,\boldsymbol{C})$ *corresponding to* λ. *Then we have*

$$\lambda_{Sp(n,C)} \downarrow^{Sp(n,C)}_{GL(1,C)\times Sp(n-1,C)} = \sum_{(\mu,\nu)} t_n^{-|\lambda-\mu|+|\mu-\nu|} \times \nu_{Sp(n-1,C)},$$

where $\downarrow^{Sp(n,C)}_{GL(1,C)\times Sp(n-1,C)}$ *denotes the restriction of the representation of* $Sp(n,\boldsymbol{C})$ *to* $GL(1,\boldsymbol{C})\times Sp(n-1,\boldsymbol{C})$ *and the summation is taken over all pairs of partitions* (μ,ν) *satisfying the following conditions:*

(1) $\lambda \supset \mu$ *and* $\lambda-\mu$ *is a horizontal strip,*

(2) $\mu \supset \nu$ *and* $\mu-\nu$ *is a horizontal strip,*

(3) $\ell(\nu) \le n-1$.

$GL(1,\boldsymbol{C})\times Sp(n-1,\boldsymbol{C})$ *is the Levi part of*

$$\left\{ \begin{pmatrix} t_n & & * \\ & Y & \\ 0 & & t_n^{-1} \end{pmatrix} ; Y \in Sp(n-1,\boldsymbol{C}), t_n \in \boldsymbol{C}^* \right\}.$$

THEOREM 4.2.

$$D(Sp(n),K;k) = \{m_1\Lambda_1 + m_2\Lambda_2 ; m_i \in \boldsymbol{Z}, m_1-|k| \ge 0 \text{ is even}, m_2 \ge 0\}.$$

PROOF. Assume that $\Lambda=(m_1,\dots,m_n)\in D(Sp(n),K;k)$. Let λ be the partition corresponding to Λ, i.e., $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)=(m_1+\dots+m_n, m_2+\dots+m_n,\dots,m_n)$. We may identify λ with Λ. By virtue of Theorem 4.1, there exists a pair of partitions (μ,ν) such that $(a)\,\nu=(0,\dots,0)$, $(b)\,(\mu,\nu)$ satisfies the conditions (1), (2), and (3) in Theorem 4.1, and $(c)\,k=-|\lambda-\mu|+|\mu-\nu|$. From (a) and (b), μ and $\lambda-\mu$ are horizontal strips, i.e., $\mu=(\mu_1,0,\dots,0)(\lambda_1\ge\mu_1\ge\lambda_2)$, and $\lambda_i=0$ for all $i\ge 3$. Moreover, from (c), we have $k=-\lambda_1-\lambda_2+2\mu_1$. Thus we see the following:

$$m_1-|k| = \begin{cases} 2(\lambda_1-\mu_1) & (k\ge 0) \\ 2(\mu_1-\lambda_2) & (k<0), \end{cases}$$

i.e., $m_1 - |k| \geq 0$ is even.

Conversely, consider an irreducible representation of $Sp(n)$ with highest weight $\Lambda = m_1\Lambda_1 + m_2\Lambda_2 (m_1 - |k| \geq 0$ is even, $m_2 \geq 0)$. Put

$$m_1 - |k| = \lambda_1 - \lambda_2 - |k| = 2m \quad (m \geq 0),$$

and

$$\mu_1 = \begin{cases} \lambda_1 - m & (k \geq 0) \\ \lambda_2 + m & (k < 0). \end{cases}$$

We take partitions $\mu = (\mu_1, 0, \ldots, 0)$ and $\nu = (0, \ldots, 0)$. Then we see the pair (μ, ν) satisfies the conditions (1), (2), and (3) in Theorem 4.1 and $-|\lambda - \mu| + |\mu - \nu| = k$. Hence we conclude that $\Lambda \in D(Sp(n), K; k)$. q.e.d.

§5. Harmonicity and isometricity of Sp(n)-equivariant maps.

Let (,) be an $\mathrm{Ad}(Sp(n))$-invariant inner product on $\mathfrak{sp}(n)$ defined by -1 times the Killing form of $\mathfrak{sp}(n)$. If we endow CP^{2n-1} with an $Sp(n)$-invariant Riemannian metric g_1 induced by (,), then an $Sp(n)$-equivariant map corresponding to an element of $D(Sp(n), K; k)$ is a harmonic map because of Proposition 2.2. However, CP^{2n-1} admits other $Sp(n)$-invariant Riemannian metrics.

We put

$$X_\alpha = \frac{E_\alpha + E_{-\alpha}}{\sqrt{2}}, \quad X_{-\alpha} = \frac{E_\alpha - E_{-\alpha}}{\sqrt{2}i}, \quad \text{for each } \alpha \in \Sigma_{\mathfrak{m}}^+ .$$

Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be subspaces of $\mathfrak{m}$ spanned by $\{X_\alpha; \alpha \in \Sigma_{\mathfrak{m}}, \alpha \neq \pm(2\sum_{1\leq k<n} \alpha_k + \alpha_n)\}$ and $\{X_\alpha; \alpha = \pm(2\sum_{1\leq k<n} \alpha_k + \alpha_n)\}$, respectively. Then the subspaces $\mathfrak{m}_1$, $\mathfrak{m}_2$ are irreducible K-submodules and not equivalent each other. Thus every $Sp(n)$-invariant Riemannian metric on CP^{2n-1} can be described as $g_x = g_1|_{\mathfrak{m}_1} + x g_1|_{\mathfrak{m}_2}$ $(x > 0)$, up to a positive constant factor.

LEMMA 5.1. *Consider an irreducible representation of Sp(n). Let* v *be a nonzero weight vector of a weight* $k\Lambda_1$ *such that it gives an Sp(n)-equivariant map. Then the vector* $E_{\pm(2\sum_{1\leq k<n} \alpha_k + \alpha_n)} v$ *is a weight vector of the weight* $(k \pm 2)\Lambda_1$ *such that it gives an Sp(n)-equivariant map or zerovector.*

PROOF. We put $\alpha_0 = 2\sum_{1\leq k<n} \alpha_k + \alpha_n$ for convenience. From the condition, we have $E_{\pm\alpha} v = 0$ for each $\alpha \in \Sigma_0$. We assume that $E_{\pm\alpha_0} v \neq 0$. Since $[E_{\pm\alpha}, E_{\pm\alpha_0}] = 0 (\alpha \in \Sigma_0)$, we have $E_{\pm\alpha} E_{\pm\alpha_0} v = E_{\pm\alpha_0} E_{\pm\alpha} v = 0$ for each $\alpha \in \Sigma_0$. Hence we

observe that $E_{\pm\alpha_0}v$ is a weight vector such that it gives an Sp(n)-equivariant map. q.e.d.

THEOREM 5.2. *For any Sp(n)-invariant Riemannian metric on* $\boldsymbol{C}P^{2n-1}$, *the Sp(n)-equivariant map corresponding to an element of* $D(Sp(n),K;k)$ *is a harmonic map.*

PROOF. In case we endow $\boldsymbol{C}P^{2n-1}$ with a metric g_1, from Proposition 2.1 and 2.2, we obtain

$$(*)\qquad \left(\sum_{\alpha\in\Sigma_{\mathfrak{m}}}\rho(X_\alpha)^2\right)v_0 = c_1 v_0 \quad \text{for some } c_1\in\boldsymbol{R}.$$

While we give $\boldsymbol{C}P^{2n-1}$ a metric g_x, a necessary and sufficient condition for a map to be a harmonic map is

$$(\sum_{\substack{\alpha\in\Sigma_{\mathfrak{m}}\\ \alpha\neq\pm\alpha_0}}\rho(X_\alpha)^2 + \sum_{\alpha=\pm\alpha_0}\tfrac{1}{x}\rho(X_\alpha)^2)v_0 = c_2 v_0 \quad \text{for some } c_2\in\boldsymbol{R},$$

where $\alpha_0 = 2\sum_{1\le k<n}\alpha_k + \alpha_n$. From (*), we claim that the condition above is equivalent to

$$\left(\sum_{\alpha=\pm\alpha_0}\rho(X_\alpha)^2\right)v_0 = \left(\sum_{\alpha=\pm\alpha_0}\rho(E_\alpha E_{-\alpha})\right)v_0 = c_3 v_0 \quad \text{for some } c_3\in\boldsymbol{R}.$$

But this holds by Lemma 5.1. q.e.d.

We shall study the isometricity of harmonic maps constructed in Theorem 5.2.

Lemma 5.3. *Consider that an irreducible representation of Sp(n) with highest weight* $m_1\Lambda_1 + m_2\Lambda_2$. *Let* w *be a weight vector of a weight* $m_1\Lambda_1$ *such that it determines an Sp(n)-equivariant map. Then we have*

(a) $E_{-\alpha_0}E_{\alpha_0}(E^j_{-\alpha_0}w) = -(m_1 - j + 1)jE^j_{-\alpha_0}w$ *for* $j = 0,\ldots,m_1,= 0$

(b) $E_{\alpha_0}E_{-\alpha_0}(E^j_{-\alpha_0}w) = -(m_1 - j)(j+1)E^j_{-\alpha_0}w$ *for* $j = 0,\ldots,m_1$,

where $\alpha_0 = 2\sum_{1\le k<n}\alpha_k + \alpha_n$.

PROOF. (a) We shall use induction on j. For $j = 0$, the claim holds because of $E_{\alpha_0}w = 0$. Assume it is true for $j-1$. For j,

$$\begin{aligned}E_{-\alpha_0}E_{\alpha_0}(E_{-\alpha_0}^{j}w) &= E_{-\alpha_0}(E_{-\alpha_0}E_{\alpha_0}+\sqrt{-1}\alpha_0)(E_{-\alpha_0}^{j-1}w)\\ &= E_{-\alpha_0}\{-(m_1-j+2)(j-1)-(m_1-2j+2)\}(E_{-\alpha_0}^{j-1}w)\\ &= -(m_1-j+1)j(E_{-\alpha_0}^{j-1}w).\end{aligned}$$

(b) From (a), we have

$$\begin{aligned}E_{\alpha_0}E_{-\alpha_0}(E_{-\alpha_0}^{j}w) &= (E_{-\alpha_0}E_{\alpha_0}+\sqrt{-1}\alpha_0)(E_{-\alpha_0}^{j}w)\\ &= \{-(m_1-j+1)j-(m_1-2j)\}(E_{-\alpha_0}^{j}w)\\ &= -(m_1-j)(j+1)(E_{-\alpha_0}^{j}w). \quad \text{q.e.d.}\end{aligned}$$

Using this lemma, we obtain the following.

PROPOSITION 5.4. *Consider an irreducible representation ρ of $Sp(n)$ with highest weight $\Lambda = m_1\Lambda_1 + m_2\Lambda_2$. Then the energy density $e(\varphi)$ of the $Sp(n)$-equivariant map $\varphi : (CP^{2n-1}, g_x) \to (CP^m, h)$ corresponding to a weight $(m_1-2j)\Lambda_1$ of ρ is given by*

$$e(\varphi)=\frac{1}{2}[m_2{}^2+(m_1+2n-1)m_2+(n-1)m_1+\frac{1}{x}\{(2j+1)m_1-2j^2\}]\,|v_0|^2,$$

where h is the Fubini-Study metric of CP^m. If $\varphi^*h = rg_x$ for some constant $r>0$, then r is given by

$$r=\frac{e(\varphi)}{2n-1}.$$

PROOF. We have

$$\begin{aligned}2e(\varphi) &= \sum_{\substack{\alpha\in\Sigma_{\text{III}}\\ \alpha\neq\pm\alpha_0}}\varphi^*h(X_\alpha,X_\alpha)+\sum_{\alpha=\pm\alpha_0}\varphi^*h\left(\frac{X_\alpha}{\sqrt{x}},\frac{X_\alpha}{\sqrt{x}}\right)\\ &= \sum_{\substack{\alpha\in\Sigma_{\text{III}}\\ \alpha\neq\pm\alpha_0}}\langle\rho(X_\alpha)v_0,\rho(X_\alpha)v_0\rangle+\frac{1}{x}\sum_{\alpha=\pm\alpha_0}\langle\rho(X_\alpha)v_0,\rho(X_\alpha)v_0\rangle\\ &= -\sum_{\substack{\alpha\in\Sigma_{\text{III}}\\ \alpha\neq\pm\alpha_0}}\langle\rho(X_\alpha)^2v_0,v_0\rangle+\left(1-\frac{1}{x}\right)\sum_{\alpha=\pm\alpha_0}\langle\rho(X_\alpha)^2v_0,v_0\rangle\\ &= -\langle\rho(\mathscr{C})v_0,v_0\rangle-(m_1-2j)^2(\Lambda_1,\Lambda_1)|v_0|^2+\left(1-\frac{1}{x}\right)\sum_{\alpha=\pm\alpha_0}\langle\rho(X_\alpha)^2v_0,v_0\rangle\\ &= (\Lambda,\Lambda+2\delta)\,|v_0|^2-\frac{(m_1-2j)^2}{2}|v_0|^2-\left(1-\frac{1}{x}\right)((2j+1)m_1-2j^2)\,|v_0|^2\end{aligned}$$

$$=\left\{m_2{}^2+(m_1+2n-1)m_2+(n-1)m_1+\frac{1}{x}((2j+1)m_1-2j^2)\right\}|v_0|^2,$$

where $\alpha_0=2\sum_{1\le k<n}\alpha_k+\alpha_n$, $\mathscr{C}$ is the Casimir operator of $Sp(n)$ with respect to an Ad($Sp(n)$)-invariant inner product (,) of $\mathfrak{g}$, and $\delta=\Lambda_1+\cdots+\Lambda_n$. We note that the eigenvalue of the Casimir operator $\rho(\mathscr{C})$ is $-(\Lambda,\Lambda+2\delta)$, by Freudenthal's formula. If $\varphi^*h=rg_x$, then we have

$$2e(\varphi)=\sum_{\substack{\alpha\in\Sigma_{\mathfrak{m}}\\ \alpha\neq\pm\alpha_0}}\varphi^*h(X_\alpha,X_\alpha)+\sum_{\alpha=\pm\alpha_0}\varphi^*h\left(\frac{X_\alpha}{\sqrt{x}},\frac{X_\alpha}{\sqrt{x}}\right)$$
$$=2(2n-1)r. \qquad \text{q.e.d.}$$

THEOREM 5.5. *Consider an irreducible representation of Sp(n) with highest weight* $m_1\Lambda_1+m_2\Lambda_2$. *Let* $\varphi:(CP^{2n-1},g_x)\to(CP^m,h)$ *be the Sp(n)-equivariant map corresponding to a weight* $(m_1-2j)\Lambda_1$ $(m_1-2j\neq0)$. *Then* φ *is an isometric immersion if the following equation holds:*

$$(*)\quad \frac{2(n-1)}{x}\{(2j+1)m_1-2j^2\}=m_2{}^2+(m_1+2n-1)m_2+(n-1)m_1.$$

In case $x=2$, g_2 *is the Fubini-Study metric. Then the equation above becomes*

$$m_2{}^2+(m_1+2n-1)m_2-2(n-1)jm_1+2(n-1)j^2=0.$$

We may rewrite Theorem 5.5 as follows.

THEOREM 5.6. *Consider the Sp(n)-equivariant map* φ *corresponding to* $\Lambda=m_1\Lambda_1+m_2\Lambda_2\in D(Sp(n),K;k)$ $(k\neq0)$. *If the equation*

$$\frac{n-1}{x}(m_1{}^2+2m_1-k^2)=m_2{}^2+(m_1+2n-1)m_2+(n-1)m_1$$

holds, then φ *is an isometric immersion. In case of* $x=2$, *the equation above becomes*

$$m_2{}^2+(m_1+2n-1)m_2-\frac{n-1}{2}(m_1{}^2-k^2)=0.$$

PROOF OF THEOREM 5.5. Assume that $\varphi^*h=rg_x$ for some constant $r>0$, then by virtue of Lemma 5.3, we have

$$r=\varphi^*h\left(\frac{X_{\alpha_0}}{\sqrt{x}},\frac{X_{\alpha_0}}{\sqrt{x}}\right)=-\frac{1}{x}\langle\rho(X_{\alpha_0}X_{\alpha_0})v_0,v_0\rangle=\frac{1}{2x}\{(2j+1)m_1-2j^2\}|v_0|^2,$$

where $\alpha_0 = 2\sum_{1\le k<n} \alpha_k + \alpha_n$. From this equation and Proposition 5.4, we have

$$\frac{1}{2x}\{(2j+1)m_1 - 2j^2\}$$
$$= \frac{1}{2(2n-1)}[m_2{}^2 + (m_1 + 2n - 1)m_2 + (n-1)m_1 + \frac{1}{x}\{(2j+1)m_1 - 2j^2\}].$$

Hence we get the equation (*).

Conversely, if the equation (*) holds, then we set

$$r = \{(2j+1)m_1 - 2j^2\}|v_0|^2 / 2x$$

and get

$$\varphi^* h(X_{\alpha_0}/\sqrt{x}, X_{\alpha_0}/\sqrt{x}) = rg_x(X_{\alpha_0}/\sqrt{x}, X_{\alpha_0}/\sqrt{x}),$$

i.e., $\varphi^* h = rg_x$ q.e.d.

REMARK.

(1) By the condition $m_1 - 2j \neq 0($ *or* $k \neq 0)$, we see a map φ is an immersion.

(2) If the map corresponding to a weight $k\Lambda_1$ is an isometric immersion, so is the map corresponding to a weight $-k\Lambda_1$. Because the equations in Theorem 5.6 remains the same by replacing k with $-k$.

(3) In case of $n = 2$, $k = 4$, and $\Lambda = 6\Lambda_1 + \Lambda_2$, we have an $Sp(n)$-equivariant, but not $SU(2n)$-equivariant, minimal immersion from $\boldsymbol{CP}^3$ to $\boldsymbol{CP}^{230}$.

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