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GEODESICS IN REDUCTIVE HOMOGENEOUS SPACES

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

By

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1. Introduction

The purpose of this paper is to give a complete description of geodesics in certain reductive homogeneous Riemannian manifolds. In case of naturally reductive homogeneous space, it is well known that geodesics are orbits of 1-parameter subgroups. On the other hand, H. C. Wang [7] studied the case of semisimple Lie groups with certain left invarient metric and determined all geodesics. In this paper we shall show a description of geodesics in certain reductive homogeneous Riemannian manifolds which include the case of naturally reductive and the case of semisimple Lie groups due to H. C. Wang. We first recall the cases of naturally reductive and semisimple Lie groups more precisely and then state our main result.

Let $M=G/K$ be a reductive homogeneous space with decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$. We identify the tangent space T_0M at the origin $o=\{K\}$ with $\mathfrak{m}$ in a natural manner.

(1) (cf. [6]) Let $M=G/K$ be a reductive homogeneous space with a G-invariant Riemannian metric g. Then the Riemannian homogeneous space $M=G/K$ is said to be *naturally reductive* if it admits the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ satisfying the condition

$$g(X, [Z, Y]_{\mathfrak{m}})_o+g([Z, X]_{\mathfrak{m}}, Y)_o=0$$

for $X, Y, Z\in\mathfrak{m}$.

Then a geodesic $\gamma(t)$ in a naturally reductive homogeneous space $M=G/K$ such that $\gamma(0)=o$, $\dot{\gamma}(0)=X$ is written by

$$\exp(tX)\cdot o\,.$$

(2) (H. C. Wang [7]) Let G be a connected semisimple Lie group, $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a Cartan decomposition of its Lie algebra, and θ the corresponding Cartain involution. Consider the left invariant Riemannian metric given by the positive definite bilinear form B_θ on $\mathfrak{g}\times\mathfrak{g}$ such that,

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$$B_\theta(X, Y) = -B(X, \theta Y)$$

where B is the Killing form of $\mathfrak{g}$, $X, Y \in \mathfrak{g}$. Then a geodesic $\gamma(t)$ in G such that $\gamma(0)=e$, $\dot{\gamma}(0)=T+X$ for $T\in\mathfrak{k}$, $X\in\mathfrak{p}$ is written by

$$\exp t(X-T) \exp 2tT .$$

In this paper, we study geodesics in reductive homogeneous spaces satisfying certain conditions, which will be denote by $(G/K, \mathfrak{g}, \mathfrak{m}^1\oplus\mathfrak{m}^2, c)$ (see section 2 for notations). In section 2 we shall see that some of Kähler C-spaces with second Betti number $b_2=1$ and connected semisimple Lie groups with certain left invariant metric are examples of these spaces. Now we can state our main theorem.

THEOREM. *Let $\gamma(t)$ be a geodesic in $(G/K, g, \mathfrak{m}^1\oplus\mathfrak{m}^2, c)$ such that $\gamma(0)=o$, $\dot{\gamma}(0)=X^1+X^2$ for $X^i\in\mathfrak{m}^i$. Then,*

$$\gamma(t)=\exp t(X^1+cX^2) \exp (1-c)tX^2\cdot o .$$

As application of this theorem, we shall show that a geodesic in $(G/K, g, \mathfrak{m}^1\oplus\mathfrak{m}^2, c)$ which intersects itself is a closed geodesic.

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2. Definition and examples

Let $M=G/K$ be a reductive homogeneous spacedelete with the $\mathrm{Ad}(K)$ invariant decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ and a G-invariant Riemannian metric g. Now, we assume that $\mathfrak{m}$ has an orthogonal decomposition $\mathfrak{m}=\mathfrak{m}^1\oplus\mathfrak{m}^2$ with respect to g_o satisfying the following conditions:

$$[\mathfrak{k}, \mathfrak{m}^i]\subset\mathfrak{m}^i \qquad \text{for } i=1, 2, \tag{2.1}$$

$$[\mathfrak{m}^1, \mathfrak{m}^1]\subset\mathfrak{k}+\mathfrak{m}^2, \tag{2.2}$$

$$[\mathfrak{m}^2, \mathfrak{m}^2]\subset\mathfrak{k}, \tag{2.3}$$

$$[\mathfrak{m}^1, \mathfrak{m}^2]\subset\mathfrak{m}^1, \tag{2.4}$$

Moreover there exists nonzero constant c in $\boldsymbol{R}$ such that

$$g([X, Y]_{\mathfrak{m}^2}, Z)_o+cg(X, [Z, Y])_o=0, \tag{2.5}$$

for each $X, Y\in\mathfrak{m}^1$, $Z\in\mathfrak{m}^2$.

We denote the above space by $(G/K, g, \mathfrak{m}^1\oplus\mathfrak{m}^2, c)$.

REMARK. Let $(\mathfrak{g}, \mathfrak{k}+\mathfrak{m}^2)$ and $(\mathfrak{k}+\mathfrak{m}^2, \mathfrak{k})$ be orthogonal symmetric pairs, and let $\mathfrak{m}^1$ be an orthogonal complement of $\mathfrak{k}+\mathfrak{m}^2$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Then $\mathfrak{m}=\mathfrak{m}^1\oplus\mathfrak{m}^2$ satisfies conditions (2.1), (2.2), (2.3) and (2.4).

EXAMPLE 2.1. Kähler C-spaces with second Betti number $b_2=1$ (cf. [4]).

Let $\mathfrak{g}^C$ and $\mathfrak{h}^C$ respectively be a complex semisimple Lie algebra and its Cartan subalgebra. Put $l=\dim_C\mathfrak{h}^C$. Δ denotes the set of nonzero roots of $\mathfrak{g}^C$ with respect to $\mathfrak{h}^C$.

Let B be the Killing form of $\mathfrak{g}^C$. For $\xi\in(h^C)^*=\mathrm{Hom}(\mathfrak{h}^C; \boldsymbol{C})$, we define $H_\xi\in\mathfrak{h}^C$ by $B(H, H_\xi)=\xi(H)$ for all $H\in\mathfrak{h}^C$. Fix a suitable lexicographic order on $(\mathfrak{h}^C)^*$. Put a fundamental root system $\Pi=\{\alpha_1, \cdots, \alpha_l\}$. By Δ^+ and Δ^- we denote the sets of positive and negative roots, respectively. For each $\alpha\in\Delta$, we select a root vector E_α so that $\{H_j=H_{\alpha_j}, (j=1, \cdots, l), E_\alpha, (\alpha\in\Delta)\}$ forms Weyl's canonical basis of $\mathfrak{g}^C$, that is, it satisfies the following

$$B(E_\alpha, E_{-\alpha})=-1\,, \qquad \text{for } \alpha\in\Delta^+\,,$$
$$[E_\alpha, E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta}\,,$$
$$N_{\alpha,\beta}=N_{-\alpha,-\beta}\in\boldsymbol{R}\,.$$

Then following (vector) space $\mathfrak{g}$ is a compact real form of $\mathfrak{g}^C$:

$$\mathfrak{g}=\sum_{j=1}^{l}\boldsymbol{R}\sqrt{-1}H_j+\sum_{\alpha\in\Delta^+}\{\boldsymbol{R}A_\alpha+\boldsymbol{R}B_\alpha\},$$

where $A_\alpha=E_\alpha+E_{-\alpha}$, $B_\alpha=\sqrt{-1}(E_\alpha-E_{-\alpha})$, $\alpha\in\Delta^+$.

We assume that $\mathfrak{g}^C$ is simple. Consider a subset $\Phi=\{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$ of Π. We define a subset $\Delta^+(\Phi)$ of Δ^+ and a complex subalgebra $\mathfrak{l}_\Phi$ of $\mathfrak{g}^C$ by

$$\Delta^+(\Phi)=\left\{\alpha=\sum_{j=1}^{l}n_j\alpha_j\in\Delta^+\,;\, n_{i_k}\geqq 0 \text{ for some } \alpha_{i_k}\in\Phi\right\},$$

$$\mathfrak{l}_\Phi=\mathfrak{h}+\sum_{\alpha\in\Delta^+}CE_{-\alpha}+\sum_{\beta\in\Delta^+-\Delta^+(\Phi)}CE_\beta\,.$$

Then the intersection $\mathfrak{k}=\mathfrak{g}\cap\mathfrak{l}_\Phi$ is a real subalgebra of $\mathfrak{g}$ expressed as

$$\mathfrak{k}=\sum_{j=1}^{l}\boldsymbol{R}\sqrt{-1}H_j+\sum_{\alpha\in\Delta^+-\Delta^+(\Phi)}\{\boldsymbol{R}A_\alpha+\boldsymbol{R}B_\alpha\}\,.$$

Let G^C be a simply connected complex Lie group with the Lie algebra $\mathfrak{g}^C$ and L_Φ a connected closed complex subgroup of G^C with the Lie subalgebra $\mathfrak{l}_\Phi$. Let G be a simply connected compact group with the Lie algebra $\mathfrak{g}$, and K be a connected closed subgroup with the Lie subalgebra $\mathfrak{k}$.

The canonical imbedding $G\rightarrow G^C$ gives a diffeomorphism of a compact homogeneous space $M=G/K$ to a simply connected complex homogeneous space G^C/L_Φ. By a result of Borel and Hirzebruch [2], we see that the second Betti number $b_2(M)=r$.

Hence we obtain a C-space G/K with $b_2=r$ from a pair $(\mathfrak{g}^C, \Phi)$, where $\mathfrak{g}^C$ is a complex simple Lie algebra and Φ is a subset of Π.

When G/K admits a G-invariant Kähler metric g, $(G/K, g)$ is a Kähler C-space.

Conversely, a Kähler C-space with $b_2=r$ can be described by a homogeneous space G/K for some G and K, with a G-invariant Kähler metric g [8].

Define a linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ as follows:

$$\mathfrak{m}=\sum_{\alpha\in\Delta^+(\Phi)}\{\boldsymbol{R}A_\alpha+\boldsymbol{R}B_\alpha\}.$$

Then we have $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ as a direct sum and $[\mathfrak{k}, \mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}]\subset\mathfrak{m}$ and $\mathfrak{k}\perp\mathfrak{m}$ with respect to B.

Since K leaves the orign of a C-space $M=G/K$ fixed, K acts on the tangent space at the origin as the linear isotropy and the adjoint representation of K on $\mathfrak{m}$.

Let a complex structure I of $\mathfrak{m}$ be defined by $IA_\alpha=B_\alpha$, $IB_\alpha=-A_\alpha$, $\alpha\in\Delta^+(\Phi)$. This gives a G-invariant complex structure on G/K and coincides with the canonical structure induced from the complex homogeneous space G^C/L_Φ.

Now, suppose that Φ consists of a single root α_i of Π, that is $\Phi=\{\alpha_i\}$. This is equivalent to that the second Betti number $b_2=1$.

By $\Delta^+(\alpha_i\,;n)$ we define a subset

$$\left\{\alpha\in\Delta^+(\Phi)\,;\ \alpha=\sum_{j=1}^{l}m_j\alpha_j,\ m_i=n\right\}$$

of $\Delta^+(\Phi)$. Then $\mathfrak{m}^n=\sum_{\alpha\in\Delta^+(\alpha_i;n)}\{\boldsymbol{R}A_\alpha+\boldsymbol{R}B_\alpha\}$ gives a linear subspace of $\mathfrak{m}$ for $n\in\boldsymbol{N}$. They satisfy the following properties:

$$[\mathfrak{k}, \mathfrak{m}^n]\subset\mathfrak{m}^n,$$

$$[\mathfrak{m}^n, \mathfrak{m}^m]\subset\mathfrak{m}^{n+m}+\mathfrak{m}^{|n-m|}, \qquad (n\neq m)$$

$$[\mathfrak{m}^n, \mathfrak{m}^n]\subset\mathfrak{k}+\mathfrak{m}^{2n}.$$

By a theorem of Borel [1], we obtain the Kähler metric g as follows:

$$g(\sum_n X^n, \sum_m Y^m)_o=\kappa\sum_n n\{-B(X^n, Y^n)\},$$

where B is Killing form of $\mathfrak{g}^C$, X^n, $Y^n\in\mathfrak{m}^n$, $\kappa>0$.

Suppose that $\Delta^+(\alpha_i, k)=\emptyset$ for $k\geqq3$. Then we have,

$$\mathfrak{m}=\mathfrak{m}^1+\mathfrak{m}^2,$$

$$g(X^1+X^2,\ Y^1+Y^2)_o=-\kappa B(X^1,\ Y^1)-2\kappa B(X^2,\ Y^2).$$

Thus we see that a Kähler C-space G/K associated with $(\mathfrak{g}^C, \{\alpha_i\})$ where $\Delta^+(\alpha_i, k)=\emptyset$ for $k\geqq 3$ satisfies conditions (2.1), (2.2), (2.3), (2.4) and (2.5).

Using our notation, we denote this space by $(G/K, g, \mathfrak{m}^1\oplus\mathfrak{m}^2, 2)$.

REMARK. In above case, $\mathfrak{m}^1$, $\mathfrak{m}^2$ are irreducible as K-module (cf. [5]). Hence, G-invariant Riemannian metric g of G/K is written by

$$g(X^1+X^2,\ Y^1+Y^2)_o=-\kappa B(X^1,\ Y^1)-\kappa c B(X^2,\ Y^2)$$

where X^i, $Y^i\in\mathfrak{m}^i$ $(i=1, 2)$, $c>0$. We denote this metric by g_c. Then, for $c>0$, the space $(G/K, g_c, \mathfrak{m}^1\oplus\mathfrak{m}^2, c)$ is an example of our spaces.

EXAMPLE 2.2. The connected semisimple Lie group.

Let G be a connected semisimple Lie group, $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a Cartan decomposition of its Lie algebra, and θ the corresponding Cartan involution. Let K be a closed subgroup with Lie algebra $\mathfrak{k}$. We put a Lie subgroup of $G\times K$ as follows:

$$K^*=\{(k,\ k)\in G\times K \mid k\in K\}.$$

Then we have $G\cong G\times K/K^*$, and

$$Lie(G\times K)=\{(X,\ Y)\mid X\in\mathfrak{g},\ Y\in\mathfrak{k}\},$$

$$Lie(K^*)=\mathfrak{k}^*=\{(Y,\ Y)\mid Y\in\mathfrak{k}\}.$$

We put subspaces of $Lie(G\times K)$ as follows:

$$\mathfrak{m}^1=\{(X,\ 0)\mid X\in\mathfrak{p}\},$$

$$\mathfrak{m}^2=\{(T,\ -T)\mid T\in\mathfrak{k}\},$$

$$\mathfrak{m}=\mathfrak{m}^1\oplus\mathfrak{m}^2.$$

Then we have $Lie(G\times K)=\mathfrak{k}^*+\mathfrak{m}$ and $\mathfrak{m}=T_o(G\times K/K^*)$. Consider the left invariant Riemannian metric g given by the positive definite bilinear from B_θ on $\mathfrak{m}\times\mathfrak{m}$ such that

$$B_\theta((X+T,\ -T),\ (Y+S,\ -S))=-B(T,\ S)+B(2X,\ 2Y)$$

where B is the Killing form of $\mathfrak{g}$, $(X, 0)$, $(Y, 0)\in\mathfrak{m}^1$, and $(T, -T)$, $(S, -S)$ $\in\mathfrak{m}^2$. It is clear that this space satisfies condition (2.1), (2.2), (2.3), (2.4) and (2.5). Hence, we can denote this space by $(G\times K/K^*, g, \mathfrak{m}^1\oplus\mathfrak{m}^2, -2)$.

3. Main theorem

THEOREM 3.1. *Let $\gamma(t)$ be a geodesic in $(G/K, \mathfrak{g}, \mathfrak{m}^1\oplus\mathfrak{m}^2, c)$ such that $\gamma(0)=o$, $\dot{\gamma}(0)=X^1+X^2$ for $X^i\in\mathfrak{m}^i$. Then,*

$$\gamma(t)=\exp t(X^1+cX^2)\exp(1-c)tX^2\cdot o\,.$$

In order to prove this theorem, we need some preparations.

Let G/K be a reductive homogeneous space. Put π denote the natural mapping of G onto G/K. For $g\in G$, we put L_g and R_g the left translation $h\mapsto gh$ and the right translation $h\mapsto hg$ of G, respectively, and τ_g be the diffeomorphism $hK\mapsto ghK$ of G/K onto itself.

For $X, Y\in\mathfrak{m}$, we defined vector fields on G as follows:

$$\tilde{X}_g=(L_g)_*X\,,\qquad \hat{Y}_g=(R_g)_*Y\,.$$

Then, by easy calculation, we obtain a following Lemma.

LEMMA 3.2. *For each $X, Y\in\mathfrak{m}$, $g\in G$,*

$$(\tau_{g^{-1}})_*\pi_*(\tilde{X}_g)=X\,. \tag{3.6}$$

$$(\tau_{g^{-1}})_*\pi_*(\hat{X}_g)=\{Ad(g^{-1})X\}_{\mathfrak{m}}\,, \tag{3.7}$$

$$[\tilde{X}, \tilde{Y}]_g=\widetilde{[X, Y]}_g\,, \tag{3.8}$$

$$[\tilde{X}, \hat{Y}]_g=0\,. \tag{3.9}$$

For $X, Z\in\mathfrak{m}$, we put curves $\sigma(t)$ and $\gamma(t)$, respectively in G, and $M=G/K$ as follows,

$$\sigma(t)=\exp tX\exp tZ\,,\qquad \gamma(t)=\pi(\sigma(t))\,.$$

Then, the vector field along $\sigma(t)$ is written by

$$\dot{\sigma}(t)=(\hat{X}+\tilde{Z})_{\sigma(t)}\,.$$

Take a normal neighborhood V of 0 in $\mathfrak{m}$, that is $\pi|_{\exp(V)}:\exp(V)\to\pi(\exp(V))$ is a diffeomorphism. We put $U=\{\sigma(t)g,\ g\in\exp(V)\}$. Then $\pi(U)$ is a neighborhood of the curve $\gamma(t)$. We extend $\dot{\gamma}$ to a vector field on the neighborhood $\pi(U)$ of $\gamma(t)$ as follows:

$$\dot{\gamma}_{\pi(\sigma(t)g)}:=\pi_*((\hat{X}+\tilde{Z})_{\sigma(t)g})$$

for $\sigma(t)g\in U$. We denote this extended vector field by same notation $\dot{\gamma}$.

PROOF OF THEOREM 3.1. Put $\sigma(t)=\exp tX\exp tZ$, $\gamma(t)=\pi(\sigma(t))$. For arbitrary $Y\in\mathfrak{m}$, we define a vector field $\tilde{Y}_\sigma$ on U by

$$\tilde{Y}_\sigma := \tilde{Y}|_U .$$

Then we have

$$g(\pi_*(\tilde{Y}_\sigma), \nabla_{\dot\gamma}\dot\gamma)_{\gamma(t)} = \dot\gamma_{\gamma(t)} g(\pi_*(\tilde{Y}_\sigma), \dot\gamma) + g(\dot\gamma, [\pi_*(\tilde{Y}_\sigma), \dot\gamma])_{\gamma(t)} - \frac{1}{2}(\pi_*(\tilde{Y}_\sigma))_{\gamma(t)} \|\dot\gamma\|_o^2, \tag{3.10}$$

where $\|\dot\gamma\|^2 := g(\dot\gamma, \dot\gamma)$.

From equations (3.6), (3.7), (3.8) and (3.9) in Lemma 3.2, we can calculate as follows.

$$\begin{aligned}
&\dot\gamma_{\gamma(t)} g(\pi_*(\tilde{Y}_\sigma), \dot\gamma) \\
&= \frac{d}{ds}\Big|_{s=0} g(\pi_*(\tilde{Y}_\sigma), \dot\gamma)_{\gamma(t+s)} \\
&= \frac{d}{ds}\Big|_{s=0} g(\pi_*(\tilde{Y}_{\sigma(t+s)}, \pi_*((\hat{X}+\tilde{Z})_{\sigma(t+s)}))_{\sigma(t+s)\cdot o} \\
&= \frac{d}{ds}\Big|_{s=0} g((\tau_{(\tau-(t+s))})_*\pi_*(\tilde{Y}_\sigma), (\tau_{\sigma(-(t+s))})_*\pi_*((\hat{X}+\tilde{Z})_{\sigma(t+s)}))_o \\
&= \frac{d}{ds}\Big|_{s=0} g(Y, \{\mathrm{Ad}\,(\sigma(t+s)^{-1})X+Z\}_{m})_o \\
&= \frac{d}{ds}\Big|_{s=0} g(Y, \{\mathrm{Ad}\,(\exp(-(t+s)Z)\exp(-(t+s)X))X+Z\}_{m})_o \\
&= \frac{d}{ds}\Big|_{s=0} g(Y, \{\mathrm{Ad}\,(\exp(-(t+s)Z))X+Z\}_{m})_o \\
&= g(Y, \{\mathrm{Ad}\,(\exp(-tZ))[X, Z]\}_{m})_o ,
\end{aligned}$$

$$\begin{aligned}
&g(\dot\gamma, [\pi_*(\tilde{Y}_\sigma), \dot\gamma])_{\gamma(t)} \\
&= g(\pi_*((\hat{X}+\tilde{Z})_{\sigma(t)}), [\pi_*(\tilde{Y}_{\sigma(t)}), \pi_*((\hat{X}+\tilde{Z})_{\sigma(t)})])_{\sigma(t)\cdot o} \\
&= g((\tau_{\sigma(t)^{-1}})_*\pi_*((\hat{X}+\tilde{Z})_{\sigma(t)}, (\tau_{\sigma(t)^{-1}})_*[\pi_*(\tilde{Y}_{\sigma(t)}), \pi_*((\hat{X}+\tilde{Z})_{\sigma(t)})])_o \\
&= g(\{\mathrm{Ad}\,(\sigma(t)^{-1})X+Z\}_{m}, (\tau_{\sigma(t)^{-1}})_*\pi_*(\widetilde{[Y, Z]}_{\sigma(t)}))_o \\
&= g(\{\mathrm{Ad}\,(\exp(-tZ))X+Z\}_{m} [Y, Z]_{m})_o ,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{2}(\pi_*\tilde{Y}_\sigma)_{\gamma(t)} \|\dot\gamma\|_o^2 \\
&= -\frac{1}{2}\frac{d}{ds}\Big|_{s=0} \|\pi_*((\hat{X}+\tilde{Z})_{\sigma(t)\exp sY})\|^2_{\sigma(t)\exp sY\cdot o} \\
&= -\frac{1}{2}\frac{d}{ds}\Big|_{s=0} \|(\tau_{\exp(-sY)\sigma(t)^{-1}})_*\pi_*((\hat{X}+\tilde{Z})_{\sigma(t)\exp sY})\|_o^2
\end{aligned}$$

$$=-\frac{1}{2}\frac{d}{ds}\bigg|_{s=0}\|\{\mathrm{Ad}(\exp(-sY)\sigma(t)^{-1})X+Z\}_{\mathfrak{m}}\|_o^2$$

$$=-\frac{1}{2}\frac{d}{ds}\bigg|_{s=0}\|\{\mathrm{Ad}\exp(-sY)\exp(-tZ))X+Z\}_{\mathfrak{m}}\|_o^2$$

$$=-g([\mathrm{Ad}(\exp(-tZ))X,\ Y]_{\mathfrak{m}},\ \{\mathrm{Ad}(\exp(-tZ))X+Z\}_{\mathfrak{m}})_o\,.$$

Hence (3.10) is

$$g(Y,\ \{\mathrm{Ad}(\exp(-tZ))[X,\ Z]\}_{\mathfrak{m}})_o$$
$$+g(\{\mathrm{Ad}(\exp(-tZ))X+Z\}_{\mathfrak{m}},\ [Y,\ Z]_{\mathfrak{m}})_o$$
$$-g([\mathrm{Ad}(\exp(-tZ))X,\ Y]_{\mathfrak{m}},\ \{\mathrm{Ad}(\exp(-tZ))X+Z\}_{\mathfrak{m}})_o\,.$$

Now, we put

$$X=X^1+cX^2 \quad\text{and}\quad Z=(1-c)X^2 \quad\text{for } X^i\in\mathfrak{m}^i\,.$$

Then, (3.10) is

$$g(Y,\ \{\mathrm{Ad}(\exp(c-1)tX^2)[X^1+cX^2,\ (1-c)X^2]\}_{\mathfrak{m}})_o$$
$$+g(\{\mathrm{Ad}(\exp(c-1)tX^2)(X^1+cX^2)+(1-c)X^2\}_{\mathfrak{m}},\ [Y,\ (1-c)X^2]_{\mathfrak{m}})_o$$
$$-g([\mathrm{Ad}(\exp(c-1)tX^2)(X^1+cX^2),\ Y]_{\mathfrak{m}},$$
$$\{\mathrm{Ad}(\exp(c-1)tX^2)(X^1+cX^2)+(1-c)X^2\}_{\mathfrak{m}})_o$$
$$=g(Y,\ [\mathrm{Ad}(\exp(c-1)tX^2)X^1,\ (1-c)X^2]_{\mathfrak{m}})_o$$
$$+g((\mathrm{Ad}(\exp(c-1)tX^2)X^1+X^2)_{\mathfrak{m}},\ ]Y,\ (1-c)X^2]_{\mathfrak{m}})_o$$
$$-g([\mathrm{Ad}(\exp(c-1)tX^2)X^1+cX^2,\ Y]_{\mathfrak{m}},$$
$$\{\mathrm{Ad}(\exp(c-1)tX^2)X^1+X^2\}_{\mathfrak{m}})_o\,. \tag{3.11}$$

Since $[\mathfrak{m}^2,\ \mathfrak{m}^1]\subset\mathfrak{m}^1$, for simplicity, we can put

$$\mathrm{Ad}(\exp(c-1)tX^2)X^1=:Z^1\in\mathfrak{m}^1\,.$$

Then, (3.11) is

$$g(Y,\ [Z^1,\ (1-c)X^2])_o+g(Z^1+X^2,\ [Y,\ (1-c)X^2]_{\mathfrak{m}})_o$$
$$-g([Z^1+cX^2,\ Y]_{\mathfrak{m}},\ Z^1+X^2)_o\,. \tag{3.12}$$

First, we put $Y=Y^1\in\mathfrak{m}^1$. From (2.2), (2.4) and (2.5), (3.12) is

$$g(Y^1,\ [Z^1,\ (1-c)X^2])_o+g(Z^1+X^2,\ [Y^1,\ (1-c)X^2]_{\mathfrak{m}})_o$$
$$-g([Z^1+cX^2,\ Y^1]_{\mathfrak{m}},\ Z^1+X^2)_o$$
$$=g(Y^1,\ [Z^1,\ (1-c)X^2])_o+g(Z^1,\ [Y^1,\ (1-c)X^2])_o$$
$$-g([Z^1,\ Y^1]_{\mathfrak{m}},\ X^2)_o-g([cX^2,\ Y^1],\ Z^1)_o$$

$$=(1-c)g(Y^1, [Z^1, X^2])_o+(1-c)g(Z^1, [Y^1, X^2])_o$$
$$-g([Z^1, Y^1]_{\mathfrak{m}}, X^2)_o-cg([X^2, Y^1], Z^1)_o$$
$$=\frac{1-c}{c}g([Y^1, Z^1]_{\mathfrak{m}}, X^2)_o+\frac{1-c}{c}g([Z^1, Y^1]_{\mathfrak{m}}, X^2)_o$$
$$-cg(Z^1, [Y^1, X^2])_o-cg([X^2, Y^1], Z^1)_o$$
$$=0 .$$

Next, we put $Y=Y^2\in\mathfrak{m}^2$. From conditions (2.3), (2.4) and (2.5), (3.12) is

$$g(Y^2, [Z^1, (1-c)X^2])_o+g(Z^1, [Y^2, (1-c)X^2]_{\mathfrak{m}})_o$$
$$-g([Z^1+cX^2, Y^2]_{\mathfrak{m}}, Z^1+X^2)_o$$
$$=-g([Z^1, Y^2], Z^1)_o$$
$$=-\frac{1}{c}g(Y^2, [Z^1, Z^1]_{\mathfrak{m}})_o$$
$$=0 .$$

Hence, for a curve $\gamma(t)=\pi(\exp t(X^1+cX^2)\cdot\exp(1-c)tX^2)$, we get

$$(\nabla_{\dot\gamma}\dot\gamma)_{\gamma(t)}=0 .$$

Q.E.D.

REMARK. From Theorem 3.1 and Example 2.2, we have the result of H.C. Wang ([7]).

Moreover, we suppose that

$$\mathrm{Ad}(K)\mathfrak{m}^i\subset\mathfrak{m}^i \qquad \text{for } i=1, 2 . \tag{3.13}$$

If K is connected, this condition (3.13) is equivariant to (2.1). Under this assumption, we get the following Corollary:

COROLLARY 3.3. *A geodesic in* $(G/K, g, \mathfrak{m}^1\oplus\mathfrak{m}^2, c)$ *which intersects itself is a closed geodesic.*

PROOF. Let $\gamma(t)=\exp t(X^1+cX^2)\exp(1-c)tX^2\cdot o$ be a geodesic in G/K such that $\gamma(0)=\gamma(L)$ for some $L\in\boldsymbol{R}\setminus\{0\}$. Then,

$$\exp L(X^1+cX^2)\exp(1-c)LX^2\in K .$$

We put $k=\exp L(X^1+cX^2)\exp(1-c)LX^2$. Then,

$$\exp L(X^1+cX^2)=k\exp(c-1)LX^2 .$$

In order to show $\dot\gamma(L)=\dot\gamma(0)=X^1+X^2$, we shall calculate $\dot\gamma(L)$ by two ways.

First we have

$$\begin{aligned}\gamma(L+t)&=\exp(L+t)(X^1+cX^2)\exp(1-c)(L+t)X^2\cdot o\\&=\exp L(X^1+cX^2)\exp t(X^1+cX^2)\exp(1-c)LX^2\exp(1-c)tX^2\cdot o\\&=k\exp(c-1)LX^2\exp t(X^1+cX^2)\exp(1-c)LX^2\exp(1-c)tX^2\cdot o\\&=k\exp(c-1)LX^2\exp t(X^1+cX^2)\exp(1-c)LX^2k^{-1}k\exp(1-c)tX^2\cdot o\\&=\exp t(\mathrm{Ad}(k\exp(c-1)LX^2)(X^1+cX^2))\exp(1-c)t(\mathrm{Ad}(k)X^2)\cdot o\\&=\exp t(\mathrm{Ad}(k\exp(c-1)LX^2)X^1+c\,\mathrm{Ad}(k)X^2)\exp(1-c)t(\mathrm{Ad}(k)X^2)\cdot o.\end{aligned}$$

Hence we have

$$\dot{\gamma}(L)=\mathrm{Ad}(k\exp(c-1)LX^2)X^1+\mathrm{Ad}(k)X^2\,. \tag{3.14}$$

Secondly

$$\begin{aligned}\gamma(L+t)&=\exp(L+t)(X^1+cX^2)\exp(1-c)(L+t)X^2\cdot o\\&=\exp t(X^1+cX^2)\exp L(X^1+cX^2)\exp(1-c)(L+t)X^2\cdot o\\&=\exp t(X^1+cX^2)k\exp(c-1)LX^2\exp(1-c)(L+t)X^2\cdot o\\&=\exp t(X^1+cX^2)k\exp(1-c)tX^2\cdot o\\&=\exp t(X^1+cX^2)\exp(1-c)t\,\mathrm{Ad}(k)X^2\cdot o\,.\end{aligned}$$

Thus we have

$$\dot{\gamma}(L)=X^1+cX^2+(1-c)\,\mathrm{Ad}(k)X^2\,. \tag{3.15}$$

By conditions (2.4) and (3.13), we have

$$\mathrm{Ad}(k\exp(c-1)LX^2)X^1\in\mathfrak{m}^1\,.$$

Hence, from (3.14), (3.15), we get

$$cX^2+(1-c)\,\mathrm{Ad}(k)X^2=\mathrm{Ad}(k)X^2\,.$$

That is

$$\mathrm{Ad}(k)X^2=X^2\,.$$

Hence, from (3.15), we obtain

$$\dot{\gamma}(L)=X^1+X^2\,.$$

Q. E. D.

References

[1] A. Borel, Kählerian coset spaces of semisimple Lie groups, Proc. Nat. Acad. Sci. U. S. A., **40** (1954), 1147-1151.

[2] A. Borel and F. Hirzebruch, Characteristic classes and homogeneos spaces I, Amer. J. Math. **80** (1958), 458-538.

[3] S. Helgason, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.
[4] M. Itoh, On curvature properties of Kähler C-spaces, J. Math. Japan, **30**, No. 1 (1978), 39-71.
[5] M. Kimura, Homogeneous Einstein metrics on certain Kähler C-spaces, Recent Topics in Differential and Analytic Geometry, Advanced Studies in Pure Mathematics 18-I, (1990), 303-320.
[6] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry II, Interscience publishers, 1969.
[7] H.C. Wang, Discrete nilpotent subgroups of Lie groups, J. Differential Geometry, **3** (1969), 481-492.
[8] H.C. Wang, Closed manifolds with homogeneous complex structure, Amer. J. Math., **76** (1954), 1-32.

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