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SUM THEOREMS FOR THE STRONG SMALL TRANSFINITE DIMENSION

By

Eduardo CUCHILLO-IBÁÑEZ and Juan TARRÉS

Abstract. We state some sum theorems for the strong small transfinite dimension in different classes of topological spaces.

Introduction

P. Borst introduced in [1] the strong small transfinite dimension, *sind*. There, he stated:

THEOREM ([1], corollary of proposition IV.5). *Let* X *be a normal space which has a locally finite closed cover* $\{F_s\}_{s \in S}$ *such that for each* F_S *it is sind* $(F_S) \neq \Delta$. *Then*

$$\operatorname{sind}(X) \neq \Delta .$$

We could say that this is a locally finite sum *weak* theorem, because in it there is no relation between the strong small transfinite dimensions of the closed sets F_S and that of the whole space X. We'll connect them obtaining the locally finite sum theorem for *sind*, in its classical formulation, in the class of strongly hereditarily normal spaces. In addition, we establish an open sum theorem in the class of regular spaces.

Preliminaries

We shall use the notation and definitions in [1], [3], [4] and [5]. For every ordinal ξ we have $\xi = \lambda(\xi) + n(\xi)$ where $\lambda(\xi)$ is a limit ordinal and $n(\xi)$ is a finite ordinal. We take the extra symbol Δ, satisfying $\Delta > \xi$ and $\Delta + \xi = \xi + \Delta = \Delta$ for each ordinal number ξ.

In order to define the strong small transfinite dimension (due to P. Borst, [1])

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we define, for a subspace Y of a topological space X, the sets:

$$\mathrm{P}_n(Y) = \cup\{U / U \text{ open in Y and Ind } [Cl_Y(U)] \le n\}$$

where $n \in N \cup \{0\}$, and $A_0[Y] = Y$. For every ordinal number ξ we obtain inductively:

$$A_\xi[Y] = Y - \bigcup_{\eta<\xi} P_\eta(Y) \text{ and } P_\xi(Y) = P_{n(\xi)}\left(\mathrm{A}_{\lambda(\xi)}[Y]\right).$$

We simplify by denoting $A_\xi[X] = A_\xi$ and $P_\xi(\mathrm{X}) = P_\xi$ for every ordinal ξ.

DEFINITION. *Let X be a topological space, then:*

$$\mathrm{sind}(X) = -1 \quad \text{iff } X = \varnothing$$

$$\mathrm{sind}(X) \le \xi \quad \text{iff } A_\xi = \varnothing$$

sind $(X) = \xi$ *iff sind* $(X) \le \xi$ *and sind* $(X) < \xi$ *does not hold*
in other case, we say that X *has not sind or sind* $(X) = \Delta$

Locally finite sum theorem

Recall that a topological space X is called *strongly hereditarily normal* (see [3], definition 2.1.2) if X is a T_1 -space and for every pair A, B of separated sets in X there exist open sets $U, V \subset X$ such that $A \subset U, B \subset V, U \cap V = \varnothing$ and U and V can be represented as the union of a point-finite family of F_σ-sets in X.

THEOREM 1. Let X be a strongly hereditarily normal space. If $\mathscr{C} = \{C_i\}_{i\in I}$ is a locally finite closed cover of X such that for each $i \in I$ sind $(C_i) \le \xi$, then

$$\mathrm{sind}(X) \le \xi.$$

PROOF. We'll obtain sind $(X) \le \xi$ by proving $A_\xi = \phi$, that is to say, $X = \bigcup_{\eta<\xi} \mathrm{P}_\eta$. So, we'll see that for each point $x \in X$ there exists an ordinal number η_0, with $\eta_0 < \xi$, such that $x \in \mathrm{P}_{\eta_0}$.

Let's take a point $x_0 \in X$ and let V be an open neighbourhood of x_0 such that intersects to a finite number of elements of the closed cover $\mathscr{C}$: $C_{i_1}, \ldots, C_{i_n}$. We'll show the existence of an ordinal $\eta_0 < \xi$, for which $x_0 \in \mathrm{P}_{\eta_0}$, by induction on n, the number of elements of $\mathscr{C}$ whose intersection with V is not empty.

i. If n=1, V only cuts C_{i_1}, whereupon

$$x_0 \in V \subset C_{i_1}.$$

As $\operatorname{sind}(C_{i_1}) < \xi$, $A_\xi[C_{i_1}] = \phi$. Since V is open in C_{i_1}, from the corollary of lemma 3 of [2] we obtain

$$A_\xi[V] = A_\xi[C_{i_1}] \cap V = \phi .$$

Anew due to the mentioned result,

$$\phi = A_\xi[V] = A_\xi \cap V .$$

Hence, $x_0 \notin \mathrm{A}_\xi = X - \bigcup_{\eta<\xi} \mathrm{P}_\eta$. Thus, there exists an ordinal $\eta_0 < \xi$ such that $x_0 \in \mathrm{P}_{\eta_0}$.

ii. When $n=2$, V only cuts two members, say C_{i_1} and C_{i_2}, of the cover $\mathscr{C}$. We'll have $V \subset C_{i_1} \cup C_{i_2}$ and suppose that

$$x_o \in C_{i_1} \cap C_{i2},$$

because if, for example, $x_0 \in C_{i_1} - C_{i_2}$, after considering the open neighbourhood of x_0

$$W = V \cap (X - C_{i_2}),$$

we'll be situated in the section i. Hence, as

$$\operatorname{sind}(C_{i_1}) \le \xi \ \text{ and } \ \operatorname{sind}(C_{i_2}) \le \xi ,$$

there exist ordinals $\alpha_1, \alpha_2 < \xi$ such that

$$x_o \in \mathrm{P}_{\alpha 1}(C_{i1}) \cap \mathrm{P}_{\alpha 2}(C_{i2})$$

Let's define the ordinal number $\rho = \max\{\alpha_1, \alpha_2\} < \xi$. We are going to prove, by transfinite induction on ρ, that

$$x_o \in \bigcup_{\eta\le\rho} \mathrm{P}_\eta .$$

ii. 1. If $\rho = 0, \alpha_1 = \alpha_2 = 0$ too. In this case

$$x_0 \in \mathrm{P}_0(C_{i1}),$$

so there exists U_1, open neighbourhood of x_0 in C_{i_1}, such that

$$\operatorname{Ind}[Cl_{C_{i_1}}(U_1)] = \operatorname{Ind}[\overline{U}_1] \le 0 .$$

Analogously, $x_0 \in \mathrm{P}_0(C_{i2})$ and there exists U_2, open neighbourhood of x_0 in C_{i_2}, such that

$$\operatorname{Ind}\ [Cl_{C_{i2}}(U_2)] = \operatorname{Ind}\ [\overline{U}_2] \le 0 .$$

Take open subsets W_1 and W_2 of X with

$$U_1 = W_1 \cap C_{i_1} \ \text{ and } \ U_2 = W_2 \cap C_{i_2} .$$

The sets

$$G_1 = W_1 \cap W_2 \cap V \cap C_{i_1}$$

and

$$G_2 = W_1 \cap W_2 \cap V \cap C_{i_2}$$

are open neighbourhoods of the point x_0 in C_{i_1} and C_{i_2}, respectively, with

$$G_1 \subset U_1 \quad \text{and} \quad G_2 \subset U_2 .$$

Let's consider

$$G = G_1 \cup G_2 =$$
$$= (W_1 \cap W_2 \cap V \cap C_{i_1}) \cup (W_1 \cap W_2 \cap V \cap C_{i_2}) =$$
$$= (W_1 \cap W_2 \cap V) \cap (C_{i_1} \cup C_{i_2}) =$$
$$= W_1 \cap W_2 \cap V .$$

G is an open neighbourhood of x_0 in X, and

$$\overline{G} = \overline{G_1 \cup G_1} \subset \overline{U_1 \cup U_2} = \overline{U}_1 \cup \overline{U}_2 .$$

Since the large inductive dimension *Ind* satisfies the finite sum theorem for closed subsets in the class of strongly hereditarily normal spaces,

$$\operatorname{Ind}[\overline{G}] \leq \operatorname{Ind}[\overline{U}_1 \cup \overline{U}_2] \leq$$
$$\leq \max \{\operatorname{Ind}[\overline{U}_1], \operatorname{Ind}[\overline{U}_2]\} \leq 0$$

whence we conclude that

$$x_0 \in \mathrm{P}_0(\mathrm{X}) = \mathrm{P}_0 .$$

ii. 2. Let's assume that the result is true for each ordinal η with $\eta < \rho$ $(\rho > 0)$, that is to say, in the conditions of this theorem, let's admit the veracity of the following sentence:
"each point $x \in X$ which possesses an open neighbourhood that only cuts two elements, C_{i_1} and C_{i_2}, of the closed cover $\mathscr{C}$, if it is

$$x_0 \in \mathrm{P}_{\eta_1}(C_{i_1}) \cap \mathrm{P}_{\eta_2}(C_{i_2})$$

with $\eta = \max\{\eta_1, \eta_2\} < \rho$, then

$$x_0 \in \bigcup_{\beta \leq \eta} \mathrm{P}_\beta ."$$

Now we'll see that this sentence is also true for the ordinal P_β . Since

$$x_0 \in P_{\alpha 1}(C_{i1}) \cap P_{\alpha 2}(C_{i2})$$

we have

$$x_0 \in A_{\lambda(\alpha_1)}[C_{i1}] \cap A_{\lambda(\alpha 2)}[C_{i2}].$$

As C_{i_l} and C_{i2} are closed in X, from lemma 3 of [2] we have the following relations:

$$x_0 \in A_{\lambda(\alpha_1)}[C_{i_l}] \subset A_{\lambda(\alpha_1)} \cap C_{i_1}$$

and

$$x_0 \in A_{\lambda(\alpha_2)}[C_{i2}] \subset A_{\lambda(\alpha_2)} \cap C_{i2}.$$

Since $\rho = \max\{\alpha_1, \alpha_2\}$, $\lambda(\rho) = \max\{\lambda(\alpha_1), \lambda(\alpha_2)\}$ and, clearly,

$$A_{\lambda(\alpha 1)} \cap A_{\lambda(\alpha 2)} = A_{\lambda(\rho)},$$

therefore

$$x_0 \in A_{\lambda(\rho)}.$$

Next we are going to prove the following relation,

$$A_{\lambda(\rho)} \cap V \subset A_{\lambda(\rho)}[C_{i_1}] \cup A_{\lambda(\rho)}[C_{i_2}]:$$

given $y \in A_{\lambda(\rho)} \cap V \subset A_{\lambda(\rho)} \cap (C_{i_l} \cup C_{i2})$, we differentiate two cases:

ii. 2. I. If $y \notin C_{i_l} \cap C_{i2}$, let's suppose that $y \in C_{i_l} - C_{i2}$,

$$y \in V \cap (X - C_{i2})$$

which is an open neighbourhood of y in X. From the corollary of lemma 3 in [2],

$$A_{\lambda(\rho)}[V \cap (X - C_{i2})] = A_{\lambda(\rho)} \cap V \cap (X - C_{i2}).$$

Since

$$V \cap (X - C_{i2}) \subset C_{i_l},$$

anew from the above-mentioned result,

$$A_{\lambda(\rho)}[V \cap (X - C_{i2})] = A_{\lambda(\rho)}[C_{i_l}] \cap V \cap (X - C_{i2}).$$

Finally, as

$$y \in A_{\lambda(\rho)} \cap V \cap (X - C_{i2}),$$

we conclude that

$$y \in A_{\lambda(\rho)}[C_{i_l}].$$

ii. 2. II. If $y \in C_{i_l} \cap C_{i2}$, assume that

$$y \notin A_{\lambda(\rho)}[C_{i1}] \cup A_{\lambda(\rho)}[C_{i2}],$$

to obtain, later on, a contradiction.

$$y \notin A_{\lambda(\rho)}[C_{il}] = C_{il} - \bigcup_{\eta<\lambda(\rho)} P_\eta(C_{il}) \ ,$$

so there exists $\eta_1 < \lambda(\rho)$ with $y \in P_{\eta_1}(C_{il})$.

$$y \notin A_{\lambda(\rho)}[C_{i2}] = C_{i2} - \bigcup_{\eta<\lambda(\rho)} P_\eta(C_{i2}) \ ,$$

so there exists $\eta_2 < \lambda(\rho)$ with $y \in P_{\eta_2}(C_{i2})$. Thus

$$y \in P_{\eta_1}(C_{il}) \cap P_{\eta_2}(C_{i2})$$

with $\eta_1, \eta_2 < \lambda(\rho) \leq \rho$. Call η_0=max$\{\eta_1, \eta_2\}$. It's obvious that $\eta_0 < \rho$ and $\lambda(\eta_0) < \lambda(\rho)$.

V is an open neighbourhood of the point y which only cuts two members, C_{il} and C_{i2}, of the closed cover $\mathscr{C}$. Furthermore, we have $y \in P_{\eta_1}(C_{il}) \cap P_{\eta_2}(C_{i2})$, being η_0= max$\{\eta_1, \eta_2\}$ an ordinal number less than ρ. It follows, from the induction hypothesis, that

$$y \in \bigcup_{\beta \leq \eta_0} P_\beta,$$

with $\lambda(\eta_0) < \lambda(\rho)$, so $\eta_0 < \lambda(\rho)$. However, the point y was such that

$$y \in A_{\lambda(\rho)} = X - \bigcup_{\eta<\lambda(\rho)} P_\eta \ ,$$

what is a contradiction. Then, we have just established the inclusion

$$A_{\lambda(\rho)} \cap V \subset A_{\lambda(\rho)}[C_{il}] \cup A_{\lambda(\rho)}[C_{i2}].$$

We had

$$x_0 \in \mathrm{P}_{\alpha_1}(C_{il}) \cap \mathrm{P}_{\alpha_2}(C_{i2}).$$

Detaching this intersection,

$$x_0 \in \mathrm{P}_{\alpha_1}(C_{il}) = \mathrm{P}_{n(\alpha_1)}\left(A_{\lambda(\alpha_1)}[C_{il}]\right),$$

so there exists U_1, open neighbourhood of x_0 in $A_{\lambda(\alpha_1)}[C_{il}]$, such that

$$\mathrm{Ind}\left[C\ell_{A_{\lambda(\alpha 1)}[C_{i_1}]}(U_1)\right] = \mathrm{Ind}\left[C\ell_{C_{i_1}}(U_1)\right] =$$

$$= \mathrm{Ind}\left[\overline{U}_1\right] \leq n(\alpha_1).$$

$$x_0 \in \mathrm{P}_{\alpha_2}(C_{i2}) = \mathrm{P}_{n(\alpha_2)}\left(A_{\lambda(\alpha_2)}[C_{i2}]\right),$$

so there exists U_2, open neighbourhood of x_0 in $A_{\lambda(\alpha_2)}[C_{i_2}]$, such that

$$\text{Ind}\left[C\ell_{A_{\lambda(\alpha_2)}[C_{i_2}]}(U_2)\right]=\text{Ind}\left[\text{Cl}_{C_{i2}}(U_2)\right]=$$

$$=\text{Ind}\ [\overline{U}_2]\leq n(\alpha_2).$$

Let's analyze the two possible options:
First. $\lambda(\alpha_1)=\lambda(\alpha_2)$. Here, $\lambda(\rho)=\lambda(\alpha_1)=\lambda(\alpha_2)$.

Let W_1 and W_2 be open subsets of X with

$$U_1=W_1\cap A_{\lambda(\rho)}[C_{i_1}] \text{ and } U_2=W_2\cap A_{\lambda(\rho)}[C_{i_2}].$$

An open neighbourhood of x_0 in $A_{\lambda(\rho)}$ is

$$G=W_1\cap W_2\cap V\cap A_{\lambda(\rho)}\subset$$

$$\subset W_1\cap W_2\cap(A_{\lambda(\rho)}[C_{i_l}]\cup A_{\lambda(\rho)}[C_{i_2}])=$$

$$=\left(W_1\cap W_2\cap A_{\lambda(\rho)}[C_{i_l}]\right)\cup\left(W_1\cap W_2\cap A_{\lambda(\rho)}[C_{i_2}]\right)\subset$$

$$\subset U_1\cup U_2.$$

Let's observe that

$$\text{Cl}_{A_{\lambda(\rho)}}(G)=\overline{G}\subset\overline{U_1\cup U_2}=\overline{U}_1\cup\overline{U}_2.$$

From the closed subspace theorem and the finite sum theorem for closed subsets, which are satisfied by the large inductive dimension *Ind* in the class of strongly hereditarily normal spaces,

$$\text{Ind}\ [\overline{G}]\leq\ \text{Ind}\ [\overline{U}_1\cup\overline{U}_2]\leq$$

$$\leq\max\{\text{Ind}[\overline{U}_1],\text{Ind}[\overline{U}_2]\}\leq\max\{\text{n}(\alpha_1),\text{n}(\alpha_2)\}.$$

Since we are in the case $\lambda(\alpha_1)=\lambda(\alpha_2)$ and $\rho=\max\{\alpha_1,\alpha_2\}$, it follows that $n(\rho)=\max\{\text{n}(\alpha_1),\text{n}(\alpha_2)\}$ and

$$\text{Ind}\ [\overline{G}]\leq n(\rho).$$

That is how

$$x_0\in \text{P}_{n(\rho)}\left(A_{\lambda(\rho)}\right)=\text{P}_{(\rho)}.$$

Second. $\lambda(\alpha_1)\neq\lambda(\alpha_2)$. Let's suppose, for example, $\lambda(\alpha_1)>\lambda(\alpha_2)$. So $\alpha_1>\alpha_2$ and $\rho=\max\{\alpha_1,\alpha_2\}=\alpha_1$. As

$$x_0\in P_{\alpha_1}(C_{i_l})\cap P_{\alpha_2}(C_{i_2}),$$

particularly,

$$x_0 \in \mathrm{P}_{\alpha_2}(C_{i_2}).$$

Being $\alpha_2 < \lambda(\rho)$, $x_0 \notin A_{\lambda(\rho)}[C_{i_2}]$, closed subset in C_{i_2}, so in X. Let W be an open neighbourhood of x_0 in X such that

$$W \cap A_{\lambda(\rho)}[C_{i_2}] = \phi.$$

Take W_1, open in X, with

$$U_1 = W_1 \cap A_{\lambda(\rho)}[C_{i_l}].$$

An open neighbourhood of x_0 in $A_{\lambda(\rho)}$ is

$$G = W_1 \cap W \cap V \cap A_{\lambda(\rho)} \subset$$
$$\subset W_1 \cap W \cap \left(A_{\lambda(\rho)}[C_{i_l}] \cup A_{\lambda(\rho)}[C_{i_2}]\right) =$$
$$= (W_1 \cap W \cap A_{\lambda(\rho)}[C_{i_l}]) \cup (W_1 \cap W \cap A_{\lambda(\rho)}[C_{i_2}]) =$$
$$= (W_1 \cap W \cap A_{\lambda(\rho)}[C_{i_l}]) =$$
$$= W \cap U_1 \subset U_1.$$

Since

$$Cl_{A_{\lambda(\rho)}}(G) = \overline{G} \subset \overline{U}_1,$$

from the closed subspace theorem for *Ind*,

$$\mathrm{Ind}[\overline{G}] \leq \mathrm{Ind}[\overline{U}_1] \leq n(\alpha_1) = n(\rho),$$

whereupon

$$x_0 \in \mathrm{P}_{n(\rho)}\left(A_{\lambda(\rho)}\right) = \mathrm{P}_\rho.$$

iii. Let's suppose that the result is truthful for either natural number m, with $m \leq n$ and $n \geq 2$, that is to say, let's admit the veracity of the following sentence: "if X is a strongly hereditarily normal space and $\mathscr{C}$ is a cover, locally finite, constituted by closed subsets whose dimension *sind* is non higher than the ordinal ξ, for each point $x \in X$ owner of an open neighbourhood that cuts at most n elements of the family $\mathscr{C}$, there exists an ordinal number η_0, with $\eta_0 < \xi$, such that

$$x \in \mathrm{P}_{\eta_0}."$$

Let's check it for $n+1$:

Let x_0 be a point of X which has an open neighbourhood V that cuts n+1 members of the closed cover $\mathscr{C}$, $C_{i_1} C_{i_2}, \cdots, C_{i_{n+1}}$. The subset $F = C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_n}$ is closed in X, thus F, with the topology induced by X, is a strongly hereditarily normal space. It's clear that

$$\mathscr{F} = \{C_{i_1}, C_{i_2}, \ldots, C_{i_n}\}$$

is a closed cover of F formed by n closed subsets whose dimension *sind* is less or equal to ξ. From the induction hypothesis,

$$\operatorname{sind}(F) \leq \xi .$$

Now consider the closed cover of X

$$\mathfrak{R} = \{F\} \cup \{C_i : i \in I - \{i_1, \ldots, i_n\}\} .$$

This is a locally finite closed cover whose elements have strong small transfinite dimension non higher to the ordinal ξ. The chosen point x_0 has an open neighbourhood V that only cuts two closed subsets of $\mathfrak{R}$, F and $C_{i_{n+1}}$. By the induction hypothesis we obtain the existence of an ordinal number η_0, with $\eta_0 < \xi$, such that

$$x_0 \in \mathrm{P}_{\eta_0}$$

and it concludes the proof. ■

COROLLARY. *Let X be a strongly hereditarily normal space. If $X = C_1 \cup \ldots \cup C_n$, with $C_1, \ldots, C_n$ closed subsets of X such that sind $(C_i) \leq \xi$, for each $i \in \{1, \ldots n\}$, then*

$$sind(X) \leq \xi .$$

Let's point out that theorem 1 is not verified by the class of normal spaces. In fact, A. R. Pears constructs in the example 4.3.4 of [5] a compact and normal space S such that

$$\operatorname{locInd}[S] = 2,$$

hence

$$\operatorname{sind}(S) = 3.$$

This space S admits a decomposition $S = S_1 \cup S_2$, where S_1 and S_2 are closed subsets of S with

$$\operatorname{locInd}[S_1] = 1 \quad \text{and} \quad \operatorname{locInd}[S_2] = 1 ,$$

that is to say

$$\operatorname{sind}[S_1] = 2 \quad \text{and} \quad \operatorname{sind}[S_2] = 2 \ ,$$

whereupon we confirm the assertion.

Open sum theorem

In order to establish the open sum theorem for the strong small transfinite dimension, let's prove the next lemma:

LEMMA. *Let X be a regular space. If Y is an open subspace of X, for either ordinal number* ξ*:*

a) $\bigcup_{\eta\le\xi} P_\eta(Y) = \bigcup_{\eta\le\xi} P_\eta \cap Y$.

b) $A_\xi[Y] = A_\xi \cap Y$.

PROOF. a) It's easy to see that lemma 2 of [2] is true for regular spaces; so we have, for each ordinal ξ,

$$\bigcup_{\eta\le\xi} P_\eta(Y) \subset \bigcup_{\eta\le\xi} P_\eta \cap Y .$$

Let's see the other inclusion by transfinite induction on the ordinal number ξ.

i. For $\xi = 0$, take $x \in P_0 \cap Y$. There exists U, open neighbourhood of x in X, such that $\operatorname{Ind}[\overline{U}] \le 0$. $U \cap Y$ is another open neighbourhood of x in X, regular space, and hence there exists V, open in X, such that

$$x \in V \subset \overline{V} \subset U \cap Y .$$

In this way, $\overline{V} \subset Y$, $Cl_Y(V) = \overline{V}$ and $V \subset \overline{V} \subset U \subset \overline{U}$. From the closed subspace theorem for *Ind*,

$$\operatorname{Ind}\ [\overline{V}] \le \operatorname{Ind}\ [\overline{U}] \le 0 .$$

Consequently, $x \in P_0(Y)$.

ii. Let's suppose that for each ordinal β, with $\beta < \xi$, it is

$$\bigcup_{\eta\le\beta} P_\eta \cap Y \subset \bigcup_{\eta\le\beta} P_\eta(Y) .$$

From the initial observation of this proof we obtain that

$$\bigcup_{\eta\le\beta} P_\eta \cap Y = \bigcup_{\eta\le\beta} P_\eta(Y),$$

for either ordinal $\beta, \beta < \xi$. Therefore, for each pair of ordinal numbers α, β, with $\beta < \alpha \leq \xi$,

$$\bigcup_{\beta<\alpha} [\bigcup_{\eta \leq \beta} P_\eta \cap Y] = \bigcup_{\beta<\alpha} [\bigcup_{\eta \leq \beta} P_\eta(Y)],$$

that is,

$$\bigcup_{\eta<\alpha} P_\eta \cap Y = \bigcup_{\eta<\alpha} P_\eta(Y)$$

for each ordinal $\alpha \leq \xi$. Taking complementaries in Y,

$$A_\alpha[Y] = A_\alpha \cap Y$$

for $\alpha \leq \xi$.

Clearly, it suffices to prove that

$$P_\xi \cap Y \subset \bigcup_{\eta \leq \xi} P_\eta(Y).$$

Let's admit $P_\xi \cap Y \neq \phi$ and take a point

$$x \in P_\xi \cap Y = P_{n(\xi)}(A_{\lambda(\xi)}) \cap Y.$$

On the one hand, $x \in Y$; on the other there exists U, open neighbourhood of x in $A_{\lambda(\xi)}$, such that

$$\text{Ind}\left[C\ell_{A_{\lambda(\xi)}}(U)\right] = \text{Ind}\ [\overline{U}] \leq n(\xi).$$

Since $\lambda(\xi) \leq \xi$, $A_{\lambda(\xi)}[Y] = A_{\lambda(\xi)} \cap Y$. As well as Y is open in X, $A_{\lambda(\xi)}[Y]$ is an open subset of $A_{\lambda(\xi)}$ which contains the point x. Through the regularity of $A_{\lambda(\xi)}$, there exists V, open neighbourhood of x in $A_{\lambda(\xi)}$, such that

$$x \in V \subset C\ell_{A_{\lambda(\xi)}}(V) = \overline{V} \subset A_{\lambda(\xi)}[Y] \cap U.$$

In this manner,

$$V \subset \overline{V} \subset A_{\lambda(\xi)}[Y],$$

$$C\ell_{A_{\lambda(\xi)}[Y]}(V) = \overline{V}$$

and

$$x \in V \subset \overline{V} \subset U \subset \overline{U}.$$

Applying the closed subspace theorem for *Ind*,

$$\operatorname{Ind}[\overline{V}] \leq \operatorname{Ind}[\overline{U}] \leq n(\xi).$$

From this, and bearing in mind that V is an open neighbourhood of x in $A_{\lambda(\xi)}[Y]$ because it is in $A_{\lambda(\xi)}$ and $V \subset \overline{V} \subset A_{\lambda(\xi)}[Y] \subset A_{\lambda(\xi)}$, we obtain

$$x \in P_{n(\xi)}\big(A_{\lambda(\xi)}[Y]\big) = P_{\xi}(Y) \subset \bigcup_{\eta \leq \xi} P_{\eta}(Y).$$

b) It follows from a). ■

THEOREM 2. *Let X be a regular space. If X admits an open cover* $\Re = \{G_i\}_{i \in I}$ *such that sind* $(G_i) \leq \xi$ *for each* $i \in I$,

$$sind(X) \leq \xi.$$

PROOF. We'll show that $A_{\xi} = \phi$:
for either $i \in I, A_{\xi}[G_i] = \phi$, because of sind $(G_i) \leq \xi$. Since $\Re$ is a cover of X,

$$A_{\xi} = A_{\xi} \cap \bigcup_{i \in I} G_i = \bigcup_{i \in I}(A_{\xi} \cap G_i).$$

From the previous lemma,

$$\bigcup_{i \in I}(A_{\xi} \cap G_i) = \bigcup_{i \in I} A_{\xi}[G_i] = \bigcup_{i \in I} \phi = \phi. \blacksquare$$

This result about the open sum permits us to know a characterization of the existence of the strong small transfinite dimension *sind* in the class of regular spaces:

COROLLARY. *For a regular space X the next sentences are equivalent:*
a) sind$(X) \leq \xi$.
b) each point $x \in X$ *has an open neighbourhood V such that sind* $(V) \leq \xi$.
c) each point $x \in X$ *has a neighbourhood V such that sind* $(V) \leq \xi$.

References

[1] P. Borst: "Infinite-dimension theory (Part II)", manuscript (1981).
[2] E. Cuchillo-Ibáñez: "A relation between the strong small transfinite dimension and the D-dimension", Q. & A. Gen. Top. **9** (1991) 81–88.
[3] R. Engelking: "Dimension Theory". North-Holland Publishing Company, Amsterdam 1978.
[4] R. Engelking: "Transfinite dimension", Surveys in General Topology, Academic Press (1980) 131–161.
[5] A. R. Pears: "Dimension Theory of General Spaces", Cambridge University Press, Cambridge 1975.

Juan Tarrés
Facultad de Ciencias Matemáticas
Dep. Geometría y Topología
Ciudad Universitaria
28040 Madrid, Spain

Eduardo Cuchillo-Ibáñez
E.T.S. Ingenieros de Montes
U.D. Matemáticas
Ciudad Universitaria
28040 Madrid, Spain