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ON EMBEDDINGS OF PERFECT GO-SPACES INTO PERFECT LOTS

By

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§1. Introduction

A *linearly ordered topological space* (abbreviated *LOTS*) is a triple $\langle X,\lambda,\leq\rangle$, where $\langle X,\leq\rangle$ is a linearly ordered set and λ is the usual interval topology defined by $\leq$. Throughout this paper, λ, $\lambda(\leq)$ or λ_X denote the usual interval topology on a linearly ordered set $\langle X,\leq\rangle$.

A *generalized ordered space* (abbreviated *GO-space*) is a triple $\langle X,\tau,\leq\rangle$, where $\langle X,\leq\rangle$ is a linearly ordered set and τ is a topology on X such that $\lambda\subset\tau$ and τ has a base of open sets each of which is order-convex, where a subset A of X is called *order-convex* if $x\in A$ for every x lying between two points of A. For a GO-space $\langle X,\tau,\leq\rangle$ and $Y\subset X$, $\tau|Y$ denotes the subspace topology $\{U\cap Y: U\in\tau\}$ on Y and $\leq|Y$ denotes the restricted ordering of $\leq$ on Y. If it will cause no confusion, we shall omit λ (or τ) and $\leq$, and say simply " X is a LOTS (GO-space)". A topological space $\langle X,\tau\rangle$, where τ is a topology on a set X, is said to be *orderable* if $\langle X,\tau,\leq\rangle$ is a LOTS for some linear ordering $\leq$ on X. Similarly, we say simply " X is an orderable space" if it will cause no confusion. A LOTS $Z=\langle Z,\lambda,\leq_Z\rangle$ is said to be a *linearly ordered extension* of a GO-space $X=\langle X,\tau,\leq_X\rangle$ if $X\subset Z$, $\tau=\lambda|X$ and $\leq_X=\leq_Z|X$. Furthermore, if X is closed (resp., dense) in the space $\langle Z,\lambda\rangle$, then Z is said to be a *linearly ordered c-extension* (resp., *d-extension*) of X. Similarly, an orderable space $Z=\langle Z,\tau_Z\rangle$ is said to be an *orderable c-* (resp., *d-*)*extension* of a GO-space $X=\langle X,\tau_X \leq\rangle$ if X is a closed (resp., dense) subset of Z and $\tau_X=\tau_Z|X$. Note that every GO-space has a compact linearly ordered d-extension ([5, (2.9)]).

Throughout this paper, we use the following notation: Let $\langle Y,\lambda,\leq\rangle$ be a LOTS. For a GO-space $\langle X,\tau,\leq\rangle$ with the same underlying set Y and the same order $\leq$, we write $X=GO_Y(R,E,I,L)$, where $I=\{x\in X:\{x\}\in\tau-\lambda\}$, $R=\{x\in X: [x,\rightarrow)\in\tau-\lambda\}-I$, $L=\{x\in X:(\leftarrow,x]\in\tau-\lambda\}-I$ and $E=X-(I\cup R\cup L)$.

The following problem naturally arises.

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PROBLEM 1.1. Let P be a topological property. Does a GO-space with P have an orderable extension with P?

Concerning this problem, metrizability and (hereditary) paracompactness have affirmative answers (see [5]). But perfectness is unknown, where a topological space is perfect if each closed subset is a G_δ-set. The following problem was posed in [3, Question 1].

PROBLEM 1.2. Does every perfect GO-space have aperfect orderable extension?

In connection with this, the following is known from [5, (5.9) and (7.2)]: The Sorgenfrey line S is a perfect GO-space, but it does not have a perfect orderable c-extension.

However, S does not answer Problem 1.2 negatively, since the LOTS $S\times\{0,1\}$ with the lexicographic ordering is a perfect linearly ordered d-extension of S.

The following problem which is a strong version of Problem 1.2 was posed in [2, "Posed problems" No. 8] or [6, Question (V)].

PROBLEM 1.3. Does every perfect GO-space have a perfect orderable d-extension?

In connection with this, a partial negative answer was given in [8]; that is, there exists a perfect GO-space which does not have any perfect linearly ordered d-extension.

In this paper, we investigate some conditions in which we have affirmative answers of Problems 1.2 and 1.3. Throughout this paper, we use the letter ω to stand for the set of all natural numbers or the countable cardinality. For undefined terminology, we refer the reader to [4].

§2. Some conditions in which problems 1.2 and 1.3 have affirmative answers

In this section, for a GO-space X,we define LOTS's $H(X)$, $L(X)$, $M(X)$and $N(X)$, and investigate some conditions in which Problems 1.2 and 1.3 have affirmative answers.

DEFINITION 2.1. Let $X=GO_Y(R,E,I,L)$ be a GO-space on a LOTS Y. Let $I_+=\{x\in I:$ there is a $y\in X$ such that $y<x$ and $(y,x)=\phi\}$, $I_-=\{x\in I:$ there is a $y\in X$ such that $x<y$ and $(x,y)=\phi\}$ and $I_0=I-(I_+\cup I_-)$. We define subsets $H(X)$, $L(X)$, $M(X)$ and $N(X)$ of $X\times[-1,1]$ as follows:

(1) $H(X)=(X\times\{0\})\cup(R\cup I_{-})\times(-1,0))\cup((L\cup I_{+})\times(0,1))\cup(I_0\times(-1,1))$.

(2) $L(X)=(X\times\{0\})\cup((R\cup I_{-})R\cup I_{-})\times\{-1\})\cup((L\cup I_{+})\times\{1\})\cup(I_0\times\{-1,1\})$.

(3) $M(X)=(X\times\{0\})\cup(R\times(-1,0))\cup(L\times(0,1))\cup(I_{-}\times\{-1\})\cup$
$\cup(I_{+}\times\{1\})\cup(I_0\times\{-1,1\})$.

(4) $N(X)=(X\times\{0\})\cup(R\times\{-1\})\cup(L\times\{1\})\cup(I_{-}\times(-1,0))\cup$
$(I_{+}\times(0,1))\cup(I_0\times(-1,1))$.

Throughout this paper, $H(X)$, $L(X)$, $M(X)$ and $N(X)$ will be ordered lexicographically and will carry the usual interval topology of the ordering. Then it is easy to see that $e_H:X\to H(X)$, $e_L:X\to L(X)$, $e_M:X\to M(X)$ and $e_N:X\to N(X)$ defined by $e_*(x)=\langle x,0\rangle$ are order-preserving homeomorphisms from X onto the subspace $X\times\{0\}$. Note that $L(X)$ is the same space as the LOTS $\tilde{X}$ defined in [8], and $L(X)$ is the minimal d-extension of X ([8, (2.1)]).

Now we obtain the following theorem which is an affirmative answer for Problem 1.2 in a restricted situation. A "σ-discrete set" means the union of countably many discrete closed sets.

THEOREM 2.2 *Let $X=GO_Y(R,E,I,L)$ be a perfect GO-space. Then $H(X)$ is perfect if and only if $R\cup L$ is a σ-discrete set of X.*

PROOF. "Only if" part: Let $H(X)$ be perfect and let $U=R\times(-1,0)$, then U is an open set in $H(X)$. Put $U=\cup\{F_n:n\in\omega\}$, where F_n is closed in $H(X)$. Let $K_n=\{x\in R:\langle x,y\rangle\in F_n$ for some $y\in(-1,0)\}$. Then $R=\cup\{K_n:n\in\omega\}$. Suppose that K_n has a cluster point p in X. Since p is not an isolated point, we may suppose that $p\in E'\cup R\cup L$, where $E'=E-\{x:x$ is an isolated point of $X\}$. We prove that $\langle p,0\rangle$ is a cluster point of F_n in $H(X)$. Let V be a neighborhood of $\langle p,0\rangle$ in $H(X)$.

Case 1: Let $p\in E'$. There exist points a, b of X such that $a<p<b$ and $W=(\langle a,0\rangle,\langle b,0\rangle)$ is contained in V, where $(\langle a,0\rangle,\ \langle b,0\rangle)$ is an interval in $H(X)$. Since an interval (a,b) in X is a neighborhood of p in X, it follows that $(a,b)\cap(K_n-\{p\})\neq\phi$. Hence $W\cap F_n\neq\phi$. Therefore, $V\cap F_n\neq\phi$.

Case 2: Let $p\in L$. There exists a point $a\in X$ such that $a<p$ and $W=(\langle a,0\rangle,\langle p,0\rangle]\subset V$. Since $(a,p]$ is a neighborhood of p in X, $(a,p]\cap(K_n-\{p\})\neq\phi$. Hence $W\cap F_n\neq\phi$, so $V\cap F_n\neq\phi$.

Case 3: Let $p\in R$. The proof is similar to Case 2.

Since $\langle p,0\rangle \notin F_n$, this contradicts the closedness of F_n. Thus K_n does not have a cluster point in X, that is, K_n is discrete, closed and $R=\cup\{K_n : n\in\omega\}$ is σ-discrete in X. Similarly, L is σ-discrete in X. Thus $R\cup L$ is σ-discrete in X.

"If" part: Let $R\cup L$ be σ-discrete in a perfect GO-space X. Let U be open in $H(X)$. First, we show that $U\cap(I\times(-1,1))$ is F_σ in $H(X)$. Since I is open in X, I is F_σ in X i.e., $I=\cup\{F_n : n\in\omega\}$, where F_n is closed in X. It is clear that $U\cap(I\times(-1,1))=\cup\{U\cap(F_n\times(-1,1)) : n\in\omega\}$. Let $x\in F_n$. Since $U\cap(\{x\}\times(-1,1))$ is homeomorphic to an open subset of $(-1,1)$, we can express as $U\cap(\{x\}\times(-1,1))=\cup\{F(x,n,k) : k\in\omega\}$, where $F(x,n,k)$ is closed in $H(X)$. Set $G(n,k)=\cup\{F(x,n,k) : x\in F_n\}$. Then $G(n,k)$ is closed in $H(X)$. In fact, let $\langle x,t\rangle\notin G(n,k)$. If $x\in X-F_n$ and $t=0$, then there is a neighborhood V of x in X such that $V\cap F_n=\phi$. Then $W=(V\times(-1,1))\cap H(X)$ is a neighborhood of $\langle x,0\rangle$ in $H(X)$ such that $W\cap G(n,k)=\phi$. If $x\in I\cup R\cup L$ and $\langle x,t\rangle\in H(X)-G(n,k)$ with $t\neq 0$, then it is easy to see that there is a neighborhood of $\langle x,t\rangle$ in $H(X)$ that does not meet $G(n,k)$. If $x\in F_n$ and $\langle x,0\rangle\in H(X)-G(n,k)$, then we can find a neighborhood of $\langle x,0\rangle$ in $H(X)$ that does not meet $G(n,k)$ since $x\in I_0\cup I_+\cup I_-$. Hence $U\cap(F_n\times(-1,1))$ is F_σ in $H(X)$. Therefore $U\cap(I\times(-1,1))$ is F_σ in $H(X)$. Next, since R is σ-discrete in X, we can write $R=\cup\{R_n : n\in\omega\}$, where each R_n is discrete, closed in X. It follows from the above argument that $U\cap(R_n\times(-1,0])$ is an F_σ-set of $H(X)$ using the discreteness of R_n. Hence $U\cap(R\times(-1,0])$ is F_σ in $H(X)$. Similarly, $U\cap(L\times[0,1))$ is an F_σ-set of $H(X)$. Finally, we show that $E\times\{0\}$ is covered by countably many closed sets of $H(X)$ that are contained in U. To see this, it is enough to notice that $U\cap(E\times\{0\})\subset U\cap(X\times\{0\})\subset U$ and $U\cap(X\times\{0\})$ is an F_σ-set of $H(X)$, because $X\times\{0\}$ is a perfect, closed subspace of $H(X)$. Therefore, U is an F_σ-set of $H(X)$ and $H(X)$ is perfect.

REMARK 2.3. In this theorem, we may take a LOTS X^* (see [5,(2.5)]) instead of $H(X)$ since X^* can be embedded in $H(X)$. For a GO-space $X=(X,\tau,\leq)$, X^* was defined in [5, (2.5)] as follows: Let $\lambda=\lambda(\leq)$ be the usual order topology on X. Define a subset X^* of $X\times Z$ (where Z is the set of all integers) by $X^*=(X\times\{0\})\cup\{\langle x,n\rangle : [x,\rightarrow)\in\tau-\lambda$ and $n\leq 0\}\cup\{\langle x,m\rangle : (\leftarrow,x]\in\tau-\lambda$ and $m\leq 0\}$.

The following theorem is an affirmative answer for Problem 1.3 in a restricted situation. We use an abbreviation "ccc" to stand for the "countable chain condition" (i.e., every disjoint collection of open sets is countable).

THEOREM 2.4. *Let Y be a LOTS satisfying the ccc, and $X = GO_Y(R,E,I,L)$ be a GO-space. Then $L(X)$ is perfect if and only if $|I| \leq \omega$, where $|I|$ denotes the cardinality of I.*

PROOF. "If" part: We shall show that $L(X)$ satisfies the ccc. Then $L(X)$ is perfect by [5, (2.10)] and [4, 3.8.A. (b)]. Let $\{U_\alpha : \alpha \in A\}$ be a family of disjoint open sets of $L(X)$. Then we show that A is countable. Let $\langle x,t\rangle \in U_\alpha$ with $x \in R \cup L \cup E$. Then $U_\alpha \cap X$ contains a nonvoid open set of Y. Hence such U_α's are countable, because Y satisfies the ccc. Since I is countable, A is countable. Therefore, $L(X)$ satisfies the ccc.

"Only if" part: Let $L(X)$ be perfect. Since $I \times \{0\}$ is open in $L(X)$, we can express as $I \times \{0\} = \cup\{F_n : n \in \omega\}$, where F_n is closed in $L(X)$. Let $x \in (I_- \cup I_0) \cap F_n$. Since $\langle x,-1\rangle \in L(X) - F_n$, there exists a neighborhood V of $\langle x,-1\rangle$ in $L(X)$ such that $V \cap F_n = \phi$. Hence there is an $a_x \in X$ such that $a_x < x$ and $(a_x, x)_X \cap F_n = \phi$, where $(a_X, x)_X$ denotes an interval in X. If $x \in I_+ \cap F_n$, then a_x is taken as the predecessor of x. Similarly, there is a $b_x \in X$ such that $x < b_x$ and $(x, b_x)_X \cap F_n = \phi$. So, for each $x \in F_n$, there exists a neighborhood (a_x, b_x) of x in Y such that $(a_x, b_x) \cap F_n = \{x\}$. Let $x \neq y$ for $x, y \in F_n$, say $x < y$. If $(a_x, b_x) \cap (a_y, b_y) \neq \phi$, then the set $(a_x, b_x) \cap (a_y, b_y)$ does not meet F_n. In this case, we choose the intervals (a_x, b_y) and (b_x, b_y) as the disjoint neighborhoods of x and y in Y, respectively. Since Y satisfies the ccc, F_n is countable. Hence I is countable.

REMARK 2.5. If a GO-space satisfies the ccc, the answer of Problem 1.3 is "yes", as was announced in [2, "Posed problems" No. 8].

THEOREM 2.6. *Let Y be a LOTS satisfying the ccc, and $X = GO_Y(R,E,I,L)$ be a GO-space. Then $M(X)$ is perfect if and only if $|R \cup L \cup I| \leq \omega$.*

PROOF. "If" part: Suppose that $|R \cup L \cup I| \leq \omega$ and Y satisfies the ccc. Then it is enough to show that $M(X)$ satisfies the ccc. Then $M(X)$ is perfect by [5, (2.10)] and [4, 3.8.A.(b)]. Let $\{U_\alpha : \alpha \in A\}$ be a family of disjoint open sets of $M(X)$. Since I is countable, $A_I = \{\alpha \in A : (I \times \{-1,0,1\} \cap U_\alpha \neq \phi\}$ is countable. Since R is countable and $(-1,0]$ satisfies the ccc, $A_R = \{\alpha \in A : (R \times (-1,0]) \cap U_\alpha \neq \phi\}$ is countable. Similarly, $A_L = \{\alpha \in A : (L \times [0,1)) \cap U_\alpha \neq \phi\}$ is countable. Set $A_E = \{\alpha \in A : (E \times \{0\}) \cap U_\alpha \neq \phi\}$ and take an element $\alpha \in A_E$. Since U_α contains a non-void open set, A_E is countable. Hence $A = A_I \cup A_R \cup A_L \cup A_E$ is countable. Therefore, $M(X)$ satisfies the ccc.

"Only if" part: Let $M(X)$ be perfect. Since $I\times\{0\}$ is open in $M(X)$, we can express as $\cup\{F_n : n\in\omega\}$, where F_n is closed in $M(X)$. Note that each F_n is not necessarily closed in Y. However, the proof of "Only if" part of Theorem 2.4 shows that I is countable. Next, the proof of "Only if" part of Theorem 2.2 shows that R and L is σ-discrete in X. Set $R=\cup\{R_n : n\in\omega\}$, where R_n is discrete closed in X. For each $x\in R_n$, we can take a neighborhood $[x,b_x)$ of x in X such that $[x,b_x)\cap R_n=\{x\}$. It is easy to see that a collection $\{(x,b_x): x\in R_n\}$ of open intervals in Y is pairwise disjoint and each member (x,b_x) is not empty. Hence R_n is countable because Y satisfies the ccc, $|R|\leq\omega$. Similarly, $|L|\leq\omega$. Therefore, it follows that $|R\cup L\cup I|\leq\omega$.

We close this section with the following theorem.

THEOREM 2.7. *Let Y be a LOTS satisfying the ccc, and $X=GO_Y(R,E,I,L)$ be a GO-space. Then $N(X)$ is perfect if and only if I satisfies the following condition*:

(C) I is a countable union of its subsets $H_n (n\in\omega)$, and for each $n\in\omega$ and $x\in R\cup L\cup E$, there are points $a,b\in X$ such that $a<x<b$ and $(a,b)\cap H_n=\phi$.

PROOF. "If" part: Suppose that $I=\cup\{H_n : n\in\omega\}$ satisfies the condition (C). Let U be an open subset of $N(X)$. Then we shall show that U is F_σ in $N(X)$ by the following three steps.

Step (1): Let U be an open subset of $I(N)=(I\times(-1,1))\cap N(X)$. Note that $I(N)$ is open in $N(X)$. Set $H_n'=H_n\cap\pi(U)$, where $\pi: X\times(-1,1)\to X$ is the projection. For each $x\in H_n'$, we set $(\{x\}\times(-1,1))\cap U=\cup\{F(x,n,k): k\in\omega\}$, where $F(x,n,k)$ is closed in $N(X)$. Then $G(n,k)=\cup\{F(x,n,k): x\in H_n'\}$ is closed in $N(X)$. We prove this as follows:

Case 1. Let $<y,t>\in N(X)$ with $y\in I-H_n'$. Then $(\{y\}\times(-1,1))\cap N(X)$ is a neighborhood of $\langle y,t\rangle$ in $N(X)$ and does not meet $G(n,k)$.

Case 2. Let $\langle y,t\rangle\in N(X)$ with $y\in R\cup L\cup E$. Then, by the condition (C), there exist $a,b\in X$ such that $a<y<b$ and $(a,b)\cap H_n=\phi$. If $a\in H_n'$ and $(a,y)\neq\phi$, there is an $a'\in X$ such that $a<a'<y$. Then $(\{a'\}\times(0,1))\cap U=\phi$ since $(a,y)\cap H_n'=\phi$. If $a\in H_n'$ and $(a,y)=\phi$, we set $a'=a$. Then $a'\in I_-$ and $(\{a'\}\times(0,1))\cap U=\phi$ since $(\{a'\}\times(0,1))\cap N(X)=\phi$. If $a\notin H_n'$, we set $a'=a$. In all cases we considered, $(\{a'\}\times(0,1))\cap G(n,k)=\phi$. Hence $(\langle a',0\rangle,\langle y,t\rangle]\cap G(n,k)$ $=\phi$. Similarly, there is a $b'\in X$ such that $y<b'\leq b$ and $[\langle y,t\rangle,\langle b',0\rangle)\cap$ $G(n,k)=\phi$. Therefore, $(\langle a',0\rangle,\langle b',0\rangle)$ is a neighborhood of $\langle y,t\rangle$ in $N(X)$ and

does not meet $G(n,k)$.

Case 3. Let $\langle y,t\rangle \in N(X)-G(n,k)$ with $y \in H_n'$. Since $F(x,n,k)$ is closed in $(\{x\}\times(-1,1))\cap N(X)$ for each $x \in H_n'$, there exists a neighborhood of $\langle y,t\rangle$ in $N(X)$ which does not meet $G(n,k)$.

Since $U = \cup\{G(n,k): n\in\omega, k\in\omega\}$, U is F_σ in $N(X)$.

Step (2): Let U be a convex open subset of $N(X)$. Then U can be considered as an interval of $N(X)$ or $N(X)^+$, where $N(X)^+$ is the Dedekind compactification of $N(X)$. We consider the following two cases: (i) U is of the form $(a,b),[a,b),(a,b],(a,\rightarrow)$, etc., where $a,b\in N(X)$; (ii) U is of the form $[a^+,b^+]\cap N(X),[a^+,\rightarrow]\cap N(X)$, etc., where a^+,b^+ are gaps of $N(X)$ and $[a^+,b^+]$ denotes an interval in $N(X)^+$; (iii) U is of the form $[a^+,b)\cap N(X)$ or $(a,b^+)\cap N(X)$.

Case (i): It is sufficient to consider the case $U=(a,b)$, because other cases are similar to and simpler than that case.

First, we prove that $N(X)$ is first countable. Let $\langle x,t\rangle \in N(X)$. Since Y satisfies the ccc, Y is perfect. Hence Y is first countable ([1, 2.1]). If x has the immediate predecessor x', we set $a_k = x'$ for all $k\in\omega$. Otherwise, there exists an increasing sequence $\{a_k : k\in\omega\}$ which converges to x. Similarly, if x has the immediate successor x'', we set $b_k = x''$ for all $k\in\omega$. Otherwise, there exists a decreasing sequence $\{b_k : k\in\omega\}$ which converges to x. Then $\{(a_k,b_k): k\in\omega\}$ is a neighborhood base at $x\in Y$.

Case 1. Let $\langle x,t\rangle \in (L\times\{0\})\cup(R\times\{-1\})$. Then $\{(\langle a_k,0\rangle,\langle x,t\rangle]: k\in\omega\}$ is a neighborhood base at $\langle x,t\rangle$ in $N(X)$.

Case 2. Let $\langle x,t\rangle \in (L\times\{1\})\cup(R\times\{0\})$. Then $\{[\langle x,t\rangle,\langle b_k,0\rangle): k\in\omega\}$ is a neighborhood base at $\langle x,t\rangle$.

Case 3. Let $x\in E$ (hence $t=0$). Then $\{(\langle a_k,0\rangle,\langle b_k,0\rangle): k\in\omega\}$ is a neighborhood base at $\langle x,0\rangle$.

Case 4. If $x\in I$, then it is clear that $N(X)$ is first countable at $\langle x,t\rangle$.

As we have shown that $N(X)$ is first countable, there exist decreasing sequence $\{a_n\}$ converging to a and an increasing sequence $\{b_n\}$ converging to b. Therefore $U=\cup\{[a_n,b_n]: n\in\omega\}$ is an F_σ-set of $N(X)$.

Case (ii): It is sufficient to consider the case $U=[a^+,b^+]\cap N(X)$, and a^+,b^+ are gaps of $N(X)$, because other cases are similar to this case. Since $U = N(X)-((\leftarrow,a^+)\cup(b^+,\rightarrow))\cap N(X)$, U is closed in $N(X)$.

(iii) This is done by mixing proofs of Cases (i) and (ii).

Step (3): Express U as the union of the collection $\{U_\alpha : \alpha\in A\}$ of all convex components of U in $N(X)$. Set $B=\{\alpha\in A: U_\alpha\subset I(N)\}$, $\Lambda=\{\alpha\in A: U_\alpha$ is not contained in $I(N)\}$ and $V=\cup\{U_\alpha:\alpha\in B\}$. Then $U = V\cup(\cup\{U_\alpha:\alpha\in\Lambda\})$, where V is open in $I(N)$ and U_α is a convex open subset of $N(X)$ for each $\alpha \in \Lambda$. Each $U_\alpha(\alpha \in \Lambda)$ contains a point $\langle x,t\rangle$ which belongs to $(E\times\{0\})\cup(L\times\{0,1\})\cup(R\times\{-1,0\})$. It follows that, for each $\alpha\in\Lambda, U_\alpha\cap(X\times\{0\})$ contains a nonvoid open set of Y. Since Y satisfies the ccc, it follows that $|\Lambda|\leq\omega$. V and U_α are F_σ in $N(X)$ as shown in Steps (1) and (2). Hence U is F_σ in $N(X)$. Thus $N(X)$ is perfect.

"Only if" part: If $N(X)$ is perfect, $I(N)=(I\times(-1,1))\cap N(X)$ is an F_σ-set of $N(X)$. Let $I(N)=\cup\{F_n : n\in\omega\}$, where each F_n is closed in $N(X)$. Then $I=\cup\{H_n : n\in\omega\}$, where $H_n=\{x\in X :< x,0 >\in F_n\}$. We shall show that $I=\cup\{H_n : n\in\omega\}$ satisfies the condition (C) as follows:

Case 1. Let $x\in L$. Since $\langle x,0\rangle\notin F_n$ and F_n is closed in $N(X)$, there exists a neighborhood V of $\langle x,0\rangle$ in $N(X)$ such that $V\cap F_n=\phi$. Hence there exists $a\in X$ such that $a<x$ and $(\langle a,0\rangle,\langle x,0\rangle]\subset V$. Therefore, $(a,x]\cap H_n=\phi$. Since $\langle x,1\rangle\notin F_n$, there exists a neighborhood W of $\langle x,1\rangle$ in $N(X)$ such that $W\cap F_n=\phi$. Hence there exists $b\in X$ such that $x<b$ and $[\langle x,1\rangle,\langle b,0\rangle)\subset W$. Hence $[x,b)\cap H_n=\phi$. Therefore, $(a,b)\cap H_n=\phi$.

Case 2. Let $x\in R$. The proof is similar to Case 1.

Case 3. Let $x\in E$. Since $\langle x,0\rangle\notin F_n$, there exists a neighborhood V of $\langle x,0\rangle$ in $N(X)$ such that $V\cap F_n=\phi$. Hence there exist $a,b\in X$ such that $a<x<b$ and $(\langle a,0\rangle,\langle b,0\rangle)\subset V$. Therefore, $(a,b)\cap H_n=\phi$.

This completes the proof of Theorem 2.7.

§3. Examples

In this section, we present several examples.

EXAMPLE 3.1. The following two examples show that the condition "ccc" is needed in Theorems 2.4 and 2.6.

(1) Let $Y=\omega_1, X=GO_Y(\phi,Y,\phi,\phi)=Y$, where ω_1 is the set of all ordinals less than ω_1. Then $L(X)=M(X)=X$ is not perfect, but $|I|=|R\cup L\cup I|=|\phi|\leq\omega$. Notice that Y does not satisfy the ccc.

(2) Let $Y=\omega_1\times[0,1)$ be a LOTS with the lexicographic order. Then Y is the long line (see [4]). Each point may be thought of as $\alpha+x$, where $\alpha\in\omega_1$ and $x\in[0,1)$. Let $X=GO_Y(\ell im\,\omega_1, Y-\omega_1, \omega_1-(\ell im\,\omega_1),\phi)$, where lim ω_1 denotes the set of all limit ordinals less than ω_1. Then it is easy to see that $M(X)=(X\times\{0\})\cup((\ell im\,\omega_1)\times(-1,0))\cup((\omega_1-(\ell im\,\omega_1))\times\{-1,1\})$ and $M(X)$ is a pairwise disjoint union of clopen metrizable spaces. Thus $M(X)$ is metrizable (hence, perfect). But $|I|=|\omega_1-(\ell im\,\omega_1)|=\omega_1>\omega$ and $|R|=|\ell im\,\omega_1|>\omega$. Notice that Y does not satisfy the ccc.

EXAMPLE 3.2. Let $Y=\omega_1\times[0,1)$ be the same space as Example 3.1 (2). Let $X=GO_Y(\omega_1, Y-\omega_1,\phi,\phi)$. Since ω_1 is the set of all ordinals less than ω_1, it follows that X is a pairwise disjoint union of clopen metrizable spaces $\{z:\alpha\le z<\alpha+1,\alpha\in\omega_1\}$, thus X is metrizable (hence, perfect). Since $N(X)=(X\times\{0\})\cup(\omega_1\times\{-1\})$contains a subspace $\omega_1\times\{-1\}, N(X)$ is not perfect. Since $I(=\phi)$ satisfies the condition (C), the ccc is needed in Theorem 2.7.

EXAMPLE 3.3. Let $K=[0,1]-\cup\{(a_n,b_n):n\in\omega\}$ be the Cantor set, $A=\{a_n:n\in\omega\}, B=\{b_n:n\in\omega\}$ and $Y=[0,1]$ be the usual unit interval. Let $X=GO_Y(A,Y-K,K-(A\cup B),B)$. Then X is a metrizable (hence, perfect) space, because $\{\mathfrak{B}(i,n):i,n\in\omega\}\cup\{\{x\}:x\in K-(A\cup B)\}$ is a σ-discrete base for X, where $\{\mathfrak{B}(i,n):n\in\omega\}$ be a σ-discrete base for $[a_i,b_i]$. But $N(X)=(X\times\{0\})\cup(A\times\{-1\})\cup(B\times\{1\})\cup((K-(A\cup B))\times(-1,1))$ is not perfect. On the contrary, suppose that $N(X)$ is perfect. Then an open set $I\times(-1,1)=(K-(A\cup B))\times(-1,1)$ of $N(X)$ is F_σ. Let $I\times(-1,1)=\cup\{F_n:n\in\omega\}$, where each F_n is closed in $N(X)$. Let $H_n=\{x\in K:<x,0>\in F_n\}$. Then $K=(\cup\{H_n:n\in\omega\})\cup(\cup\{\{a_n,b_n\}:n\in\omega\})$ is a countable union of subsets of K. For a while, we consider the usual topology on K. Since $K=(\cup\{Cl_K H_n:n\in\omega\})\cup(\cup\{\{a_n,b_n\}:n\in\omega\}$ is a countable union of closed subsets of K, by the Baire Category Theorem, there is an $n\in\omega$ such that $Cl_K H_n$ contains a non-void open set U of K. We may assume that $U=U'\cap K$, where U' is an open interval in $\mathbf{R}$. We shall show that there exists a point $a_i\in A\cap U'$. Since $U'\cap K\neq\phi$, there is an $x\in U'\cap K$. If $x\in B$, then there is an $a_i\in A$ such that $x<a_i$ and $a_i\in U'$ since U' is an open interval containing x and K is the Cantor set. Similarly, if $x\in K-(A\cup B)$, then there is an $a_i\in A$ such that $a_i<x$ and $a_i\in U'$. Hence there exists an $a_i\in A\cap U'$. Since $a_i\in U\subset Cl_K H_n$. a_i is a cluster point of H_n in K, and hence $\langle a_i,-1\rangle\in N(X)$ is a cluster point of F_n in $N(X)$. This contradicts the closedness of F_n. Therefore, $N(X)$ is not perfect.

It follows from Theorem 2.7 that I does not satisfy the condition (C).

On the other hand, $I=K-(A\cup B)$ is a closed set of X. Therefore this example shows that, in Theorem 2.7, the statement " I satisfies the condition (C)" can not be weakened by " I is F_σ in X".

EXAMPLE 3.4. Let $\mathbf{R}$ and $\mathbf{Q}$ be the set of all real numbers and all rational numbers, respectively. Let K be the Cantor set and $T=\cup\{K+q:q\in\mathbf{Q}\}$ where $K+q=\{x+q:x\in K\}$. Let $X=GO_{\mathbf{R}}(\mathbf{R}-T,\phi,T,\phi)$. Since T satisfies the condition (C), $N(X)$ is perfect by Theorem 2.7. However, $L(X)$ is not perfect by Theorem 2.4. We do not know whether this example has a perfect orderable d-extension.

(This example was announced in [7].)

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