TSUKUBA J. MATH.
Vol. 19 No. 2 (1995), 395–407

GRADED COALGEBRAS AND MORITA-TAKEUCHI CONTEXTS

By

S. DĂSCĂLESCU, C. NĂSTĂSESCU, S. RAIANU and F. Van OYSTAEYEN

0. Introduction

Viewing a G-graded k-coalgebra over the field k as a right kG-comodule coalgebra it is possible to use a Hopf algebraic approach to the study of coalgebras graded by an arbitrary group that was started in [NT].

Let $C=\bigoplus_{g\in G} C_g$ be a G-graded coalgebra. The graded C-comodules may be viewed as comodules over the smash product $C\rtimes kG$, the general definition of which was given in [M]. Coalgebras graded by an arbitrary group have been considered in [FM] in order to introduce the notion of G-graded Hopf algebras. On the other hand, M. Takeuchi introduced in [T] the sets of pre-equivalence data connecting categories of comodules over two coalgebras (we call such a set a Morita-Takeuchi context). The main result of this note is a coalgebra version of a result established by M. Cohen, S. Montgomery in [CM] for group-graded rings: for a graded coalgebra C the coalgebras C_1 and $C\rtimes kG$ are connected by a Morita-Takeuchi context in which one of the structure maps is injective. Most of the results in this note are consequences of the foregoing. As a first application we find that a coalgebra C is strongly graded if and only if the other structure map of the context is also injective. The final section provides analogues of the Cohen-Montgomery duality theorems: if C is a coalgebra graded by the finite group G of order n, then G acts on the smash coproduct as a group of automorphisms of coalgebras and $(C\rtimes kG)\rtimes kG^*$ is coalgebra isomorphic to the comatrix coalgebra $M^c(n, C)$. If G is a finite group of order n, acting on the coalgebra D as a group of coalgebra automorphisms, then the smash coproduct $D\rtimes kG^*$ is strongly graded by G and moreover: $(D\rtimes kG^*)\rtimes kG\cong M^c(n, D)$. The second duality theorem is again a direct consequence of the Morita-Takeuchi context mentioned above.

Received November 25, 1993.

1. Graded Coalgebras and the Smash Coproduct

Throughout this paper k is a field. We use Sweedler's "sigma" notation [S] and further notation and conventions in [T], [D]. Let G be a group with identity element 1. Recall that a k-coalgebra (C, Δ, ε) is graded by G if C is a direct sum of k-subspaces, $C=\bigoplus_{\sigma\in G} C_{\sigma}$, such that $\Delta(C_{\sigma})\subset\sum_{xy=\sigma} C_x\otimes C_y$, for all $\sigma\in G$, and $\varepsilon(C_{\sigma})=0$ for $\sigma\neq 1$. A right C-comodule M with structure map $\rho: M\to M\otimes C$ is a graded C-comodule if $M=\bigoplus_{\sigma\in G} M_{\sigma}$ as k-subspaces, such that $\rho(M_{\sigma})\subset\sum_{xy=\sigma} M_x\otimes C_y$ for all $\sigma\in G$. For graded right C-comodules M and N a graded comodule morphism is a C-comodule morphism $f: M\to N$ such that $f(M_{\sigma})\subset N_{\sigma}$ for $\sigma\in G$. The category of graded right C-comodules, denoted by gr^C, is a Grothendieck category, cf. [NT]. The main purpose of this section is to develop a Hopf algebraic approach to the graded theory. First we recall, see [S] or [A], some definitions.

1.1. DEFINITION. Let H be a bialgebra over the field k, A a k-algebra and $(C, \Delta_C, \varepsilon_C)$ a k-coalgebra. Then:

i. A is said to be a (right) H-module algebra if A is a right H-module such that $(ab)\cdot h=\sum(a\cdot h_1)(b\cdot h_2)$ and $1_A\cdot h=\varepsilon(h)1_A$ for any $h\in H$, and $a, b\in A$.

ii. C is a right H-comodule coalgebra if C is an H-comodule by $c\mapsto\sum c_{(0)}\otimes c_{(1)}$ such that we have:

$$\sum c_{1(0)}\otimes c_{2(0)}\otimes c_{1(1)}c_{2(1)}=\sum c_{(0)1}\otimes c_{(0)2}\otimes c_{1(1)},$$

$$\sum\varepsilon_C(c_{(0)})c_{(1)}=\varepsilon_C(c)1_H \quad \text{for all } c\in C$$

iii. C is a (left) H-module coalgebra if C is a left H-module such that: $\Delta_C(h\cdot c)=\sum h_1\cdot c_1\otimes h_2\cdot c_2$, $\varepsilon_C(h\cdot c)=\varepsilon_H(h)\varepsilon_C(c)$ for $c\in C$, $h\in H$.

In the sequel we shall not refer to "right" of "left" as in the above definitions, the choice of "sides" shall remain fixed throughout.

For any group G the group algebra kG has a bialgebra structure defined by $\Delta(g)=g\otimes g$ and $\varepsilon(g)=1$ for all $g\in G$. The next result establishes the connection between G-graded coalgebras and kG-comodule coalgebras.

1.2. PROPOSITION. *A coalgebra C graded by G many in a natural way be viewed as a kG-comodule coalgebra; conversely every kG-comodule coalgebra is a G-graded coalgebra.*

PROOF. For a G-graded C the map $\rho: C\to C\otimes kG$, $c\mapsto c\otimes\sigma$ for all $\sigma\in G$,

$c\in C_\sigma$, defines a kG-comodule coalgebra structure on C. Conversely, if C is a kG-comodule coalgebra then any $c\in C$ has a unique presentation $\rho(c)=\sum_{g\in G} c_g\otimes g$. Put $C_g=\{c_g, c\in C\}$, $g\in G$; C_g is a k-subspace of C. From $(I\otimes\varepsilon)\rho(c)=c\otimes 1$ we derive that $c=\sum_{g\in G} c_g$ and $C=\sum_{g\in G} C_g$. For $c\in C$, $g\in G$ we have that $c\in C_g$ if and only if $\rho(c)=c\otimes g$. If $\sum_{g\in G} c_g=0$ for some $c_g\in C_g$ then by applying ρ we obtain $\sum c_g\otimes g=0$ or $c_g=0$ for all $g\in G$. Therefore $C=\bigoplus_{g\in G} C_g$. Consider $c\in C_\sigma$ and $\Delta(c)=\sum c_1\otimes c_2$ with homogeneous c_1's and c_2's. From 1.1 we retain that $\sum c_1\otimes c_2\otimes\sigma$ equals $\sum c_1\otimes c_2\otimes \deg c_1\cdot\deg c_2$, or in other words $\Delta(c)$ is the sum of all terms with $\sigma=\deg c_1\cdot\deg c_2$, establishing that C is a G-graded coalgebra. □

We say that the group G acts on the coalgebra D whenever there is a group morphism $\varphi: G\to \mathrm{Aut}(D)$, the latter denoting the set of all coalgebra automorphisms of D with group structure defined as follows: if $f, g\in \mathrm{Aut}(D)$, $f\cdot g=f\circ g$.

1.3. PROPOSITION. *If G acts on the coalgebra D then D has the structure of a kG-module coalgebra; conversely any kG-module coalgebra has a natural G-action.*

PROOF. Suppose that $\varphi: G\to \mathrm{Aut}(D)$ determines that G acts on D then the map $kG\otimes D\to D$, $g\otimes d\mapsto\varphi(g)(d)$ defines a kG-module structure on D as desired. Conversely, if D is a kG-module coalgebra then we may define a G-action on D by $\varphi: G\to \mathrm{Aut}(D)$, $\varphi(g)(d)=g\cdot d$ for $g\in G$, $d\in D$. □

1.4. REMARK. Let, for a finite group G, kG^* be the dual bialgebra for the finite dimensional bialgebra kG. If the finite group G acts on the coalgebra D then D is also a kG^*-comodule coalgebra. If $\{p_g, g\in G\}$ is the dual basis of $\{g, g\in G\}$ then $\{p_g, g\in G\}$ is a system of orthogonal idempotents of kG^*. The coalgebra structure of kG^* is given in the usual way by: $\Delta(p_g)=\sum_{xy=g} p_x\otimes p_y$, $\varepsilon(p_g)=\delta_{g,1}$.

The right comodule structure of D is given by $\rho: D\to D\otimes kG^*$, $\rho(d)=\sum_{g\in G}(g\cdot d)\otimes p_g$.

In the sequel, the smash coproduct plays a central part. For a bialgebra H and an H-module coalgebra C the smash-coproduct $C\rtimes H$ is defined as the k-space $C\otimes H$ with $\Delta: C\rtimes H\to (C\rtimes H)\otimes(C\rtimes H)$ given by $\Delta(c\rtimes h)=\sum(c_1\rtimes c_{2(1)}\cdot h_2)\otimes(c_{2(0)}\rtimes h_1)$, and $\varepsilon: C\rtimes H\to k$ given by $\varepsilon(c\rtimes h)=\varepsilon_C(c)\varepsilon_H(h)$.

1.5. PROPOSITION. *$C \rtimes H$ with Δ and ε as above is a coalgebra.*

PROOF. This is just the right hand version of Theorem 2.11 of [M], a proof is given in Proposition 2.3 of [FM]. □

The smash coproduct is useful in general but has particular interest in some special cases frequently considered:

i. Graded smash coproduct

If the coalgebra C is graded by G then the coalgebra structure of $C \rtimes kG$ is given by: $\Delta(c \rtimes g)=\sum(c_1 \rtimes \deg c_2 \cdot g)\otimes(c_2 \otimes g)$, for any homogeneous $c \in C$ and $g \in G$ (where we assumed, as we will always do in the sequel, that we have used the homogeneous decomposition $\sum c_1 \otimes c_2$), whereas for all $c \in C$, $g \in G$ we have that $\varepsilon(c \rtimes g)=\varepsilon_C(c)$.

ii. If the finite group G acts on the coalgebra D, i.e. D is a kG^*-comodule coalgebra, then the coalgebra structure of $D \rtimes kG^*$ is given by:

$$\Delta(d \rtimes p_g)=\sum_{uv=g}(d_1 \rtimes p_v)\otimes(v \cdot d_2 \rtimes p_u),$$

and

$$\varepsilon(d \rtimes p_g)=\varepsilon_D(d)\delta_{g,1}, \qquad \text{for all } d \in D,\ g \in G.$$

Note that the graded smash coproduct appears in a natural way when one studies graded comodules. Recall that a k-Abelian category is k-equivalent to a category of comodules $\mathcal{M}^c$ over some coalgebra C if and only it it is of finite type (Theorem 5.1 of [T]). The coalgebra giving the category as a category of comodules may, in general, be a somewhat mystical object. However for a G-graded coalgebra C the k-Abelian category of graded comodules, say gr^C, is of finite type and it is therefore, equivalent to a category of comodules over the coalgebra given in the following.

THEOREM 1.6. *If C is a coalgebra graded by G then the categories gr^C* and *and $\mathcal{M}^{C \rtimes kG}$ are isomorphic.*

PROOF. Take $M \in \mathrm{gr}^C$ with $\rho: M \to M \otimes C$, $\rho(m)=\sum m_0 \otimes m_1$. We make M into a right $C \rtimes kG$-comodule by defining $\rho': M \to M \otimes (C \rtimes Kg)$, $m \mapsto \sum m_0 \otimes (m_1 \rtimes (\deg m)^{-1})$ for homogeneous $m \in M$. A morphism $f: M \to N$ of G-graded C-comodules is also a morphism of $C \rtimes kG$-comodules and we have defined a functor $T: \mathrm{gr}^C \to \mathcal{M}^{C \rtimes kG}$.

Conversely, starting from an $M \in \mathcal{M}^{C \rtimes kG}$ we obtain on M a right C-comodule structure and a right kG-comodule structure because the linear maps $\alpha: C\rtimes kG \to C$, $c\rtimes g \mapsto c$, and $\beta: C\rtimes kG\to kG$, $c\rtimes g\mapsto \varepsilon_C(c)g^{-1}$ for $c\in C$, $g\in G$, are coalgebra morphisms. As in the proof of Proposition 1.2 it follows that $M=\bigoplus_{g\in G} M_g$ and a straightforward verification learns that M becomes a graded C-comodule. Now, for $M, N\in \mathcal{M}^{C\rtimes kG}$ and a morphism of $C\rtimes kG$-comodules $f: M\to N$ it follows that f is also a morphism of G-graded C-comodules when M and N are viewed as such. This defines the functors $S: \mathcal{M}^{C\rtimes kG}\to \mathrm{gr}^C$ and it is easily seen that T and S are isomorphisms of categories and inverse to each other.

1.7. REMARKS. 1. If the coalgebra C is graded by a finite group G, then the dual algebra C^* is graded by G with $C^*_g=\{f\in C^*, f(C_x)=0 \text{ for all } x\neq g\}$. Hence C^* is a kG^*-module algebra and we may construct the smash product $C^*\#kG^*$ with multiplication given by: $(c^*\#h^*)(d^*\#g^*)=\sum (c^*(d^*\cdot h^*_1)\#g^*h^*_2$, for all $c^*, d^*\in C\cdot$ and $h^*, g^*\in kG^*$. It is easy to see that the algebra $C^*\#kG^*$ is algebra-isomorphic to the dual algebra of $C\rtimes kG$.

2. If G acts on the coalgebra D via $\varphi: G\to \mathrm{Aut}(D)$, then the group morphism $\bar{\varphi}: G\to \mathrm{Aut}(D^*)$ given by $\bar{\varphi}(g)(d^*)=d^*\varphi(g)$ for $g\in G$, $d^*\in D^*$, defines an action of G on the algebra D^*. Note that $\mathrm{Aut}(D^*)$ is a group with respect to $\sigma\cdot\tau=\tau\circ\sigma$ for $\sigma, \tau\in \mathrm{Aut}(D^*)$. Thus D is a kG-module coalgebra and D^* is a kG-module algebra. If G is finite then D is a kG^*-comodule coalgebra and the dual algebra of the smash coproduct $D\rtimes kG^*$ is isomorphic to the skew group ring $D^*\#kG$.

2. The Morita-Takeuchi Context Associated to a Graded Coalgebra

The Morita-theorems for categories of comodules have been proved by M. Takeuchi in [T]; we call a set of pre-equivalence data as in [T] a Morita-Takeuchi context.

2.1. DEFINITION. A Morita-Takeuchi context $(C, D, {}_CP_D, {}_DQ_C, f, g)$ consists of coalgebras C and D, bicomodules ${}_CP_D$, ${}_DQ_C$ and bicolinear maps $f: C\to P\square_D Q$, $g: D\to Q\square_C P$ making the following diagrams commute:

$$\begin{array}{ccc} P & \xrightarrow{\cong} & P\square_D D \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle I\square g} \\ C\square_C P & \xrightarrow[f\square I]{} & P\square_D Q\square_C P \end{array} \qquad \begin{array}{ccc} Q & \xrightarrow{\cong} & Q\square_C C \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle I\square f} \\ D\square_D Q & \xrightarrow[g\square I]{} & Q\square_C P\square_D Q \end{array}$$

The context is called **strict** if f and g are injective, hence isomorphisms. In this case the categories $\mathcal{M}^C$ and $\mathcal{M}^D$ of comodules over C, resp. D, are equivalent categories.

The following remark extends a corresponding one for Morita contexts given in [CRW].

2.2. PROPOSITION. *Let $(C, D, {}_CP_D, {}_DQ_C, f, g)$ be a Morita-Takeuchi context such that f is injective. Then $\mathcal{M}^C$ is equivalent to a quotient category of $\mathcal{M}^D$.*

PROOF. Theorem 2.5 of [T] yields that f is an isomorphism and the exact functor $S=-\square_D Q : \mathcal{M}^D \to \mathcal{M}^C$, has a right adjoint $T=-\square_C P: \mathcal{M}^C \to \mathcal{M}^D$ such that the natural transformation $f^{-1}: ST \to Id$ is an isomorphism. By a result of P. Gabriel (cf. [G] or Proposition 15.18 of [F]) we have: $\ker S=\{X\in\mathcal{M}^D, X\square_D Q=0\}$ is a localizing subcategory of $\mathcal{M}^D$ and S induces an equivalence from the quotient category $\mathcal{M}/\mathrm{Ker}\, S$ to $\mathcal{M}^C$. □

2.3. COROLLARY. *Let $(C, D, {}_CP_D, {}_DQ_C, f, g)$ be a Morita-Takeuchi context such that f is injective then g is injective (i.e. the context is strict) if and only if ${}_DQ$ is faithfully coflat.* □

PROOF. By Proposition 2.2 the injectivity of g is equivalent to S being an equivalence, again equivalent to $\mathrm{Ker}\, S=\{0\}$ or ${}_DQ$ being faithfully coflat. □

Before establishing the main result of this section let us point out that there is a natural way to associate a graded coalgebra to a given Morita-Takeuchi context. Indeed, if we have a Morita-Takeuchi context $(C, D, {}_CP_D, {}_DQ_C, f, g)$ let $x\mapsto\sum x_{-1}\otimes x_0$, resp. $x\mapsto\sum x_{(0)}\otimes x_{(1)}$, be the left, resp. right, comodule structure of P, resp. Q. The image of $u\in C$ (resp. D) under f (resp. g) in $P\square_D Q$ (resp. $Q\square_C P$) will be denoted by $\sum f(u)_1\otimes f(u)_2$. (resp. $\sum g(u)_1\otimes g(u)_2$).

Put $\Gamma=\begin{pmatrix} C & P \\ Q & D\end{pmatrix}=\left\{\begin{pmatrix} c & p \\ q & d\end{pmatrix}, c\in C,\ d\in D,\ p\in P,\ q\in Q\right\}$.

We make Γ into a coalgebra by defining $\Delta: \Gamma\to\Gamma\otimes\Gamma$ as follows:

$$\Delta\begin{pmatrix} c & 0 \\ 0 & 0\end{pmatrix}=\Sigma\begin{pmatrix} c_1 & 0 \\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix} c_2 & 0 \\ 0 & 0\end{pmatrix}+\Sigma\begin{pmatrix} 0 & f(c)_1 \\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix} 0 & 0 \\ f(c)_2 & 0\end{pmatrix}$$

$$\Delta\begin{pmatrix} 0 & 0 \\ 0 & d\end{pmatrix}=\Sigma\begin{pmatrix} 0 & 0 \\ 0 & d_1\end{pmatrix}\otimes\begin{pmatrix} 0 & 0 \\ d_2 & 0\end{pmatrix}+\Sigma\begin{pmatrix} 0 & 0 \\ g(d)_1 & 0\end{pmatrix}\otimes\begin{pmatrix} 0 & g(d)_2 \\ 0 & 0\end{pmatrix}$$

$$\Delta\begin{pmatrix} 0 & p \\ 0 & 0\end{pmatrix}=\Sigma\begin{pmatrix} p_{-1} & 0 \\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix} 0 & p_0 \\ 0 & 0\end{pmatrix}+\Sigma\begin{pmatrix} 0 & p_{(0)} \\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix} 0 & 0 \\ 0 & p_{(1)}\end{pmatrix}$$

$$\Delta\begin{pmatrix} 0 & 0 \\ q & 0\end{pmatrix}=\Sigma\begin{pmatrix} 0 & 0 \\ 0 & q_{-1}\end{pmatrix}\otimes\begin{pmatrix} 0 & 0 \\ q_0 & 0\end{pmatrix}+\Sigma\begin{pmatrix} 0 & 0 \\ q_{(0)} & 0\end{pmatrix}\otimes\begin{pmatrix} q_{(1)} & 0 \\ 0 & 0\end{pmatrix}$$

for $c\in C$, $d\in D$, $p\in P$, $q\in Q$, and extended linearly, $\varepsilon: \Gamma\to k$ given by $\varepsilon\begin{pmatrix} c & p \\ q & d\end{pmatrix}=\varepsilon_C(c)+\varepsilon_D(d)$. Moreover Γ is $\boldsymbol{Z}$-graded by putting $\Gamma_0=\begin{pmatrix} C & 0 \\ 0 & D\end{pmatrix}$, $\Gamma_{-1}=\begin{pmatrix} 0 & 0 \\ Q & 0\end{pmatrix}$ and $\Gamma_1=\begin{pmatrix} 0 & P \\ 0 & 0\end{pmatrix}$, $\Gamma_k=0$ for $k\neq -1, 0, 1$.

Let $C=\bigoplus_{\sigma\in G} C_\sigma$ be a coalgebra, graded by G. Recall from [NT] that C_1 is a coalgebra with comultiplication $\Delta_1: C_1\to C_1\otimes C_1$ given by $\Delta_1(c)=\sum \pi(c_1)\otimes \pi(c_2)=\sum \pi(c_1)\otimes c_2=\sum c_1\otimes\pi(c_2)$ for all $c\in C_1$, where $\pi: C\to C_1$ is the natural projection. The co-unit of C_1 is just ε_C restricted to C_1. Since π is a coalgebra map, C becomes a left C_1-comodule via the structure map $\rho_1^l: C\to C_1\otimes C$, $c\mapsto\sum\pi(c_1)\otimes c_2$ (c homogeneous) and it becomes a right C_1-comodule via $\rho_1^r: C\to C\otimes C_1$, $c\mapsto \sum c_1\otimes\pi(c_2)$ (c homogeneous). Now C is a graded right C-comodule, so by Theorem 1.6 C is a right $C\rtimes kG$-comodule via the map

$$\rho_2^r: C\to C\otimes(C\rtimes kG),\ c\mapsto \sum c_1\otimes(c_2\rtimes(\deg c)^{-1})$$

for c homogeneous. For any homogeneous $c\in C$, we have $(I\otimes\rho_2^r)\rho_1^l(c)=(\rho_1^l\otimes I)\rho_2^r(c)=\sum\pi(c_1)\otimes c_2\otimes(c_3\rtimes(\deg c)^{-1})$; thus C becomes a left C_1, right $C\rtimes kG$-bicomodule. In a similar way C becomes a left $C\rtimes kG$, right C_1-bicomodule where the left $C\rtimes kG$-comodule-structure of C is given by $\rho_2^l(c)=\sum(c_1\rtimes\deg c_2)\otimes c_2$, for any homogeneous $c\in C$.

Define $f: C_1\to C\square_{C\rtimes kG}C$, $c\mapsto\sum c_1\otimes c_2=\Delta_C(c)$. Observe that for any $c\in C_1$ we obtain:

$$\begin{aligned}\sum\rho_2^r(c_1)\otimes c_2&=\sum c_1\otimes c_2(\deg c_2)^{-1}(\deg c_1)^{-1}\otimes c_3\\&=\sum c_1\otimes c_2\otimes\deg c_3\otimes c_3\\&=\sum c_1\otimes\rho_2^l(c_2)\end{aligned}$$

so the definition of f above is satisfactory. Moreover, f is a morphism of left and right C_1-comodules as is easily verified. Note also that f is injective because it is the restriction of the comultiplication of C to C_1.

Next define $g: C\rtimes kG\to C\square_{C_1}C$, $c\rtimes x\mapsto\sum c_1\otimes\pi_{x^{-1}}(c_2)$ for $x\in G$ and homogeneous $c\in C$, where π_x denotes the projection from C to C_x. In order to have that g is well-defined it is necessary that: $\sum(c_1)_1\otimes\pi((c_1)_2)\otimes\pi_{x^{-1}}(c_2)=\sum c_1\otimes \pi(\pi_{x^{-1}}((c_2)_1))\otimes\pi_{x^{-1}}((c_2)_2)$. However the left hand side is obtained from $\sum c_1\otimes c_2\otimes c_3$ by collecting the terms with $\deg c_2=1$ and $\deg c_3=x^{-1}$; on the other hand the right hand sum is an expression of the same thing. Moreover g is a morphism of right (and left) $C\rtimes kG$-comodules; this follows from: $\sum_{\deg c_2=x^{-1}}(c_1\otimes$

$(c_2)_1)\otimes((c_2)_2 \rtimes x)=\sum_{\deg(c_1)_2=x^{-1}(\deg c_2)^{-1}}((c_1)_1\otimes(c_1)_2)\otimes(c_2 \rtimes x)$ because both members are actually equal to: $\sum_{\deg c_2 \deg c_3=x^{-1}}(c_1\otimes c_2)\otimes(c_3 \rtimes x)$. The other assertion (left) follows in a similar way.

2.4. THEOREM. *With notation as above*: $(C_1, C\rtimes kG, {}_{C_1}C_{C\rtimes kG}, {}_{C\rtimes kG}C_{C_1}, f, g)$ *is a Morita-Takeuchi context. The map f is injective hence an isomorphism.*

PROOF. The only thing left to be proved is that f and g do satisfy the compatibility conditions, i.e. the following diagrams are commutative:

$$\begin{array}{ccc} C & \xrightarrow[\cong]{\theta} & C\square_{C\rtimes kG}C\rtimes kG \\ \cong\downarrow\phi & & \downarrow I\square g \\ C_1\square_{C_1}C & \xrightarrow[f\square I]{} & C\square_{C\rtimes kG}C\square_{C_1}C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow[\cong]{\theta'} & C\square_{C_1}C \\ \cong\downarrow\phi' & & \downarrow I\square f \\ (C\rtimes kG)\square_{C\rtimes kG}C & \xrightarrow[g\square I]{} & C\square_{C_1}C\square_{C\rtimes kG}C\,. \end{array}$$

Now for $c\in C_x$ we have: $(I\square g)\theta(c)=(I\square g)(\sum c_1\otimes(c_2\rtimes x^{-1}))=\sum c_1\otimes c_2\otimes\pi_x(c_3)=\sum_{\deg c_3=x} c_1\otimes c_2\otimes c_3$, and also $(f\square I)(\phi(c))=(f\square I)(\sum \pi(c_1)\otimes c_2)=(f\square I)(\sum_{\deg c_1=1} c_1\otimes c_2)=(f\square I)(\sum_{\deg c_2=x} c_1\otimes c_2)=\sum_{\deg c_3=x} c_1\otimes c_2\otimes c_3$.

That proves commutativity of the first diagram. For the second diagram we just compute: $(I\square f)\theta'(c)=(I\square f)(\sum c_1\otimes\pi(c_2))=(I\square f)(\sum_{\deg c_2=1} c_1\otimes c_2)=(I\square f)(\sum_{\deg c_1=x} c_1\otimes c_2)=\sum_{\deg c_1=x} c_1\otimes c_2\otimes c_3$ and also $(g\square I)\phi'(c)=(g\square I)(\sum(c_1\rtimes \deg c_2)\otimes c_2)=\sum_{\deg c_2=(\deg c_3)^{-1}} c_1\otimes c_2\otimes c_3=\sum_{\deg c_2 \deg c_3=1} c_1\otimes c_2\otimes c_3=\sum_{\deg c_1=x} c_1\otimes c_2\otimes c_3$. □

2.5. COROLLARY. *Jf $C=\bigoplus_{\sigma\in G} C_\sigma$ is a graded coalgebra then $\mathcal{M}^{C_1}$ is equivalent to a quotient category of gr^C.*

PROOF. A consequence of Theorem 1.6, Theorem 2.4 and Proposition 2.2. □

Recall that a G-graded coalgebra $C=\bigoplus_{\sigma\in G} C_\sigma$ is said to be strongly graded if the canonical k-linear map $\gamma_{u,v}: C_{u,v}\to C_u\otimes C_v$, $c\mapsto\sum \pi_u(c_1)\otimes\pi_v(c_2)$, is injective for all $u, v\in G$ (see [NT]). The next result establishes that strongly graded coalgebras may be characterized using the Morita-Takeuchi context from Theorem 2.4 just like in the case of group-graded rings (see [CM]).

2.6. COROLLARY. *Let $C=\bigoplus_{\sigma\in G} C_\sigma$ be a G-graded coalgebra, then the following assertions are equivalent*:

1. *C is strongly G-graded*
2. *The context given in Theorem 2.4. is strict*
3. *C is faithfully coflat as a left $C\rtimes kG$-comodule.*

PROOF. 2.⇒1. Take $u, v\in G$ and $c\in C_{uv}$ such that we have: $\gamma_{u,v}(c)=\sum \pi_u(c_1)\otimes\pi_v(c_2)=0$. Then $g(c\rtimes v^{-1})=\sum c_1\otimes\pi_v(c_2)=\sum\pi_u(c_1)\otimes\pi_u(c_2)=0$, hence $c\rtimes v^{-1}=0$ and $c=0$.

1.⇒2. Let $\alpha=\sum c_i\rtimes x_i\in C\rtimes kG$ with c_i homogeneous of degree σ_i. Suppose that for $i\neq j$ we have $(\sigma_i, x_i)\neq(\sigma_j, x_j)$. If $g(\alpha)=0$ then $\sum_{i,(c_i)}(c_i)_1\otimes\pi_{x_i^{-1}}((c_i)_2)=0$, therefore $\sum_{i,(c_i)}\pi_{\sigma_i x_i}((c_i)_1)\otimes\pi_{x_i^{-1}}((c_i)_2)=0$. On the other hand: $\pi_{i\sigma x_i}((c_i)_1)\otimes\pi_{x_i^{-1}}(c_i)_2)\in C_{\sigma_i x_i}\otimes C_{x_i^{-1}}$. Since $C\otimes C=\bigoplus_{u,v\in G}C_u\otimes C_v$ we obtain for fixed i, the relation: $\sum_{(c_i)}\pi_{\sigma_i x_i}((c_i)_1)\otimes\pi_{x_i^{-1}}((c_i)_2)=0$. The latter yields $\gamma_{\sigma_i x_i, x_i^{-1}}(c_i)=0$ and therefore $c_i=0$ for every choice of i, i. e. $\alpha=0$ follows.

2.⇔3. Follows from Corollary 2.3. □

As a further application we reobtain Theorem 5.3 of [NT] which is a coalgebra version of a well-known result of E. Dade.

2.7. COROLLARY. *The graded coalgebra C is strongly graded if and only if the induced functor $-\square_{C_1}C: \mathcal{M}^{C_1}\to\mathrm{gr}^C$ is an equivalence of categories.*

2.8. REMARK. The functor $(-)_1: \mathrm{gr}^C\to\mathcal{M}^{C_1}$, $M\mapsto M_1$, is naturally isomorphic to the functor $-\square_{C\rtimes kG}G$ since they are both left adjoints of the induced functor $-\square_{C_1}C$ (see [NT] Proposition 4.1, [T] Remark 2.4). Therefore the localizing category implicit in Corollary 2.5 is just $\mathrm{Ker}(-)_1=\mathrm{Ker}(-\square_{C\rtimes kG}C)$.

As a final application of these techniques let us include a short proof of Corollary 6.4 in [NT].

2.9. COROLLARY. *If C is a strongly graded coalgebra for the group G then G is a finite group.*

PROOF. If G is infinite we could select a non-zero homogeneous $c\in C$ and $x\in G$ such that $x\neq\deg(c_2)^{-1}$ for all c_2. Then $g(c\rtimes x)=0$, but that would contradict injectivity of g. □

3. Duality.

For a quasi-finite right C-comodule M, the so-called coalgebra of "co-endomorphisms" of M has been defined in [T., 1.17] and it is denoted by $e_{-C}(M)$. Unfortunately this coalgebra is not easy to use because of the rather complex comultiplication, so it will be useful to give a nicer description of $e_{-C}(M)$ in some particular situation, e. g. in case M is a finitely cogenerated free-comodule (that is, $M\cong X\otimes C$, for some finite dimensional k-vectorspace X, with the obvious comodule structure).

Let C be a coalgebra, X an n-dimensional k-space with basis $\{x_1, \cdots, x_n\}$. Consider the $n\times n$ comatrix coalgebra $M^C(n, k)$ which is a k-space with basis $\{x_{ij}, 1\leqq i, j\leqq n\}$ and Δ, ε given as follows: $\Delta(x_{ij})=\sum_p x_{ip}\otimes x_{pj}$, $\varepsilon(x_{ij})=\delta_{ij}$.

The $n\times n$ comatrix coalgebras over C, denoted by $M^c(n, C)$ is defined to be the tensor product of coalgebra $C\otimes M^c(n, k)$. We endow $C\otimes X$ with a left C-and a right $M^c(n, C)$-bicomodule structure as follows. The left C-comodule structure is given by by the map: $\rho_1^l: C\otimes X\rightarrow C\otimes C\otimes X$, $c\otimes x\mapsto\sum c_1\otimes c_2\otimes x$. The right $M^c(n, C)$-comodule structure is given by the map: $\rho_2^r: C\otimes X\rightarrow C\otimes X\otimes M^c(n, C)$, $c\otimes x_i\mapsto\sum_p c_1\otimes x_p\otimes c_2\otimes x_{pi}$.

In a similar way $C\otimes X$ is a left $M^c(n, C)$-right C-bicomodule via the structure maps:

$$\rho_1^r: C\otimes X\rightarrow C\otimes X\otimes C,\ c\otimes x\mapsto\sum c_1\otimes x\otimes c_2$$

$$\rho_2^l: C\otimes X\rightarrow M^c(n, C)\otimes C\otimes X,\ c\otimes x_i\mapsto\sum_p c_1\otimes x_{ip}\otimes c_2\otimes x_p$$

Define $f: C\rightarrow(C\otimes X)\square_{M^c(n,C)}(C\otimes X)$, $c\mapsto\sum_{i,(c)}(c_i\otimes x_i)\otimes(c_2\otimes x_i)$, which is obviously injective and C-bicolinear. Define $g: M^c(n, C)\rightarrow(C\otimes X)\square_C(C\otimes X)$, $c\otimes x_{ij}\mapsto\sum(c_1\otimes x_i)\otimes(c_2\otimes x_j)$ which is also injective and $M^c(n, C)$-bicolinear. One easily verifies the following relations:

$$(I\square f)\rho_1^r(c\otimes x_i)=(g\square I)\rho_2^l(c\otimes x_i)=\sum_p c_1\otimes x_i\otimes c_2\otimes x_p\otimes c_3\otimes x_p$$

$$(f\square I)\rho_1^l(c\otimes x_i)=(I\square g)\rho_2^r(c\otimes x_i)=\sum_p c_1\otimes x_p\otimes c_2\otimes x_p\otimes c_3\otimes x_i$$

According to results of [T] we immediately obtain:

3.1. PROPOSITION. *$(C, M^c(n, C), C\otimes X, C\otimes X, f, g)$ is a strict Morita-Takeuchi context. In particular we have coalgebra isomorphisms:*

$$e_{C-}(C\otimes X)\cong M^c(n, C)\cong e_{-C}(C\otimes X)$$

3.2. THEOREM. *Let G be a finite group acting on the coalgebra D, then $D\rtimes kG^*$ is a strongly graded coalgebra and there exist coalgebra isomorphisms:*

$$(D\rtimes kG^*)\rtimes kG\cong e_{D-}(D\rtimes kG^*)\cong M^c(n, D)$$

where $n=|G|$.

PROOF. The map $\rho: D\otimes kG^*$, $d\mapsto\sum_g(g\cdot d)\otimes p_g$, makes D into a kG^*-comodule. The comultiplication of $D\rtimes kG^*$ is given by $\Delta(d\rtimes p_x)=\sum_{uv=x}(d\rtimes p_v)\otimes(vd_2\rtimes p_u)$. This establishes that $D\rtimes kG^*$ is a graded coalgebra of type G with grading given by $(D\rtimes kG^*)_g=D\rtimes p_{g^{-1}}$. The canonical morphism $D\rtimes p_1\rightarrow$

$(D\rtimes p_{\sigma^{-1}})\otimes(D\rtimes p_{\sigma})$, $d\rtimes p_1\mapsto\sum(d_1\rtimes p_{\sigma^{-1}})\otimes(\sigma^{-1}d_2\rtimes \rho_{\sigma})$, is clearly injective. Thus $D\rtimes kG^*$ is a strongly graded coalgebra, and $(D\rtimes kG^*)_1=D\rtimes p_1\cong D$. Applying the Morita-Takeuchi context (constructed in Section 2) to $D\rtimes kG^*$, we have a strict context and so it provides us with coalgebra isomorphisms:

$$(D\rtimes kG^*)\rtimes kG\cong e_{(D\rtimes p_1)}\text{-}(D\rtimes kG^*)\cong e_{D}\text{-}(D\rtimes kG^*)\,.$$

The left $(D\rtimes p_1)$-structure of $D\rtimes kG^*$ is given by $d\rtimes p_x\mapsto\sum(d_1\rtimes p_1)\otimes(d_2\rtimes p_x)$, and this yields exactly the left D-comodule structure of $D\otimes X$ where $X=kG^*$ is a k-space of dimension n. Proposition 3.1 yields the second isomorphism. □

A similar result holds for graded coalgebras (or coactions).

3.3. THEOREM. *Let C be a coalgebra graded by the finite gronp G. Then G acts on the coalgebra $C\rtimes kG$ and there are coalgebra isomorphisms:*

$$(C\rtimes kG)\rtimes kG^*\cong e_C\text{-}(C\rtimes kG)\cong M^c(n,\,C)$$

PROOF. An action of G on the coalgebra $C\rtimes kG$ is given by $h\cdot(c\rtimes g)=c\rtimes gh^{-1}$, $g,\,h\in G$ and $c\in C$. Thus $C\rtimes kG$ becomes a kG^*-comodule coalgebra via the map:

$$c\rtimes g\mapsto\sum_y y\cdot(c\rtimes g)\otimes p_y=\sum_y(c\rtimes gy^{-1})\otimes p_y\,.$$

The comultiplication of $(C\rtimes kG)\rtimes kG^*$ is given by

$$\Delta((c\rtimes x)\rtimes p_g)=\sum_{uv=g}((c_1\rtimes \deg c_2\cdot x)\rtimes p_v)\otimes((c_2\rtimes xv^{-1})\rtimes p_u)$$

for any $x,\,g\in G$ and homogeneous $c\in C$. Now let $\{e_{x,y},\ x,\,y\in G\}$ be a basis for $M^c(n,\,k)$. Define a map $F\colon(C\rtimes kG)\rtimes kG^*\to M^c(n,\,C)$, $(c\rtimes x)\rtimes p_g\mapsto c\otimes e_{\alpha,\beta}$ where $\alpha=\deg c\cdot x$, $\beta=xg^{-1}$ for $x,\,g\in G$ and homogeneous $c\in C$. Let us check that F is a coalgebra morphism. Indeed,

$$\begin{aligned}\Delta(F((c\rtimes x)\rtimes p_g))&=\Delta(c\otimes e_{\alpha,\beta})\\&=\sum_{z,(c)}(c_1\otimes e_{\alpha,z})\otimes(c_2\otimes e_{z,\beta})\end{aligned}$$

and also

$$\begin{aligned}(F\otimes F)(\Delta((e\rtimes x)\rtimes p_g)&=\sum_{uv=g}(c_1\otimes e_{\deg c_1\deg c_2x\,\deg c_2xv^{-1}})\otimes(c_2\otimes e_{\deg c_2xv^{-1},\,xv^{-1},\,u^{-1}})\\&=\sum_v(c_1\otimes e_{\alpha,\deg c_2xv^{-1}})\otimes(c_2\otimes e_{\deg c_2xv^{-1},\beta})\,.\end{aligned}$$

Since $\{\deg c_2xv^{-1},\ v\in G\}=G$, both sums are equal. Now, consider $(c\rtimes x)\rtimes p_g\in(C\rtimes kG)\rtimes kG^*$ for $x,\,g\in G$ and c homogeneous. Write ε for the co-unit of $(C\rtimes kG)\rtimes kG^*$ and ε' for the co-unit of $M^c(n,\,C)$. Then we have:

$$\varepsilon((c \rtimes x) \rtimes p_g) = \varepsilon_C(c)\delta_{\deg c, 1}\delta_{g, 1}$$

$$\begin{aligned}\varepsilon'(c \otimes e_{\alpha, \beta}) &= \varepsilon_C(c)\delta_{\deg c, 1}\delta_{\deg cx, xg^{-1}} \\ &= \varepsilon_C(c)\delta_{\deg c, 1}\delta_{x, xg^{-1}} = \varepsilon_C(c)\delta_{\deg c, 1}\delta_{1, g^{-1}} \\ &= \varepsilon_C(c)\delta_{\deg c, 1}\delta_{g, 1}\,.\end{aligned}$$

Therefore F is a coalgebra map as claimed. Now define $H: M^c(n, C) \to (C \rtimes kG) \rtimes kG^*$ by putting $H(c \otimes c_{u,v}) = (c \rtimes (\deg c)^{-1}u) \rtimes p_{v^{-1}(\deg c)^{-1}u}$, for $u, v \in G$ and homogeneous $c \in C$. Again H is a coalgebra morphism because:

$$\Delta(H(c \otimes e_{u,v})$$

$$= \sum_{zt = v^{-1}(\deg c)^{-1}u} ((c_1 \rtimes \deg c_2(\deg c)^{-1}u) \rtimes p_t) \otimes ((c_2 \rtimes (\deg c)^{-1}ut^{-1}(\rtimes p_z)$$

$$(H \otimes H)(\Delta(c \otimes e_{u,v}))$$

$$= (H \otimes H)(\textstyle\sum_h (c_1 \otimes e_{u,h}) \otimes (c_2 \otimes e_{h,v}))$$

$$= \textstyle\sum_h ((c_1 \rtimes (\deg c_1)^{-1}u) \rtimes p_{h^{-1}(\deg c_1)^{-1}u}) \otimes ((c_2 \rtimes (\deg c_2)^{-1}h) \rtimes p_{v^{-1}(\deg c_2)^{-1}h})\,.$$

For fixed c_1 and u we have that $\{h^{-1}(\deg c_1)^{-1}u),\ h \in G\} = G$ and if we write $t = h^{-1}(\deg c_1)^{-1}u$, $z = v^{-1}(\deg c_2)^{-1}h$, then the above sums are clearly equal as desired. The fact that H preserves the co-unit too is obvious. Finally it is clear that $F \cdot H$ and $H \cdot F$ are the identities so that we do arrive at a coalgebra isomorphism. The isomorphism involving $e_{c\_}(C \rtimes kG)$ is obvious because of Proposition 3.1 (the left C-comodule structure of $C \rtimes kG$ is given bp $c \rtimes g \mapsto \sum c_1 \otimes (c_2 \rtimes g)$). □

3.4. COROLLORY. *There exists a strict Morita-Tekeuchi context connecting C and $(C \rtimes kG) \rtimes kG^*$.*

PROOF. $C \rtimes kG$ is a left C-comodule that is a quasi-finite injective cogenerator (in view of Proposition 3.1 and [T]). Moreover $C \rtimes kG$ is a right $(C \rtimes kG) \rtimes kG^*$-comodule via $c \rtimes g \mapsto \sum_u (c_1 \rtimes \deg c_2 gu) \otimes (c_2 \rtimes gu) \rtimes p_{u^{-1}}$, for $g \in G$ and homogeneous $c \in C$. Hence $C \rtimes kG$ is a $C-(C \rtimes kG) \rtimes kG^*$-bicomodule. The assertion now follows from [T, Theorem 3.5 iv]. □

3.5. REMARKS. The Morita-Takeuchi context of the above corollary may be given in detail. This may have an independent interest because it provides another proof of Theorem 3.3 and provides a hint for establishing a more general duality result we do not dwell upon here. The second bicomodule is also $C \rtimes kG$ with right C-comodule structure given by the map: $c \rtimes g \mapsto \sum (c_1 \rtimes \deg c_2 g) \otimes c_2$ (for homogeneous c) and left $(C \rtimes kG) \rtimes kG^*$-comodule structure given by: $c \rtimes g \mapsto \sum_h (c_1 \rtimes \deg c_2 g) \rtimes p_h \otimes (c_2 \rtimes gh)$ (for homogeneous c) we have $f: C \to (C \rtimes kG) \square_{(C \rtimes kG) \rtimes kG^*} (C \rtimes kG)$, $f(c) = \sum_h (c_1 \rtimes \deg c_2 h) \otimes (c_2 \rtimes h_2)$ for homogeneous $c \in C$, $g: (C \rtimes kG) \rtimes kG^* \to (C \rtimes kG) \square_C (C \rtimes kG)$, $g((c \rtimes g) \rtimes p_h) = \sum (c_1 \rtimes \deg c_2 g) \otimes (c_2 \rtimes gh)$, for homogeneous $c \in C$. It is also easily seen that f and g are injective maps.

Acknowledgement. We thank Akira Masuoka for bringing paper [T] to our attention.

References

[A] E. Abe, Hopf Algebras, Cambridge University Press, 1977.

[CM] M. Cohen, S. Montgomery, Group Graded Rings, Smash Products and Group Actions, Trans. AMS, **282** (1984), 237-258.

[CRW] M. Cohen, S. Raianu and S. Westreich, Semiinvariants of Hopf Algebra Actions, Israel J. Math., **88** (1994), 279-306.

[D] Y. Doi, Homological Coalgebra, J. Math. Soc. Japan, **33** (1981), 31-50.

[F] C. Faith, Algebras, Rings, Modules and Categories, Vol. I, Springer Verlag, Berlin, 1973.

[FM] D. Fischman and S. Montgomery, Biproducts of Braided Hopf Algebras and a Double Centralizer Theorem for Lie Color Algebras, preprint.

[G] P. Gabriel, Des Catégories Abeliennes, Bull. Soc. Math. France, **90** (1962), 323-448.

[M] R.K. Molnar, Semi-Direct Products of Hopf Algebras, J. Algebra, **47** (1977), 29-51.

[NT] C. Năstăsescu and B. Torrecillas, Graded Coalgebras, Tsukuba Math. J., **17** (1993), 461-479.

[S] M. Sweedler, Hopf Algebras, Benjamin, 1969.

[T] M. Takeuchi, Morita Theorem for Categories of Comodules, J. of Fac. of Sci., Univ. of Tokyo, Vol. 24, no. 3, 1977, 629-644.

S. Dăscălescu, C. Năstăsescu, S. Raianu
Facultatea de Matematică
Str. Academiei 14
RO-70109, Bucharest, 1
Romania

F. Van Oystaeyen
Dept. of Mathematics
UIA, Universiteitsplein 1
B-2610 Antwerpen, Wilrijk
Belgium

Added in proof. A general duality result for crossed coproducts was proved by S. Dāsoālesae, S. Raianu, Y. Zhang in "Finite Hopf-Galois coextensions, crossed coproducts and duality", to appear in J. Algelma.