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A REMARK ON MINIMAL MODELS

By

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Abstract. We prove the following theorem: Let T be superstable and let A any set. Then there is no minimal model over A which has an infinite set of indiscernibles over A.

0. Introduction

A model M is said to be *minimal* if there is no proper elementary submodel of M. We consider the size of indiscernible sets in a minimal model. Shelah showed that if a theory T is totally transcendental then there is no infinite indiscernible set in a minimal model of T (see [3, IV, Theorem 4.21]). On the other hand, in [2] Marcus constructed a minimal (and prime) structure with an infinite indiscernible set. His structure is stable but not superstable. Our aim here is therefore to extend the above statement to a superstable theory.

Shelah's proof is as follows: Let M be a model having an infinite indiscernible set I. Pick any $a\in I$ and let $J=I-\{a\}$. Since T is totally transcendental, there is $N\prec M$ which is primary (and hence atomic) over J. By indiscernibility of I, we have $a\notin N$. Hence M is not minimal.

Our proof is similar to his one. However, for the general case, we do not necessarily have the existence of primary models. So, instead of N above, we take in M a maximal set E which includes J but is independent from a. We call such E *a tp(a)-envelope of J in M* (see Definition 1.2 for the exact definition). First we show that if T is superstable, E is an elementary submodel of M (Lemma 1.4). It follows that M is not minimal, and hence we can obtain our theorem. At the end of the paper, we give a stable structure having an infinite indiscernible set (Example 1.7). The way of the construction is essentially same as Marcus's one [2].

1. The size of indiscernible sets

1.1. NOTATION. We fix a (possibly uncountable) stable theory T. We usually work in a big model $\boldsymbol{C}$ of T. Our notations are fairly standard.

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$A, B, \cdots$ are used to denote small subsets of $\boldsymbol{C}$. $\bar{a}, \bar{b}, \cdots$ are used to denote finite sequences of elements in $\boldsymbol{C}$. $\varphi, \psi, \cdots$ are used to denote formulas (with parameters). $p, q, \cdots$ are used to denote types (with parameters). The nonforking extension of a stationary types p to the domain A is denoted by $p \mid A$. The type of a over A is denoted by $tp(a/A)$. $R^\infty(p)$ (resp. $R^\infty(\varphi)$) is the infinity rank of a type p (resp. a formula φ). We simply write $R^\infty(a/A)$ instead of $R^\infty(tp(a/A))$. The set of realizations of a type p (resp. a formula φ) in a model M is denoted by p^M (resp. φ^M).

1.2. DEFINITION. Let M be a model and $A \subset B \subset M$. Let $p \in S(A)$ be stationary. Then a *p-envelope* of B in M is a maximal set E such that $B \subset E \subset M$ and any element of $(p \mid B)^M$ is independent from E over A.

1.3. REMARK. The notion of "envelopes" was introduced in [1], and was defined in the context of totally categorical theories. Our definition is a generalization of that in [1].

1.4. LEMMA. *Let T be superstable. Let M be a model and $A \subset M$. Let $p \in S(A)$ be stationary. Suppose that M contains some infinite Morley sequence I of p. Then a p-envelope of $I \cup A$ in M is an elementary submodel of M.*

PROOF. For the simplicity of the notation, we may assume that $A = \emptyset$. Take any p-envelope E of I in M. If $(p \mid I)^M = \emptyset$, then $E = M$. So we assume that $(p \mid I)^M \neq \emptyset$. Assume by way of contradiction that E is not an elementary submodel of M. Then, by the Tarski criterion, there is a consistent formula $\varphi(x, \bar{e}_0) \in L(E)$ such that $\varphi^M \cap E = \emptyset$. By superstability, pick an element b of φ^M such that $R^\infty(b/E)$ is minimal.

CLAIM. *Any $a \in (p \mid I)^M$ is independent from b over E.*

PROOF. Assume otherwise. Then there is an element a of $(p \mid I)^M$ such that $tp(a/Eb)$ forks over E. Take a formula $\theta(x, \bar{e}_1) \in tp(b/E)$ such that $R^\infty(b/E) = R^\infty(\theta)$. Now $tp(a/Eb)$ forks over $\emptyset$, so there is $\bar{e} \in E$ such that $tp(a/\bar{e}b)$ forks over $\emptyset$. Then we may assume that $\bar{e}_0, \bar{e}_1 \subset \bar{e}$. Note that $tp(a/\bar{e})$ does not fork over $\emptyset$ (because $\bar{e} \in E$). It follows that $tp(b/\bar{e}a)$ forks over $\bar{e}$. So we can get a formula $\psi(x, \bar{e}, a) \in tp(b/\bar{e}a)$ such that, if $\models \psi(b', \bar{e}, a)$ then $tp(b'/\bar{e}a)$ forks over $\bar{e}$. Let $\Gamma(a, \bar{e})$ denote $(\exists x)(\varphi(x, \bar{e}_0) \wedge \psi(x, \bar{e}, a) \wedge \theta(x, \bar{e}_1))$. Now the weight of $\bar{e}$ is finite since $R^\infty(\bar{e}) < \infty$. Therefore we can pick $a' \in I$ such that $tp(a'/\bar{e})$ does not fork over $\emptyset$. Remember that $tp(a/\bar{e})$ does not forks over $\emptyset$. It follows that $tp(a/\bar{e}) = tp(a'/\bar{e})$. Hence $\Gamma(a', \bar{e})$ holds. Therefore there is an element $b'\in\varphi^M$ such that $R^\infty(b'/\bar{e})\leqq R^\infty(b/E)$ and $tp(b'/\bar{e}a')$ forks over $\bar{e}$. Thus $R^\infty(b/E)\geqq R^\infty(b'/\bar{e})>R^\infty(b'/\bar{e}a')\geqq R^\infty(b'/E)$. Moreover $R^\infty(b'/E)\neq 0$ because b' satisfies φ. But this contradicts the minimality of $R^\infty(b/E)$. Hence the claim holds.

Thus any $a\in(p \mid I)^M$ is independent from bE over I. But this contradicts that E is an envelope. Hence E is an elementary submodel. This completes the proof of the lemma. □

1.5. EXAMPLE. Let $Per(\omega)$ denote the set of permutations of ω which move only a finite number of elements. For each $i<\omega$, define a function $\pi_i: Per(\omega)\to\omega$ such that $\pi_i(\sigma)=\sigma(i)$. Let $A=\omega\cup Per(\omega)$. Consider the structure $M=(A; \omega, Per(\omega), \{\pi_i\}_{i<\omega})$. Then ω is a Morley sequence of $tp(0)$. Note that for any $\sigma\in Per(\omega)$, $\omega\subset dcl(\sigma)$ (=the definable closure of σ). Therefore $\omega-\{0\}$ is the $tp(0)$-envelope of $\omega-\{0\}$ in M. However $\omega-\{0\}$ is not a model. Moreover $T=Th(M)$ is not superstable (since the weight of σ is infinite). This example shows that we need, in lemma 1.4, the assumption that T is superstable.

1.6. THEOREM. *Let T be superstable and let A any set. Then there is no minimal model over A which has an infinite set of indiscernibles over A.*

PROOF. Suppose that M has an infinite set I of indiscernibles over some set A. We can assume that I is already an infinite Morley sequence of some $p\in S(A)$ because $\kappa(T)$ is countable. Pick any $a\in I$. By lemma 1.4, a p-envelope E of $(I-\{a\})\cup A$ in M is an elementary submodel of M. It is clear that $a\notin E$. Hence M is not minimal. A contradiction. □

1.7. EXAMPLE (see [2]). Theorem 1.6 can not be extended to a stable theory. We construct a minimal structure with an infinite indiscernible set. Recall the structure $M=(A; \omega, Per(\omega), \{\pi_i\}_{i<\omega})$ (see Example 1.5). Note that this structure is not minimal. But by modifying the construction, we can obtain a minimal one: For each $n<\omega$, we define inductively P_n and $\{\pi_a^n: a\in P_n\}$ which satisfy the following properties:

(i) $P_0=\omega$, and $\pi_a^0=\pi_a$ $(a\in P_0)$;

(ii) $P_{n+1}=Per(P_n)$ $(n<\omega)$;

(iii) $\pi_a^{n+1}: P_{n+1}\to P_n$ is a function such that $\pi_a^{n+1}(\sigma)=\sigma(a)$ $(a\in P_n,\ n<\omega)$.

Let $A^*=\bigcup\{P_n: n<\omega\}$. Consider the structure $M^*=(A^*; \{P_n: n<\omega\}, \{\pi_a^n: a\in P_n,\ n<\omega\})$. Then for each $n<\omega$, if $\sigma\in P_{n+1}$ then we have $P_n\subset dcl(\sigma)$. Hence M^* is a minimal model (Proof: Take any $N\prec M^*$ and $a\in M^*$. Then there is some n such that $a \in P_n$. Now $P_{n+1} \cap N \neq \emptyset$, so we can pick some $\sigma \in P_{n+1} \cap N$. Therefore $a \in dcl(\sigma) \subset N$, so $a \in N$. It follows that $N = M^*$). It is easy to see that $P_0 = \omega$ is an infinite indiscernible set. Moreover M^* is not superstable, since M is interpreted in M^* (Recall that M is not superstable).

References

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