TSUKUBA J. MATH.
Vol. 19 No. 2 (1995), 285–304

ON THE GAUSS MAP OF RULED SURFACES IN A 3-DIMENSIONAL MINKOWSKI SPACE

By

Soon Meen CHOI

§1. Introduction.

Relative to Takahashi's theorem [9] for minimal submanifolds, the idea of submanifolds of finite type in a Euclidean space was introduced by Chen [2] and the theory is recently greatly developed. Let $x: M\rightarrow \boldsymbol{R}^{n+1}$ be an isometric immersion of n-dimensional Riemannian manifold into an $(n+1)$-dimensional Euclidean space $\boldsymbol{R}^{n+1}$ and Δ the Laplacian on M. As a generalization of Takahashi's theorem for the case of hypersurfaces, Garay [4] considered the hypersurface satisfying the condition $\Delta x=Ax$, where A denotes the constant diagonal matrix of order $n+1$.

On the other hand, let $x: M\rightarrow \boldsymbol{R}^{m}$ be an isometric immersion of a compact oriented n-dimensional Riemannian manifold into $\boldsymbol{R}^{m}$. For a generalized Gauss map $G: M\rightarrow G(n, m)\subset \boldsymbol{R}^{N}$ $\left(N=\binom{m}{n}\right)$ of x, where $G(n, m)$ is the Grassmann manifold consisting of all oriented n-planes through the origin of $\boldsymbol{R}^{m}$, Chen and Piccinni [3] characterized the submanifold satisfying the condition $\Delta G=\lambda G$ $(\lambda\in \boldsymbol{R})$. For a hypersurface M in $\boldsymbol{R}^{n+1}$ and a unit vector field ξ normal to M, we can regard $\xi(p)$ $(p\in M)$ as a point in an n-dimensional unit sphere $S^{n}(1)$ by translating parallelly to the origin in the ambient space $\boldsymbol{R}^{n+1}$. The map ξ of M into $S^{n}(1)$ is called a *Gauss map* of M in $\boldsymbol{R}^{n+1}$. Recently for the Gauss map of a surface in $\boldsymbol{R}^{3}$ the following theorem is proved by Baikoussis and Blair [1].

THEOREM. *The only ruled surfaces in* $\boldsymbol{R}^{3}$ *whose Gauss map* ξ *satisfies*

$$\Delta\xi=A\xi, \qquad A\in Mat\ (3, \boldsymbol{R}) \tag{1.1}$$

are locally the plane and the circular cylinder.

It seems to be interesting to investigate the Lorentz version of the above theorem. Now, let $\boldsymbol{R}_{1}^{m+1}$ be an $(m+1)$-dimensional Minkowski space with standard coordinate system $\{x_{A}\}$ whose line element ds^{2} is given by $ds^{2}=-(dx_{0})^{2}+$

Partially supported by TERC-KOSEF.
Received August 20, 1993.

$\sum_{i=1}^{m}(dx_i)^2$. Let $S_1^m(c)$ (resp. $H^m(c)$) be an m-dimensional de Sitter space (resp. a hyperbolic space) of constant curvature c in $\boldsymbol{R}_1^{m+1}$. We denote by $M^m(\varepsilon)$ a de Sitter space $S_1^m(1)$ or a hyperbolic space $H^m(-1)$, according as $\varepsilon=1$ or $\varepsilon=-1$. Let M be a space-like or time-like surface in $\boldsymbol{R}_1^3$ and ξ a unit vector field normal to M. Then, for any point p in M, we can regard $\xi(p)$ as a point in $H^2(-1)$ or $S_1^2(1)$ by translating parallelly to the origin in the ambient space $\boldsymbol{R}_1^3$, according as the surface M is space-like or time-like. The map ξ of M into $M^2(\varepsilon)$ is called a *Gauss map* of M into $\boldsymbol{R}_1^3$. Then we prove the following

THEOREM. *The only space-like or time-like ruled surfaces in $\boldsymbol{R}_1^3$ whose Gauss map $\xi: M\to M^2(\varepsilon)$ satisfies* (1.1) *are locally the following spaces*:

i. $\boldsymbol{R}_1^2$, $S_1^1\times\boldsymbol{R}^1$ *and* $\boldsymbol{R}_1^1\times S^1$ *if* $\varepsilon=1$,

ii. $\boldsymbol{R}^2$ *and* $H^1\times\boldsymbol{R}^1$ *if* $\varepsilon=-1$.

In §2 we define a space-like or time-like ruled surface M in $\boldsymbol{R}_1^3$. Roughly speaking, non-degenerate ruled surfaces are divided into two types: Cylindrical surfaces, non-cylindrical surfaces. The main theorem is proved for each case in §3 and §4, §5.

The author would like to express her gratitude to Professor Hisao Nakagawa for his useful advice.

§2. Ruled surfaces.

First of all, we recall one of fundamental properties in a 3-dimensional Lorentz vector space. Let $V=V^3$ be a 3-dimensional vector space with scalar product $\langle\,,\rangle$ of index 1. Then V is called a *Lorentz vector space*. In the rest of this paper, we shall identify a vector X with a transpose ${}^tX$ of X. For any vectors $X=(X_A)$ and $Y=(Y_A)$ in a Lorentz vector space V the scalar product of X and Y is defined by $\langle X, Y\rangle=-X_0Y_0+X_1Y_1+X_2Y_2$, which is called a *Lorentz product*. Let V be a 3-dimensional Lorentz vector space with Lorentz product $\langle\,,\rangle$. Then a Lorentz cross product $X\times Y$ is defined by

$$(-X_1Y_2+X_2Y_1,\ X_2Y_0-X_0Y_2,\ X_0Y_1-X_1Y_0).$$

Then it is easily seen that the Lorentz cross product satisfies the following.

LEMMA 2.1.

(2.1) $$X\times Y=0 \Leftrightarrow X \text{ and } Y \text{ are linearly dependent},$$

(2.2) $$X\times Y=-Y\times X\,,$$

(2.3) $\langle X\times Y, X\rangle=\langle X\times Y, Y\rangle=0$,

(2.4) $\langle X\times Y, Z\rangle=\langle Y\times Z, X\rangle$,

(2.5) X *or* Y : *time-like* $\Rightarrow X\times Y$: *space-like,*

(2.6) $\langle X\times Y, X\times Y\rangle=\langle X, Y\rangle^2-\langle X, X\rangle\langle Y, Y\rangle$.

A time-like or null vector in the Lorentz vector space V is said to be *causal.* For the Lorentz vector space the next two lemmas are given. See Greub [6].

LEMMA 2.2. *There are no causal vectors in* V *orthogonal to a time-like vector.*

LEMMA 2.3. *Two null vectors are orthogonal if and only if they are linearly dependent.*

Throughout this paper, we assume that all objects are smooth and all surfaces are connected, unless otherwise mentioned. Now, we define a ruled surface in $\boldsymbol{R}_1^3$. Let I and J be open intervals containing 0 in the real line $\boldsymbol{R}$. Let $\alpha=\alpha(u)$ be a curve on J into $\boldsymbol{R}_1^3$ and $\beta=\beta(u)$ a vector field along α orthogonal to α. A *ruled surface* M in $\boldsymbol{R}_1^3$ is defined as a semi-Riemannian surface swept out by the vector field β along the curve α. Then M always has a parametrization

(2.7) $$x(u, v)=\alpha(u)+v\beta(u), \qquad u\in J,\ v\in I,$$

where we call α a *base curve* and β a *director curve.* In particular, if β is constant, then it is said to be *cylindrical,* and if it is not so, then the surface is said to be *non-cylindrical.* Since our discussion is local, we may assume that we always have $\beta'(u)\neq 0$ in the non-cylindrical case. That is, the direction of the rulings is always changing.

The natural basis $\{x_u, x_v\}$ along the coordinate curves are given by

$$x_u=dx\left(\frac{\partial}{\partial u}\right)=\alpha'+v\beta', \qquad x_v=dx\left(\frac{\partial}{\partial v}\right)=\beta .$$

Accordingly we see

$$g(x_u, x_v)=g(\alpha', \alpha')+2vg(\alpha', \beta')+v^2g(\beta', \beta'),$$

$$g(x_u, x_v)=0,$$

$$g(x_v, x_v)=g(\beta, \beta).$$

Since M is a semi-Riemannian surface, it suffices to consider the case that α is a space-like or time-like curve and β is a unit space-like or time-like vector field. The ruled surface M is said to be of *type I* or *type II*, according as the base curve α is space-like or time-like. First, we divide the ruled surface of type I into three types. In the case that β is space-like, it is said to be of *type* I_{+}^{0} *or* I_{+}, according as β' is null or non-null. If β is time-like, it is said to be of *type* I_{-}. Since we have $g(\beta, \beta')=0$, if M is of *type* I_{-}, then β' is to be space-like by Lemma 2.2. On the other hand, for the ruled surface of type II, it is also said to be *of type* II_{+}^{0} *or* II_{+}, according as β' is null or β' is non-null. Notice that in the case of type II the director curve β always is space-like. Then the ruled surface of type I_{+} or I_{+}^{0} (resp. I_{-}, II_{+} or II_{+}^{0}) is space-like (resp. time-like).

Thus we can consider these kinds of ruled surfaces in $\boldsymbol{R}_1^3$.

Let M be a space-like or time-like hypersurface in $\boldsymbol{R}_1^{m+1}$ with local coordinate system $\{x_i\}$. For the components g_{ij} of the Riemannian metric g on M we denote (g^{ij}) (resp. $\mathfrak{g}$) the inverse matrix (resp. the determinant) of the matrix (g_{ij}). Then the Laplacian Δ on M is given by

$$\Delta=-\frac{1}{\sqrt{|\mathfrak{g}|}}\sum\frac{\partial}{\partial x^i}\left(\sqrt{|\mathfrak{g}|}g^{ij}\frac{\partial}{\partial x^j}\right). \tag{2.8}$$

In particular, for a Gauss map ξ of a hypersurface M in $\boldsymbol{R}_1^{m+1}$, it satisfies

$$\Delta\xi=m\ \mathrm{grad}\ H+\varepsilon S\xi \tag{2.9}$$

where grad H denotes the gradient of the mean curvature H and S denotes the trace of the square of the shape operator.

§ 3. Cylindrical ruled surfaces.

In this section we are concerned with cylindrical ruled surfaces. Let M be a cylindrical ruled surface swept out by the vector field β along the base curve α in $\boldsymbol{R}_1^3$. That is, $\alpha=\alpha(u)$ is a space-like or time-like smooth curve and $\beta=\beta(u)$ is a space-like or time-like unit constant vector along α orthogonal to α. Then the cylindrical ruled surface M is only of type I_{+}, I_{-} or II_{+}. And M is parametrized by

$$x=x(u, v)=\alpha(u)+v\beta, \qquad u\in J,\ v\in I.$$

It is space-like, provided that the base curve α is space-like and the director curve β is space-like. In the other case, the surface is time-like. Let ξ be a unit normal to M. It is defined by $f^{-1}\alpha'\times\beta$, where f is the norm of the vector $\alpha'\times\beta$. Then we get $g(\xi, \xi)=\varepsilon(=\pm 1)$. Let $M^2(\varepsilon)$ be a 2-dimensional space form as follows:

$$M^2(\varepsilon)=\begin{cases} S_1^2(1) \text{ in } \boldsymbol{R}_1^3, & \varepsilon=1; \\ H^2(-1) \text{ in } \boldsymbol{R}_1^3, & \varepsilon=-1. \end{cases}$$

Then, for any point x in M, $\xi(x)$ can be regarded as a point in $M^2(\varepsilon)$ and the map $\xi: M\to M^2(\varepsilon)$ is the Gauss map of M into $M^2(\varepsilon)$.

We give here examples of ruled surface of type I_+ and II_+ whose Gauss map satisfies

$$\Delta\xi=A\xi, \qquad A\in Mat(3, \boldsymbol{R}). \tag{3.1}$$

EXAMPLE 3.1. A hyperbolic cylinder

$$H^1(c)\times\boldsymbol{R}=\left\{(x_0, x_1, x_2)\in\boldsymbol{R}_1^3 \mid -x_0^2+x_1^2=\frac{1}{c}=-r^2,\ r>0\right\}$$

is a cylindrical ruled surface of type I_+ with base curve $\alpha(u)=(r\cosh u/r, r\sinh u/r, 0)$ and director curve $\beta(u)=(0, 0, 1)$. The Gauss map is given by

$$\xi=\left(-\sinh\frac{u}{r},\ -\cosh\frac{u}{r},\ 0\right),$$

and the Laplacian $\Delta\xi$ of the Gauss map ξ can be expressed as

$$\Delta\xi=-\frac{1}{r^2}\xi.$$

Hence the hyperbolic cylinder satisfies (3.1) with

$$A=\begin{pmatrix} -\frac{1}{r^2} & 0 & a_{13} \\ 0 & -\frac{1}{r^2} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}.$$

EXAMPLE 3.2. A Lorentz circular cylinder

$$S_1^1(c)\times\boldsymbol{R}=\left\{(x_0, x_1, x_2)\in\boldsymbol{R}_1^3 \mid -x_0^2+x_1^2=\frac{1}{c}=r^2,\ r>0\right\}$$

is a cylindrical ruled surface of type II_+ with base curve $\alpha(u)=(r\sinh u/r, \cosh u/r, 0)$ and director curve $\beta(u)=(0, 0, 1)$. The Gauss map is given by

$$\xi=\left(-\cosh\frac{u}{r},\ -\sinh\frac{u}{r},\ 0\right),$$

and the Laplacian $\Delta\xi$ of the Gauss map ξ can be expressed as

$$\Delta\xi=\frac{1}{r^2}\xi.$$

Hence the Lorentz circular cylinder satisfies (3.1) with

$$A=\begin{pmatrix} \frac{1}{r^2} & 0 & a_{13} \\ 0 & \frac{1}{r^2} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}.$$

PROPOSITION 3.1. *The only cylindrical ruled surfaces of type* I_+ *(resp.* II_+*) in* $\boldsymbol{R}_1^3$ *whose Gauss map satisfies the condition* (3.1) *are locally the plane and the hyperbolic cylinder (resp. the Minkowski plane and the Lorentz circular cylinder).*

PROOF. Let M be a cylindrical ruled surface of type I_+ or II_+ parametrized by

$$x=x(u, v)=\alpha(u)+v\beta ,$$

where β is a unit space-like constant vector along the curve α orthogonal to it. That is, it satisfies $g(\alpha', \beta)=0$, $g(\beta, \beta)=1$. Acting a Lorentz transformation, we may assume that $\beta=(0, 0, 1)$ without loss of generality. Then α may be regarded as the plane curve $\alpha(u)=(\alpha_0(u), \alpha_1(u), 0)$ parametrized by arc-length;

$$g(\alpha', \alpha')=-\alpha_0'^2+\alpha_1'^2=-\varepsilon .$$

The Gauss map ξ is given by $\xi=(-\alpha_1', -\alpha_0', 0)$. It is the space-like or time-like unit normal to M, according as $\varepsilon=1$ or -1. Since the induced semi-Riemannian metric g is given by $g_{11}=\varepsilon$, $g_{12}=0$ and $g_{22}=1$, the Laplacian of ξ is given by $\Delta\xi=(-\varepsilon\alpha_1''', -\varepsilon\alpha_0''', 0)$ from (2.8). Thus, from the condition (3.1) we have the following system of differential equations:

$$(3.2)\qquad \begin{cases} \varepsilon\alpha_1'''=a_{11}\alpha_1'+a_{12}\alpha_0' , \\ \varepsilon\alpha_0'''=a_{21}\alpha_1'+a_{22}\alpha_0' , \\ 0 \quad =a_{31}\alpha_1'+a_{32}\alpha_0' , \end{cases}$$

where $A=(a_{ij})$ is the constant matrix.

Now, in order to prove this proposition we may solve this equation and obtain the solution α_0 and α_1. First we consider that the surface M of type I_+, i.e., the plane curve α is space-like ($\varepsilon=-1$). So we get $g(\alpha', \alpha')=-\alpha_0'^2+\alpha_1'^2=1$. Accordingly we can parametrize as follows:

$$(3.3)\qquad \alpha_0'=\sinh\theta, \qquad \alpha_1'=\cosh\theta,$$

where $\theta=\theta(u)$. Differentiating (3.3), we obtain

$$(3.4)\qquad \begin{aligned} \alpha_0''&=\theta'\cosh\theta, & \alpha_0'''&=\theta''\cosh\theta+\theta'^2\sinh\theta, \\ \alpha_1''&=\theta'\sinh\theta, & \alpha_1'''&=\theta''\sinh\theta+\theta'^2\cosh\theta. \end{aligned}$$

By (3.2), (3.3) and (3.4) we have

$$-(\theta'' \sinh\theta+\theta'^2 \cosh\theta)=a_{11}\cosh\theta+a_{12}\sinh\theta,$$

$$-(\theta'' \cosh\theta+\theta'^2 \sinh\theta)=a_{21}\cosh\theta+a_{22}\sinh\theta,$$

which give

(3.5) $$\theta''=(a_{11}-a_{22})\sinh\theta\cosh\theta+a_{12}\sinh^2\theta-a_{21}\cosh^2\theta,$$

(3.6) $$\theta'^2=(a_{21}-a_{12})\sinh\theta\cosh\theta+a_{22}\sinh^2\theta-a_{11}\cosh^2\theta.$$

Differentiating (3.6), we get

$$2\theta'\theta''=\theta'\{(a_{21}-a_{12})(\cosh^2\theta+\sinh^2\theta)+2(a_{22}-a_{11})\sinh\theta\cosh\theta\}.$$

Substituting (3.5) into this equation, we get

(3.7) $$\theta'\{4(a_{11}-a_{22})\sinh\theta\cosh\theta+(3a_{12}-a_{21})\sinh^2\theta+(a_{12}-3a_{21})\cosh^2\theta\}=0.$$

We suppose that $\theta'\neq 0$. By (3.2) and (3.7) we get

(3.8) $$a_{11}=a_{22}, \qquad a_{12}=a_{21}=a_{31}=a_{32}=0,$$

because $\sinh\theta\cosh\theta$, $\sinh^2\theta$ and $\cosh^2\theta$ are linearly independent functions of $\theta=\theta(u)$. Combining the above equations with (3.6) gives

$$\theta=\pm\frac{1}{r}u+b,$$

where

$$-\frac{1}{r^2}=a_{11}=a_{22}, \qquad r>0,\ b\in\boldsymbol{R}.$$

Accordingly we have

$$\alpha_0=\pm r\cosh\theta+c_0, \qquad c_0\in\boldsymbol{R},$$

$$\alpha_1=\pm r\sinh\theta+c_1, \qquad c_1\in\boldsymbol{R}.$$

This representation gives us to

$$-(\alpha_0-c_0)^2+(\alpha_1-c_1)^2=-r^2, \qquad r>0.$$

We denote by $H^1(r, (c_0, c_1))$ the hyperbolic circle centered at (c_0, c_1) with radius r in the Minkowski plane $\boldsymbol{R}_1^2$ (the (x_0x_1)-plane). By the above equation the curve α is contained in $H^1(r, (c_0, c_1))$ and hence the ruled surface M is contained in the hyperbolic cylinder $H^1\times\boldsymbol{R}$.

On the other hand, let J_0 be a set $\{u\in J \mid \theta'(u)=0\}$. We claim that if J_0 is not empty, then J_0 is to be J itself. In fact, we suppose that $J_0\neq J$, i.e., $J-J_0\neq\emptyset$. Then (3.8) is satisfied on $J-J_0$. Since A is constant matrix, (3.8) is satisfied on J. So, (3.5) leads that $\theta''=0$ on J, i.e., θ' is constant on J.

By assumption, there exists $u_0 \in J_0$ and $\theta'(u_0)=0$. Thus θ' is zero on J, a contradiction. So in this case θ is constant on J, and hence we obtain that the normal vector ξ is the time-like constant vector by (3.3). It shows that M is contained in $\boldsymbol{R}^2$.

Next we are concerned with the cylindrical ruled surface M of type II_+, i.e., the plane curve α is time-like ($\varepsilon=1$). Then the surface M is time-like and we get $g(\alpha', \alpha')=-\alpha_0'^2+\alpha_1'^2=-1$. Accordingly we can parametrize as follows: $\alpha_0'=\cosh\theta$, $\alpha_1'=\sinh\theta$, where $\theta=\theta(u)$. By the similar discussion to that of the above ruled surface of type I_+ we can get

$$\theta'\{4(a_{11}-a_{22})\sinh\theta\cosh\theta+(3a_{12}-a_{21})\cosh^2\theta+(a_{12}-3a_{21})\sinh^2\theta\}=0. \tag{3.9}$$

We suppose that $\theta'\neq 0$. By (3.2) and (3.9) we get

$$a_{11}=a_{22}, \qquad a_{12}=a_{21}=a_{31}=a_{32}=0,$$

which yields that

$$\theta=\pm\frac{1}{r}u+b, \qquad \frac{1}{r^2}=a_{11}=a_{22}, \qquad r>0,\ b\in\boldsymbol{R}.$$

Accordingly we have

$$\alpha_0=\pm r\sinh\theta+c_0, \qquad c_0\in\boldsymbol{R},$$

$$\alpha_1=\pm r\cosh\theta+c_1, \qquad c_1\in\boldsymbol{R}.$$

This representation gives us to

$$-(\alpha_0-c_0)^2+(\alpha_1-c_1)^2=r^2, \qquad r>0.$$

We denote by $S_1^1(r, (c_0, c_1))$ the pseudo-circle centered at (c_0, c_1) with radius r in the Minkowski plane $\boldsymbol{R}_1^2$ (the (x_0x_1)-plane). By the above equation the curve α is contained in $S_1^1(r, (c_0, c_1))$ and hence the ruled surface M is contained in the Lorentz circular cylinder $S_1^1\times\boldsymbol{R}$.

On the other hand, if a set $\{u\in J\mid\theta'(u)=0\}$ is not empty, then θ is constant on J by the similar discussion to that about the surface of type I_+. So we get that the normal vector ξ is the space-like constant vector. It shows that M is contained in $\boldsymbol{R}_1^2$. □

Next, we consider a cylindrical ruled surface of type I_- in $\boldsymbol{R}_1^3$. We first give an example of the ruled surface of type I_- whose Gauss map satisfies (3.1).

EXAMPLE 3.3. A circular cylinder of index 1

$$\boldsymbol{R}_1^1\times S^1(c)=\left\{(x_0, x_1, x_2)\in\boldsymbol{R}_1^3 \mid x_1^2+x_2^2=\frac{1}{c}=r^2,\ r>0\right\}$$

is a cylindrical ruled surface of type I_- with base curve $\alpha(u)=(0, r\cos u/r, r\sin u/r)$ and director curve $\beta(u)=(1, 0, 0)$. The Gauss map is given by

$$\xi=\left(0, \cos\frac{u}{r}, \sin\frac{u}{r}\right),$$

and the Laplacian $\Delta\xi$ of Gauss map ξ can be expressed as

$$\Delta\xi=\frac{1}{r^2}\xi.$$

Hence the circular cylinder of index 1 satisfies (3.1) with

$$A=\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & \frac{1}{r^2} & 0 \\ a_{31} & 0 & \frac{1}{r^2} \end{pmatrix}.$$

PROPOSITION 3.2. *The only cylindrical ruled surfaces of type I_- in $\boldsymbol{R}_1^3$ whose Gauss map satisfies* (3.1) *are locally the Minkowski plane and the circular cylinder of index* 1.

PROOF. Let M be a cylinderical ruled surface of type I_-. Then M is parametrized by

$$x=x(u, v)=\alpha(u)+v\beta,$$

where β is a unit time-like constant vector along the space-like curve α orthogonal to it. That is, it satisfies $g(\alpha', \beta)=0$, $g(\beta, \beta)=-1$. Acting a Lorentz transformation, we may assume that $\beta=(1, 0, 0)$ without loss of generality. Then α is the plane curve $\alpha(u)=(0, \alpha_1(u), \alpha_2(u))$ parametrized by arc-length;

$$g(\alpha', \alpha')=\alpha_1'^2+\alpha_2'^2=1. \tag{3.10}$$

The Gauss map ξ is given by $\xi=(0, \alpha_2', -\alpha_1')$. It is the space-like unit normal to M. The Laplacian of ξ is given by $\Delta\xi=(0, -\alpha_2''', \alpha_1''')$. Thus, from the condition (3.1) we have the following system of differential equations:

$$\begin{cases} 0 = a_{12}\alpha_2'-a_{13}\alpha_1', \\ \alpha_2'''=a_{22}\alpha_2'-a_{23}\alpha_1', \\ \alpha_1'''=a_{32}\alpha_2'-a_{33}\alpha_1'. \end{cases} \tag{3.11}$$

Now, we solve this equation and obtain the solution α_1 and α_2. From (3.10) we can parametrize as follows:

$$\alpha_1'=\cos\theta, \qquad \alpha_2'=\sin\theta, \tag{3.12}$$

where $\theta=\theta(u)$. Then, differentiating (3.12), we obtain

$$\alpha_1''=-\theta'\sin\theta,\qquad \alpha_1'''=-\theta''\sin\theta-\theta'^2\cos\theta,$$
(3.13)
$$\alpha_2''=\theta'\cos\theta,\qquad \alpha_2'''=\theta''\cos\theta-\theta'^2\sin\theta.$$

By (3.11), (3.12) and (3.13) we have

$$-(\theta''\cos\theta-\theta'^2\sin\theta)=a_{22}\sin\theta-a_{23}\cos\theta,$$

$$-(\theta''\sin\theta+\theta'^2\cos\theta)=a_{32}\sin\theta-a_{33}\cos\theta,$$

which give

$$\theta''=-a_{32}\sin^2\theta+a_{23}\cos^2\theta-(a_{22}-a_{33})\sin\theta\cos\theta, \tag{3.14}$$

$$\theta'^2=a_{22}\sin^2\theta+a_{33}\cos^2\theta-(a_{23}+a_{32})\sin\theta\cos\theta. \tag{3.15}$$

Differentiating (3.15), we get

$$2\theta'\theta''=\theta'\{2(a_{22}-a_{33})\sin\theta\cos\theta-(a_{23}+a_{32})(\cos^2\theta-\sin^2\theta)\}.$$

Substituting (3.14) into this equation, we get

$$\theta'\{4(a_{22}-a_{33})\sin\theta\cos\theta+(a_{23}+3a_{32})\sin^2\theta-(3a_{23}+a_{32})\cos^2\theta\}=0. \tag{3.16}$$

We suppose that $\theta'\neq 0$. Then by (3.11) and (3.16) we get

$$a_{12}=a_{13}=a_{23}=a_{32}=0,\qquad a_{22}=a_{33},$$

which yields that $\theta=\pm u/r+b$, $1/r^2=a_{22}=a_{33}$, $r>0$, $b\in\boldsymbol{R}$. Accordingly we have

$$\alpha_1=\pm r\sin\theta+c_1,\qquad c_1\in\boldsymbol{R},$$

$$\alpha_2=\mp r\cos\theta+c_2,\qquad c_2\in\boldsymbol{R}.$$

This representation gives us to

$$(\alpha_1-c_1)^2+(\alpha_2-c_2)^2=r^2,\qquad r>0.$$

We denote by $S^1(r,(c_1,c_2))$ the circle centered at (c_1,c_2) with radius r in the plane $\boldsymbol{R}^2$ (the (x_1x_2)-plane). By the above equation the curve α is contained in $S^1(r,(c_1,c_2))$ and hence the ruled surface M is contained in the Lorentz circular cylinder $\boldsymbol{R}_1^1\times S^1$.

On the other hand, if a set $\{u\in J\mid\theta'(u)=0\}$ is not empty, then θ is constant on J by the similar discussion to that in Proposition 3.1. So we get that the normal vector ξ is the space-like constant vector. It shows that M is contained in $\boldsymbol{R}_1^2$. □

§4. Non-cylindrical ruled surfaces of type I_+, I_- or II_+.

In this section we are concerned with non-cylindrical ruled surfaces of type I_+, I_- or II_+ in the 3-dimensional Minkowski space $\boldsymbol{R}_1^3$. Let M be a non-cylindrical ruled surface of type I_+, I_- or II_+ with the base curve α and the director curve β. That is, $\alpha=\alpha(u)$ is a space-like or time-like curve and $\beta=\beta(u)$ is a space-like or time-like unit vector field along α orthogonal to α. Then M is parametrized by

$$x=x(u, v)=\alpha(u)+v\beta(u), \qquad u\in J,\ v\in I, \tag{4.1}$$

where $g(\beta, \beta)=\varepsilon_2=\pm 1$ and $g(\alpha', \beta)=0$. Here we can regard β as a curve in $M^2(\varepsilon_2)$ parametrized by arc-length u, i.e., $g(\beta', \beta')=\varepsilon_3=\pm 1$. And we have the natural frame $\{x_u, x_v\}$ given by

$$x_u=\alpha'+v\beta', \qquad x_v=\beta. \tag{4.2}$$

Let ξ be a unit normal to M. It is defined by $f^{-1}x_u\times x_v$, where f is a positive smooth function defined by $f^2=\varepsilon_4 g(x_u, x_u)$. Then we get

$$g(\xi, \xi)=\varepsilon=-\varepsilon_2\varepsilon_4(=\pm 1).$$

Accordingly ξ can be regarded as a Gauss map of M into the 2-dimensional space form $M^2(\varepsilon)$.

THEOREM 4.1. *The only non-cylindrical ruled surfaces of type I_+ (resp. I_- or II_+) in $\boldsymbol{R}_1^3$ whose Gauss map satisfies*

$$\Delta\xi=A\xi, \qquad A\in Mat(3, \boldsymbol{R}) \tag{4.3}$$

are locally the plane (resp. the Minkowski plane).

PROOF. Let M be a non-cylindrical ruled surface of type I_+, I_- or II_+ parametrized by

$$x=x(u, v)=\alpha(u)+v\beta(u), \qquad u\in J,\ v\in I,$$

where β is a curve in $M^2(\varepsilon)$ parametrized by arc-length. The Gauss map $\xi: M\rightarrow M^2(\varepsilon)$ of the surface M is given by

$$\xi=f^{-1}(x_u\times x_v)=f^{-1}(\alpha'+v\beta')\times\beta.$$

We define smooth functions h, k and vector fields X, Y as follows:

$$\begin{aligned} &h=g(\alpha', \beta'), \qquad k=g(\alpha', \alpha')/2, \\ &X=\alpha'\times\beta, \qquad Y=\beta'\times\beta. \end{aligned} \tag{4.4}$$

Then we have

$$f^2=-\varepsilon\varepsilon_2(\varepsilon_3 v^2+2hv+2k),\tag{4.5}$$
$$g(X, X)=-2\varepsilon_2 k,\quad g(X, Y)=-\varepsilon_2 h,\quad g(Y, Y)=-\varepsilon_2\varepsilon_3,$$

where we have used (2.6). Then ξ is represented as $\xi=f^{-1}(X+vY)$. It is easy to show that the Laplacian Δ of M can be expressed as

$$\Delta=\varepsilon\varepsilon_2\Big(-\frac{f_u}{f^3}\frac{\partial}{\partial u}+\frac{1}{f^2}\frac{\partial^2}{\partial u^2}\Big)-\varepsilon_2\Big(\frac{f_v}{f}\frac{\partial}{\partial v}+\frac{\partial^2}{\partial v^2}\Big).\tag{4.6}$$

Since we get

$$\frac{\partial\xi}{\partial u}=-\frac{f_u}{f^2}(X+vY)+\frac{1}{f}(X'+vY'),$$
$$\frac{\partial^2\xi}{\partial u^2}=-\frac{ff_{uu}-2f_u^2}{f^3}(X+vY)-2\frac{f_u}{f^2}(X'+vY')+\frac{1}{f}(X''+vY''),$$
$$\frac{\partial\xi}{\partial v}=-\frac{f_v}{f^2}(X+vY)+\frac{1}{f}Y,$$
$$\frac{\partial^2\xi}{\partial v^2}=-\frac{ff_{vv}-2f_v^2}{f^3}(X+vY)-2\frac{f_v}{f^2}Y,$$

we obtain by (4.6)

$$\varepsilon_2\Delta\xi=\Big(-\varepsilon\frac{ff_{uu}-3f_u^2}{f^5}+\frac{ff_{vv}-f_v^2}{f^3}\Big)(X+vY)$$
$$-3\varepsilon\frac{f_u}{f^4}(X'+vY')+\varepsilon\frac{1}{f^3}(X''+vY'')+\frac{f_v}{f^2}Y.$$

By the assumption (4.3) and the above equation we get the partial differential equation

$$\begin{aligned}&\{-\varepsilon(ff_{uu}-3f_u^2)+f^2(ff_{vv}-f_v^2)\}(X+vY)\\&\quad-3\varepsilon ff_u(X'+vY')+\varepsilon f^2(X''+vY'')+f^3f_vY\\&=\varepsilon_2 f^4A(X+vY).\end{aligned}\tag{4.7}$$

By (4.5) we have

$$\varepsilon ff_u=-\varepsilon_2(h'v+k'),\qquad \varepsilon ff_v=-\varepsilon_2(\varepsilon_3 v+h),$$
$$\varepsilon(ff_{uu}+f_u^2)=-\varepsilon_2(h''v+k''),\qquad ff_{vv}+f_v^2=-\varepsilon\varepsilon_2\varepsilon_3.$$

Using the above equations, we can eliminate f_{uu} and f_{vv} in (4.7), and then f_u and f_v. Then we have the following equation:

$$\begin{aligned}&\{(h''v+k'')+4\varepsilon_2 f^{-2}(h'v+k')^2-\varepsilon\varepsilon_3 f^2-2\varepsilon_2(\varepsilon_3 v+h)^2\}(X+vY)\\&\quad+3(h'v+k')(X'+vY')+\varepsilon\varepsilon_2 f^2(X''+vY'')-\varepsilon f^2(\varepsilon_3 v+h)Y\\&=f^4A(X+vY),\end{aligned}$$

which can be regarded as the polynomial with the variable f:

$$\begin{aligned}&-A(X+vY)f^{6}+\{-\varepsilon\varepsilon_{3}(X+vY)+\varepsilon\varepsilon_{2}(X''+vY'')-\varepsilon(\varepsilon_{3}v+h)Y\}f^{4}\\ &+[\{(h''v+k'')-2\varepsilon_{2}(\varepsilon_{3}v+h)^{2}\}(X+vY)+3(h'v+k')(X'+vY')]f^{2}\\ &+4\varepsilon\varepsilon_{2}(h'+vk')^{2}(X+vY)=0.\end{aligned} \tag{4.8}$$

By the definition of the function f (4.8) becomes the polynomial with the variable v whose coefficients are functions of variable u. Then, by the coefficients of v^6 and v^7, we have

$$AX=0, \qquad AY=0, \tag{4.9}$$

where A is the matrix, and X and Y are vectors. Suppose that A is non-singular. Then (4.9) means that $X=Y=0$, which implies that $\xi=0$, a contradiction. Accordingly we see that the matrix A is singular.

Next, consider the coefficients of the other powers of v in (4.8) and using (4.9) we obtain

$$Y''=0, \tag{4.10}$$

$$\varepsilon_{2}X''+\varepsilon_{3}X+4\varepsilon_{2}\varepsilon_{3}hY''-3\varepsilon_{2}\varepsilon_{3}h'Y'-(h+\varepsilon_{2}\varepsilon_{3}h'')Y=0, \tag{4.11}$$

$$\begin{aligned}&4\varepsilon_{2}\varepsilon_{3}hX''-3\varepsilon_{2}\varepsilon_{3}h'X'+(4h-\varepsilon_{2}\varepsilon_{3}h'')X+4\varepsilon_{2}(h^{2}+\varepsilon_{3}k)Y''\\ &-3\varepsilon_{2}(\varepsilon_{3}k'+2hh')Y'-(2\varepsilon_{3}h^{2}+2\varepsilon_{2}hh''-4\varepsilon_{2}h'^{2}+4k+\varepsilon_{2}\varepsilon_{3}k'')Y=0,\end{aligned} \tag{4.12}$$

$$\begin{aligned}&4\varepsilon_{2}(h^{2}+\varepsilon_{3}k)X''-3\varepsilon_{2}(\varepsilon_{3}k'+2hh')X'\\ &+(6\varepsilon_{3}h^{2}+4\varepsilon_{2}h'^{2}-2\varepsilon_{2}hh''-\varepsilon_{2}\varepsilon_{3}k'')X+8\varepsilon_{2}hkY''\\ &-6\varepsilon_{2}(hk'+kh')Y'+2(4\varepsilon_{2}h'k'-6\varepsilon_{3}hk-\varepsilon_{2}hk''-\varepsilon_{2}kh'')Y=0,\end{aligned} \tag{4.13}$$

$$\begin{aligned}&8\varepsilon_{2}hkX''-6\varepsilon_{2}(hk'+kh')X'+2(2h^{3}+4\varepsilon_{2}h'k'-\varepsilon_{2}hk''-\varepsilon_{2}kh'')X\\ &+4\varepsilon_{2}k^{2}Y''-6\varepsilon_{2}kk'Y'+2(2\varepsilon_{2}k'^{2}-\varepsilon_{2}kk''-4\varepsilon_{3}k^{2}-2h^{2}k)Y=0,\end{aligned} \tag{4.14}$$

$$4\varepsilon_{2}k^{2}X''-6\varepsilon_{2}kk'X'+2(2kh^{2}+2\varepsilon_{2}k'^{2}-\varepsilon_{2}kk''-2\varepsilon_{3}k^{2})X-4hk^{2}Y=0. \tag{4.15}$$

From (4.10) we have $Y=u\boldsymbol{a}+\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors. We claim that $Y=\boldsymbol{b}(\neq 0)$, i.e., $\boldsymbol{a}=0$. In fact, since $g(Y, Y)=-\varepsilon_2\varepsilon_3$ by (4.5), we have

$$u^{2}g(\boldsymbol{a}, \boldsymbol{a})+2ug(\boldsymbol{a}, \boldsymbol{b})+g(\boldsymbol{b}, \boldsymbol{b})=-\varepsilon_{2}\varepsilon_{3},$$

from which we conclude

$$g(\boldsymbol{a}, \boldsymbol{a})=0, \qquad g(\boldsymbol{a}, \boldsymbol{b})=0, \qquad g(\boldsymbol{b}, \boldsymbol{b})=-\varepsilon_{2}\varepsilon_{3}.$$

Since the vector Y is defined by $\beta'\times\beta$, we get $g(Y, \beta)=0$ and $g(Y, \beta')=0$, from which imply that

$$\frac{d}{du}g(Y,\ \beta)=g(\boldsymbol{a},\ \beta)=0,\quad \frac{d}{du}g(\boldsymbol{a},\ \beta)=g(\boldsymbol{a},\ \beta')=0.$$

Since $g(\beta,\ \beta')=0$, it suffices to consider the following three cases. First of all, if β and β' are space-like, then from $g(\boldsymbol{b},\ \boldsymbol{b})=-1$, $g(\boldsymbol{a},\ \boldsymbol{b})=0$ and Lemma 2.2 $\boldsymbol{a}$ is space-like. On the other hand, if β is space-like (resp. time-like) and β' is time-like (resp. space-like), then Lemma 2.2 implies that $\boldsymbol{a}$ is space-like. Since $g(\boldsymbol{a},\ \boldsymbol{a})=0$, we get $\boldsymbol{a}=0$, i.e., we have $Y=\boldsymbol{b}(\neq 0)$. This yields that $g(\beta,\ \boldsymbol{b})=0$, which means that β is contained in the plane passing through the origin in $\boldsymbol{R}_1^3$. Without loss of generality, we may suppose that $\boldsymbol{b}=(b_0,\ b_1,\ 0)$ and $g(\boldsymbol{b},\ \boldsymbol{b})=-b_0{}^2+b_1{}^2=-\varepsilon_2\varepsilon_3$. Then we get

$$Y=(Y_0,\ Y_1,\ Y_2)=(b_0,\ b_1,\ 0).$$

Now, from (4.11) we have $\varepsilon_2 X''+\varepsilon_3 X-(h+\varepsilon_2\varepsilon_3 h'')Y=0$. If we put $Z=X-\varepsilon_3 hY$, then we have

(4.16) $$Z''+\varepsilon_2\varepsilon_3 Z=0,$$

(4.17) $$g(Z,\ Z)=\varepsilon_2(\varepsilon_3 h^2-2k),$$

where we have used (4.4) and (4.5). Using $Y_2=0$ and (4.16), we see that the x_2-component of (4.12) is given by

(4.18) $$h''X_2+3h'X_2'=0,$$

where $X=(X_0,\ X_1,\ X_2)$. Using (4.16) and (4.18), we have from (4.13)

(4.19) $$(2\varepsilon_3 h^2-4k+4\varepsilon_2 h'^2-\varepsilon_2\varepsilon_3 k'')X_2-3\varepsilon_2\varepsilon_3 k'X_2'=0.$$

By making use of (4.16), (4.18) and (4.19), equations (4.14) and (4.15) can be written as

(4.20) $$h'(k'-\varepsilon_3 hh')X_2=0,$$

(4.21) $$(k'^2-2\varepsilon_3 h'^2 k)X_2=0.$$

Now, using the equation (4.17)$\sim$(4.21), we will prove that $Z=0$ on J. We first prove that X_2 vanishes on J. In fact, we suppose that there exists $u_1\in J$ such that $X_2(u_1)\neq 0$. Let J_1 be an open interval containing u_1 in $\{u\in J\,|\,X_2(u)\neq 0\}$. Then, from (4.20) and (4.21), we obtain

(4.22) $$h'(k'-\varepsilon_3 hh')=0 \quad \text{on } J_1,$$

(4.23) $$k'^2-2\varepsilon_3 kh'^2=0 \quad \text{on } J_1.$$

Differentiating (4.23), we get

(4.24) $$k''^2+k'k'''-\varepsilon_3(k''h'^2+4k'h'h''+2kh''^2+2kh'h''')=0 \quad \text{on } J_1.$$

Let J_1^0 be a set $\{u\in J_1 | h'(u)\neq 0\}$ and J_1^1 a complement of J_1^0. On J_1^0 we get $g(Z, Z)=0$ by (4.22) and (4.23). By (4.18) and (4.23) we have that $h''=0$ and $k'=0$ on J_1^1. Since we have $k''=0$ on J_1^1 by (4.24), (4.19) leads that $\varepsilon_3 h^2-2k=0$, i.e., $g(Z, Z)=0$ on J_1^1. Since ξ and β are orthonormal vectors and both orthogonal to Z on J_1, if the plane spanned by ξ and β is space-like (resp. time-like), then the vector Z is time-like or 0 (resp. space-like) and hence $Z=0$ on J_1. This means that $X_2=0$ on J_1, a contradiction. Thus $X_2=0$ on J, i.e., Z is contained in the x_0x_1-plane. We claim that X and Y are linearly dependent on J. In fact, if there exists $u_1\in J$ such that $X(u_1)$ and $Y(u_1)$ are linearly independent, then there exists a positive number ε such that X and Y are linearly independent on $J_\varepsilon=(u_1-\varepsilon, u_1+\varepsilon)$. The plane spanned by X and Y is to be x_0x_1-plane on J_ε. Since $g(X, \beta)=0$ and $g(Y, \beta)=0$, β is parallel to the x^2-axis on J_ε, i.e., $\beta=\gamma(u)e_2$ on J_ε, where $e_2=(0, 0, 1)$. Thus we have $\boldsymbol{b}=\beta'\times\beta=0$ on J_ε, a contradiction. Thus $X=qY$, where q is a non-zero smooth function on J. By the definition we have $(\alpha'-q\beta')\times\beta=0$. Since $\alpha'-q\beta'$ and β are orthogonal, we have $\alpha'-q\beta'=0$. From (4.4), we get $h=q\varepsilon_3$. Hence $Z=X-\varepsilon_3 hY=0$ on J.

By the definition we see $(\alpha'-\varepsilon_3 h\beta')\times\beta=0$. Since $\alpha'-\varepsilon_3 h\beta'$ and β are orthogonal, we have by (2.1)

$$\alpha'-\varepsilon_3 h\beta'=0.$$

By the definition of ξ we obtain $\xi=f^{-1}(\varepsilon_3 h+v)\boldsymbol{b}=\pm\boldsymbol{b}$. It means that if M is contained in $\boldsymbol{R}^2$ or $\boldsymbol{R}_1^2$, according as $\varepsilon=-1$ or $\varepsilon=1$. This completes the proof. □

REMARK. As is seen from the proof above, Theorem 4.1 holds under the condition that each entry of A is a smooth function of u. But it is not valid provided that entries are smooth functions of u and v.

We can consider an example which satisfies the condition (4.3), where an entry of A is a function of v.

EXAMPLE 4.1. A helicoid of 2nd kind with a base curve $\alpha(u)=(0, 0, u)$ and a director curve $\beta(u)=(\sinh u, \cosh u, 0)$ is the non-cylindrical ruled surface of type I_+. The Gauss map is given by

$$\xi=\frac{1}{\sqrt{1-v^2}}(\cosh u, \sinh u, v).$$

The Laplacian $\Delta\xi$ of Gauss map ξ can be expressed as

$$\Delta\xi=\frac{-2}{(1-v^2)^2}\xi, \qquad |v|<1.$$

EXAMPLE 4.2. A helicoid with a base curve $\alpha(u)=(u, 0, 0)$ and a director curve $\beta(u)=(0, -\sin u, \cos u)$ is the non-cylindrical ruled surface of type II_+. The Gauss map is given by

$$\xi=\frac{1}{\sqrt{1-v^2}}(v, -\cos u, -\sin u).$$

The Laplacian $\Delta\xi$ of Gauss map ξ can be expressed as

$$\Delta\xi=\frac{-2}{(1-v^2)^2}\xi, \qquad |v|<1.$$

REMARK. Since a helicoid and a helicoid of 2nd kind are both maximal surfaces in $\boldsymbol{R}_1^3$, it is seen by (2.9) that the Gauss maps satisfy $\Delta\xi=f(u, v)\xi$. But, in these example, $f(u, v)$ depends only on v.

§5. Ruled surfaces of type I_+^0 or II_+^0.

In this section we are concerned with non-cylindrical ruled surfaces of type I_+^0 or II_+^0 in the 3-dimensional Minkowski space $\boldsymbol{R}_1^3$. Let M be a ruled surface of type I_+^0 or II_+^0 with base curve α and director curve β. Then the surface M in $\boldsymbol{R}_1^3$ is parametrized by

$$x=x(u, v)=\alpha(u)+v\beta(u), \qquad u\in J,\ v\in I, \tag{5.1}$$

where $g(\beta, \beta)=1$, $g(\alpha', \beta)=0$ and β' is null. So β can be regarded as a null spherical curve in $S_1^2(1)$ parametrized by u. For such ruled surface M we have the natural frame $\{x_u, x_v\}$ given by

$$x_u=\alpha'+v\beta', \qquad x_v=\beta. \tag{5.2}$$

Let ξ be a unit normal to M. It is defined by $f^{-1}x_u\times x_v$, where f is a positive smooth function defined by $f^2=-\varepsilon g(x_u, x_u)$. Then we get

$$g(\xi, \xi)=\varepsilon.$$

Accordingly ξ can be regarded as a Gauss map of M into the 2-dimensional space form $M^2(\varepsilon)$.

THEOREM 5.1. *There are no ruled surfaces of type I_+^0 or II_+^0 in $\boldsymbol{R}_1^3$ whose Gauss maps satisfies*

$$\Delta\xi=A\xi, \qquad A\in Mat(3, \boldsymbol{R}). \tag{5.3}$$

PROOF. Let M be a ruled surface of type I_+^0 or II_+^0 parametrized by

$$x=x(u, v)=\alpha(u)+v\beta(u), \qquad u\in J,\ v\in I,$$

where $g(\alpha', \alpha')=\varepsilon_1$, $g(\alpha', \beta)=0$ and $g(\beta, \beta)=1$. The Gauss map $\xi: M\to M^2(\varepsilon)$ of the surface M is given by

$$\xi=f^{-1}(x_u\times x_v)=f^{-1}(\alpha'+v\beta')\times\beta.$$

We define a smooth function h and vector fields X, Y as follows:

$$h=g(\alpha', \beta'),\qquad X=\alpha'\times\beta,\qquad Y=\beta'\times\beta.$$

Then the vector Y is null. In fact, by (2.6) and the definition of Y, we get $g(Y, Y)=-g(\beta', \beta')g(\beta, \beta)=0$. Accordingly we have that $Y=0$ or null. But $Y=0$ if and only if β' is parallel to β, a contradiction. Hence Y is null. Since the vector β' is null and orthogonal to Y, there is a non-zero smooth function a such that $Y=a\beta'$ from Lemma 2.3. By the property of the Lorentz cross product, we have $Y'=\beta''\times\beta=a'\beta'+a\beta''$, which implies $g(a'\beta'+a\beta'', \beta'')=0$. Because β' and β'' are orthogonal, β'' is the null or zero vector. Thus there is a smooth function b such that $\beta''=b\beta'$ and we get

$$Y'=bY,\qquad Y''=(b'+b^2)Y. \tag{5.4}$$

It is easy to show that the Laplacian Δ of M can be expressed as

$$\Delta=\varepsilon\Big(-\frac{f_u}{f^3}\frac{\partial}{\partial u}+\frac{1}{f^2}\frac{\partial^2}{\partial u^2}\Big)-\Big(\frac{f_v}{f}\frac{\partial}{\partial v}+\frac{\partial^2}{\partial v^2}\Big). \tag{5.5}$$

Accordingly we get

$$\Delta\xi=\Big(-\varepsilon\frac{ff_{uu}-3f_u^2}{f^5}+\frac{ff_{vv}-f_v^2}{f^3}\Big)(X+vY)$$
$$-3\varepsilon\frac{f_u}{f^4}(X'+vY')+\varepsilon\frac{1}{f^3}(X''+vY'')+\frac{f_v}{f^2}Y.$$

By the assumption (5.3) and the above equation we get the partial differential equation

$$\begin{aligned}&\{-\varepsilon(ff_{uu}-3f_u^2)+f^2(ff_{vv}-f_v^2)\}(X+vY)\\ &\quad-3\varepsilon ff_u(X'+vY')+\varepsilon f^2(X''+vY'')+f^3f_vY\\ &=f^4A(X+vY).\end{aligned} \tag{5.6}$$

Since we have $f^2=-\varepsilon(2hv+\varepsilon_1)$, we obtain

$$ff_u=-\varepsilon h'v,\qquad ff_v=-\varepsilon h,$$
$$ff_{uu}+f_u^2=-\varepsilon h''v,\qquad ff_{vv}+f_v^2=0.$$

Using the above equations, we can eliminate f_{uu} and f_{vv} in (5.6), and then f_u and f_v. Then we have the following equation:

$$\{h''v+4\varepsilon f^{-2}(h'v)^2-2h^2\}(X+vY)+3h'v(X'+vY')$$
$$+\varepsilon f^2(X''+vY'')-\varepsilon f^2hY-f^4A(X+vY)=0,$$

which can be regarded as the polynomial with the variable f:

$$-A(X+vY)f^6+\varepsilon\{Y''v+(X''-hY)\}f^4$$
(5.8) $$+\{(h''v-2h^2)(X+vY)+3(h'Y'v^2+h'X'v)\}f^2$$
$$+4\varepsilon(h'v)^2(X+vY)=0.$$

From the equation $f^2=-\varepsilon(2hv+\varepsilon_1)$ and (5.8) we can calculate the coefficients of v^4. Then we have

(5.9) $$h^3AY=0.$$

Next, considering the coefficients of the other powers of v in (5.8) we obtain

(5.10) $$8h^3AX+12\varepsilon_1h^2AY+4h^2Y''+2(2h'^2-hh'')Y-6hh'Y'=0,$$

(5.11) $$12\varepsilon_1h^2AX+6hAY+4h^2X''-6hh'X'+(4h'^2-2hh'')X$$
$$+4\varepsilon_1hY''-3\varepsilon_1h'Y'+\varepsilon_1h''Y=0,$$

(5.12) $$6hAX+\varepsilon_1AY+4\varepsilon_1hX''-3\varepsilon_1h'X'$$
$$+(4h^3-\varepsilon_1h'')X+Y''-2\varepsilon_1h^2Y=0,$$

(5.13) $$\varepsilon_1AX+X''+2\varepsilon_1h^2X-hY=0.$$

Now, we prove that the function h vanishes on J. In fact, suppose that $h\neq 0$ on J. Then there exists $u_0\in J$ such that $h(u_0)\neq 0$. Let J_0 be the open interval containing u_0 in $\{u\in J\,|\,h'(u)\neq 0\}$. Then, from (5.9), we get $AY=0$ on J_0, where A is the matrix and Y is the vector. By (5.4) and (5.10) we have $AX\equiv 0 \pmod{Y}$ on J_0. Then (5.13) implies

(5.14) $$X''+2\varepsilon_1h^2X\equiv 0 \pmod{Y} \quad \text{on } J_0.$$

Using (5.12) and (5.14) we have

(5.15) $$3\varepsilon_1h'X'+(\varepsilon_1h''+4h^3)X\equiv 0 \pmod{Y} \quad \text{on } J_0.$$

Using (5.11), (5.14) and (5.15) we get

(5.16) $$h'^2X\equiv 0 \pmod{Y} \quad \text{on } J_0.$$

We know here that the differentiation of the function h is identically zero on J_0. In fact, if we suppose that $h'\neq 0$ on J_0, then there exists $u_1\in J_0$ such that $h'(u_1)\neq 0$. From (5.16), $X(u_1)\equiv 0 \pmod{Y}$. Thus there exists a non-zero smooth function c on the open interval J_1 containing u_1 in $\{u\in J_0\,|\,h'(u)\neq 0\}$ such that

$X=cY$. Thus we have $\xi=f^{-1}(c+v)Y$ on J_1. This means that ξ is null, a contradiction. Accordingly, (5.15) yields that $h^3X\equiv 0 \pmod{Y}$ on J_0. This is a contradiction. Thus the function h is always zero on J, i.e., $g(\alpha', \beta')=0$ on J. If M is the surface of type II_+^0, then since α is time-like and $h=0$, Lemma 2.2 means that β' is not causal, a contradiction. On the other hand, we suppose that M is the surface of type I_+^0. Then we know that $\alpha'=0$. In fact, the differentiating $g(\alpha', \beta)=0$ and $g(\alpha', \alpha')=1$, we obtain that α'' is orthogonal to α' and β. Since α' and β are space-like and orthogonal, α'' is time-like or 0. Differentiating $g(\alpha', \beta')=0$ and using the property $\beta''=b\beta'$, we get $g(\alpha'', \beta')=0$. If α'' is time-like, Lemma 2.2 means that β' is not causal, a contradiction. Accordingly, we have $\alpha''=0$. This shows that there are constant vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\alpha(u)=u\boldsymbol{a}+\boldsymbol{b}$. Namely, the base curve α is the space-like straight line in $\boldsymbol{R}_1^3$.

Since the vector $X=\alpha'\times\beta$ is unit time-like and $g(X, X')=0$, Lemma 2.2 leads that $X'=\alpha'\times\beta'$ is space-like. On the other hand, because α' and β' are orthogonal and β' is null, by (2.6) we have $g(X', X')=0$. Hence $X'=0$, i.e., β' is parallel to α', a contradiction.

Thus it completes the proof. □

REMARK. As is seen from the proof above, Theorem 5.1 holds under the condition that each entry of A is a smooth function of u. But it is not valid provided that entries are smooth functions of u and v.

We can consider an example which doesn't satisfy the condition (5.3).

EXAMPLE 5.1. A conjugate of Enneper's surface of 2nd kind with $\alpha(u)=(u^3/24, u^3/24-u, u^2/4-1)$ and $\beta(u)=(-u/2, -u/2, -1)$ is the non-cylindrical ruled surface of type I_+^0. The Gauss map is given by

$$\xi=\frac{-1}{\sqrt{1+v}}\left(\frac{u^2}{8}+\frac{v}{2}+1, \frac{u^2}{8}+\frac{v}{2}, \frac{u}{2}\right).$$

The Laplacian $\Delta\xi$ of Gauss map ξ can be expressed as

$$\Delta\xi=\frac{-1}{2(1+v)^2}\xi, \qquad v>-1.$$

EXAMPLE 5.2. A ruled surface with a base curve $\alpha(u)=(u^3/24+u, u^3/24, u^2/4)$ and a director curve $\beta(u)=(u/2, u/2, 1)$ is the non-cylindrical ruled surface of type II_+^0. The Gauss map is given by

$$\xi=\frac{-1}{\sqrt{1+v}}\left(-\frac{u^2}{8}+\frac{v}{2}, \frac{u^2}{8}+\frac{v}{2}+1, -\frac{u}{2}\right).$$

The Laplacian $\Delta\xi$ of Gauss map ξ can be expressed as

$$\Delta\xi=\frac{-1}{2(1+v)^2}\xi, \qquad v>-1.$$

References

[1] C. Baikoussis and D.E. Blair, On the Gauss map of ruled surfaces, Glasgow Math. J. **34** (1992), 355-359.

[2] B.Y. Chen, On submanifolds of finite type, Soochow J. Math. **9** (1983), 65-81.

[3] B.Y. Chen and P. Piccinni, Submanifolds with finite type Gauss map, Bull. Austral. Math. Soc. **35** (1987), 161-186.

[4] O.J. Garay, An extension of Takahashi's theorem, Geometriae Dedicate **34** (1990), 105-112.

[5] L.K. Graves, Codimension one isometric immersions between Lorentz spaces, Trans. Amer. Math. Soc. **252** (1979), 367-392.

[6] W. Greub, Linear algebra, Springer-Verlag, New York, 1963.

[7] O. Kobayashi, Maximal surface in the 3-dimensional Minkowski space L^3, Tokyo J. Math. **6**, **No. 2** (1983), 297-309.

[8] M. Spivak, A comprehensive introduction to differential geometry III, Interscience, New York, 1969.

[9] T. Takahashi, Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan **18** (1966), 380-385.

Topology and Geometry Research Center
Kyungpook National University
Taegu 702-701
Korea