

## GORENSTEIN BALANCE OF HOM AND TENSOR

By

Edgar E. ENOCHS and Overtoun M.G. JENDA

By Auslander and Bridger ([1], Proposition 3.8c)), a finitely generated left  $R$ -module  $C$  has Gorenstein dimension 0 if and only if there is an exact sequence

$$\dots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \dots$$

of finitely generated projective left  $R$ -modules such that  $C = \ker(P^0 \rightarrow P^1)$  and such that the dual sequence

$$\dots \longrightarrow (P^1)^* \longrightarrow (P^0)^* \longrightarrow (P^{-1})^* \longrightarrow \dots$$

is also exact.

In attempting to dualize the notion of Gorenstein dimension we called such modules  $C$  Gorenstein projective modules (see [8]) and then defined Gorenstein injective modules.

Auslander and Buchweitz showed that over certain rings all finitely generated modules have Gorenstein projective precovers (over Cohen-Macaulay rings these are their maximal Cohen-Macaulay approximations).

An application of their argument shows that over an  $n$ -Gorenstein ring all modules have Gorenstein injective preenvelopes.

Over a ring where these precovers and preenvelopes exist, we can apply methods of relative homological algebra (see Eilenberg and Moore [5]) and compute derived functors.

We can then raise the question of balance in the sense of Enochs and Jenda [6]. We can now show that  $\text{Hom}(-, -)$  is right balanced by Gorenstein projective and injective modules on a suitable category. Similarly we show that  $-\otimes-$  is left balanced by finitely generated Gorenstein projective modules (left and right).

### 1. Gorenstein Injective and Projective Resolutions.

In the following, module will mean left  $R$ -module for some ring  $R$  (unless otherwise specified).

DEFINITION 1.1. A left  $R$ -module  $G$  is said to be Gorenstein injective if there is an exact sequence

$$\dots \longrightarrow E^{-1} \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \dots$$

of injective left  $R$ -modules such that  $G = \ker(E^0 \rightarrow E^1)$  and such that for every injective left  $R$ -module  $E$ , the sequence

$$\dots \text{Hom}(E, E^{-1}) \longrightarrow \text{Hom}(E, E^0) \longrightarrow \text{Hom}(E, E^1) \longrightarrow \dots$$

is also exact.

REMARK 1.2. In [8] it was argued that if  $R$  is left noetherian, every direct summand of a Gorenstein injective module is Gorenstein injective. In ([1], 3.11(b)) it was shown that a similar result for finitely generated Gorenstein projective modules holds.

LEMMA 1.3. *If  $G$  is a Gorenstein injective left  $R$ -module and  $L$  is a left  $R$ -module of finite projective dimension then  $\text{Ext}^i(L, G) = 0$  for all  $i \geq 1$ .*

PROOF. Letting  $\dots E^{-1} \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$  be as above and letting  $G^{-j} = \ker(E^{-j} \rightarrow E^{-j+1})$  for  $j \geq 0$  we see that

$$\text{Ext}^i(L, G) = \text{Ext}^{i+j}(L, G^{-j})$$

for each such  $j$ . But for  $j \geq \text{proj. dim } L$ ,  $\text{Ext}^{i+j}(L, G^{-j}) = 0$  when  $i \geq 1$ . A similar argument gives

LEMMA 1.4 (Auslander, Bridger [1]). *If  $C$  is a finitely generated Gorenstein projective module and  $L$  has finite injective dimension, then  $\text{Ext}^i(C, L) = 0$  for all  $i \geq 1$ .*

DEFINITION 1.5 (Iwanaga [10]). A ring  $R$  is said to be i.e.  $n$ -Gorenstein if it is left and right noetherian and if it has self-injective dimension at most  $n$  on both sides.

Over an  $n$ -Gorenstein ring  $R$  there are a plentiful supply of Gorenstein injective modules. This follows from

LEMMA 1.6 ([8], Theorem 4.2). *If  $R$  is an  $n$ -Gorenstein ring and*

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \dots \longrightarrow E^{n-1} \longrightarrow G \longrightarrow 0$$

is a partial injective resolution of  $N$ , then  $G$  is Gorenstein injective.

REMARK. If  $G$  is a finite group, it is easy to see that the group ring  $ZG$

is 1-Gorenstein, for

$$0 \longrightarrow ZG \longrightarrow QG \longrightarrow Q/ZG \longrightarrow 0$$

is an exact sequence of left or right  $ZG$ -modules.

But  $QG = Q \otimes_Z ZG \cong \text{Hom}_Z(ZG, Q)$  (see [3]) is an injective  $ZG$ -module. Similarly  $Q/ZG$  is injective.

From this we can show that a  $ZG$ -module  $M$  is Gorenstein injective if and only if it is divisible as a  $Z$ -module.

The condition is necessary since every injective  $ZG$ -module is divisible and  $M$  is a quotient of an injective module. Conversely, if  $M$  is divisible,  $\text{Hom}_Z(ZG, M)$  is injective. But  $\text{Hom}_Z(ZG, M) \cong ZG \otimes_Z M$  and there is surjection  $ZG \otimes_Z M \rightarrow M$ . Hence by the preceding lemma,  $G$  is Gorenstein injective.

A similar argument shows that a finitely generated  $ZG$ -module  $C$  is Gorenstein projective if and only if it is a free  $Z$ -module.

DEFINITION 1.7 (Enochs [7]). If  $N$  and  $G$  are left  $R$ -modules and  $G$  is Gorenstein injective, then a linear map  $\phi : N \rightarrow G$  is called a Gorenstein injective preenvelope of  $N$  if the diagram

$$\begin{array}{ccc} N & \xrightarrow{\phi} & G \\ & \searrow & \vdots \\ & & G' \end{array}$$

can be completed to a commutative diagram whenever  $N \rightarrow G'$  is a map into a Gorenstein injective module  $G'$ .

If furthermore,

$$\begin{array}{ccc} N & \xrightarrow{\phi} & G \\ & \searrow \phi & \vdots \\ & & G \end{array}$$

can only be completed by automorphisms of  $G$ , then  $\phi : N \rightarrow G$  is called a Gorenstein injective envelope of  $N$ .

Dually, a Gorenstein projective precover and cover  $\phi : C \rightarrow M$  is defined (with  $C$  and  $M$  finitely generated and  $C$  Gorenstein projective).

We note that since injective modules are Gorenstein injective and every modules  $N$  is contained in an injective module, every Gorenstein injective preen-

velope  $N \rightarrow G$  is an injection. Similarly, Gorenstein projective covers  $C \rightarrow M$  must be surjective.

**PROPOSITION 1.8.** *If  $R$  is  $n$ -Gorenstein, every  $R$ -module  $N$  has a Gorenstein injective preenvelope.*

**PROOF.** We use an argument dual to that in the proof of Theorem 1.1 of [2]. Let  $X$  be the class of Gorenstein injective modules and let  $w$  be the class of injective modules. Note that by [10] every injective module has finite projective dimension. So in the language of [2],  $w$  is a generator for  $X$ . Lemma 1.6 above then gives all that is needed for the dual argument to carry through. Hence for each  $R$ -module  $N$  there is an exact sequence

$$0 \longrightarrow N \longrightarrow G \longrightarrow L \longrightarrow 0$$

with  $G$  Gorenstein injective and  $L$  of finite projective dimension.

Then if  $G'$  is Gorenstein injective,  $\text{Ext}^1(L, G') = 0$  so  $\text{Hom}(G, G') \rightarrow \text{Hom}(N, G') \rightarrow 0$  is exact. This means that  $N \rightarrow G$  is a Gorenstein injective preenvelope.

**REMARK 1.9.** There is an alternate proof of this result in [8] (see Theorem 7.2). We note that there, under our hypotheses, we get a somewhat stronger uniqueness result than that in [2] for the  $X$ -approximations.

Now we use Theorem 1.8 of [2] and get that if  $R$  is  $n$ -Gorenstein, then for every finitely generated module  $M$  there is an exact sequence

$$0 \longrightarrow L \longrightarrow C \longrightarrow M \longrightarrow 0$$

with  $C$  finitely generated and Gorenstein projective and  $L$  having finite projective dimension (and so finite injective dimension by [10]).

Since then by Lemma 1.4  $\text{Ext}^1(C, L) = 0$  whenever  $C'$  is finitely generated and Gorenstein projective, we get that  $C \rightarrow M$  is a Gorenstein projective precover of  $M$ .

**DEFINITION 1.10.** An exact sequence

$$0 \longrightarrow N \longrightarrow G^0 \longrightarrow G^1 \longrightarrow G^2 \longrightarrow \dots$$

is called a Gorenstein injective resolution of the module  $N$  if each  $G^i$  is Gorenstein injective and if  $N \rightarrow G^0$ ,  $\ker(G^i \rightarrow G^{i+1}) \rightarrow G^i$  for  $i \geq 0$  are all Gorenstein injective preenvelopes (equivalently, whenever  $G$  is a Gorenstein injective module, the sequence

$$\cdots \text{Hom}(G^1, G) \longrightarrow \text{Hom}(G^0, G) \longrightarrow \text{Hom}(N, G) \longrightarrow 0$$

is exact).

In a similar way we define a Gorenstein projective resolution

$$\cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow M \longrightarrow 0$$

of the finitely generated module  $M$  (with all the  $C_i$  finitely generated too).

From Proposition 1.8 above we see that if  $R$  is  $n$ -Gorenstein then every module  $N$  admits a Gorenstein injective resolution. Also from the above we see that every finitely generated module  $M$  admits a Gorenstein projective resolution. Such resolutions are unique up to homotopy.

**PROPOSITION 1.11.** *If  $R$  is  $n$ -Gorenstein and  $G$  is an  $R$ -module then the following are equivalent :*

- 1)  $G$  is Gorenstein injective
- 2)  $\text{Ext}^i(L, G) = 0$  for all modules  $L$  with  $\text{pd } L < \infty$  and all  $i \geq 1$
- 3)  $\text{Ext}^1(L, G) = 0$  for all modules  $L$  with  $\text{pd } L < \infty$ .

**PROOF.** 1) $\Rightarrow$ 2) by 2) $\Rightarrow$ 3) is trivial.

3) $\Rightarrow$ 1). By the proof of Proposition 1.8 we have an exact sequence  $0 \rightarrow G \rightarrow \bar{G} \rightarrow L \rightarrow 0$  with  $\bar{G}$  Gorenstein injective and  $\text{pd } L < \infty$ . By hypotheses this sequence splits so as a direct summand of a Gorenstein injective module  $G$  is also Gorenstein injective.

## 2. Balance of Hom $(-, -)$ .

For a ring  $R$  we now let  $\mathcal{F}\mathcal{G}$  Gor Proj, Gor Inj,  $\mathcal{F}\mathcal{G}$  and Mod denote (respectively) the categories of finitely generated Gorenstein projective modules, Gorenstein injective modules, finitely generated modules and all modules.

We now use the language of [6] and attempt to justify our choice of terminology (Gorenstein projective and injective) by proving :

**THEOREM 2.1.** *If  $R$  is  $n$ -Gorenstein then  $\text{Hom}(-, -)$  is right-balanced by  $\mathcal{F}\mathcal{G}$  Gor Proj  $\times$  Gor Inj on  $\mathcal{F}\mathcal{G} \times$  Mod.*

**PROOF.** We only need prove that if  $N \rightarrow G$  is any Gorenstein injective preenvelope of  $N$  and if  $0 \rightarrow N \rightarrow G \rightarrow L \rightarrow 0$  is exact, then  $\text{Hom}(C, -)$  leaves the sequence exact for any finitely generated Gorenstein projective module  $C$ , and also the corresponding dual result.

Since we know there is some such preenvelope with  $\text{inj. dim } L < \infty$  we

first prove the result under this assumption. By [10],  $\text{inj. dim } L \leq n$ . Let  $0 \rightarrow L \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^n \rightarrow 0$  be an injective resolution of  $L$ . Since  $C$  is Gorenstein projective, there is an exact sequence

$$0 \longrightarrow C \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \dots \longrightarrow P^{n-1} \longrightarrow P^n$$

with the  $P^i$ 's finitely generated and projective. Let  $D = \text{im}(P^{n-1} \rightarrow P^n)$ . Any map  $C \rightarrow L$  then gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & P^0 & \longrightarrow & \dots & \longrightarrow & P^{n-1} & \longrightarrow & D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & E^0 & \longrightarrow & \dots & \longrightarrow & E^{n-1} & \longrightarrow & E^n & \longrightarrow & 0 \end{array}$$

But  $D \rightarrow E^n$  can be extended to  $P^n \rightarrow E^n$  which in turn can be lifted to a map  $P^n \rightarrow E^{n-1}$ . Then the usual way of constructing homotopies gives an extension  $P^0 \rightarrow L$ . But  $P^0 \rightarrow L$  can be lifted to a map  $P^0 \rightarrow G$  which by restriction gives the required lifting  $C \rightarrow G$ .

In the general case, if  $0 \rightarrow N \rightarrow G \rightarrow L \rightarrow 0$  and  $0 \rightarrow N \rightarrow G' \rightarrow L' \rightarrow 0$  are both exact and given by Gorenstein injective preenvelopes of  $N$ , then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & G' & \longrightarrow & L' & \longrightarrow & 0 \end{array}$$

giving a homotopy equivalence of the two complexes. But then

$$0 \longrightarrow \text{Hom}(C, N) \longrightarrow \text{Hom}(C, G) \longrightarrow \text{Hom}(C, L) \longrightarrow 0$$

and

$$0 \longrightarrow \text{Hom}(C, N) \longrightarrow \text{Hom}(C, G') \longrightarrow \text{Hom}(C, L') \longrightarrow 0$$

are homotopically equivalent.

Hence if one has zero homology, so does the other, i.e. if one sequence is exact so is the other.

As a consequence we get that  $\text{Hom}(C, -)$  leaves any Gorenstein resolution  $0 \rightarrow N \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$  exact. A dual argument gives that when  $G$  is Gorenstein injective then  $\text{Hom}(-, G)$  leaves any Gorenstein projective resolution  $\dots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0$  (by finitely generated Gorenstein projective modules  $C_i$ ) exact.  
vpp

### 3. Dimensions.

A left  $R$ -module  $N$  is said to have Gorenstein injective dimension  $\leq n$  ( $G\text{-id } N \leq n$ ) if there is a Gorenstein injective resolution

$$0 \longrightarrow N \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \dots \longrightarrow G^n \longrightarrow 0$$

of  $N$ . In a similar manner we define the Gorenstein projective dimension of a finitely generated module to be  $\leq n$  ( $G\text{-pd } M \leq n$ ).

**THEOREM 3.2.** *If  $R$  is left and right noetherian, then the following are equivalent :*

- 1)  $R$  is  $n$ -Gorenstein.
- 2) for every  $R$ -module  $N$  (left or right),  $N$  has a Gorenstein injective preenvelope and  $G\text{-id } N \leq n$ .
- 3) every  $n$ th cosyzygy of an  $R$ -module (left or right) is Gorenstein injective.
- 4) every finitely generated  $R$ -module  $M$  (left or right) has a Gorenstein projective precover and  $G\text{-pd } M \leq n$ .
- 5) every  $n$ th syzygy in a projective resolution of a finitely generated  $R$ -module (left or right) by finitely generated projective modules is Gorenstein projective.

**PROOF.** 1) $\Rightarrow$ 2). Let  $0 \rightarrow N \rightarrow G^0 \rightarrow G^1 \rightarrow \dots \rightarrow G^{n-1}$  be a partial Gorenstein injective resolution of  $N$ . We argue that  $G = \text{Coker}(G^{n-2} \rightarrow G^{n-1})$  (or  $\text{Coker}(N \rightarrow G^0)$  if  $n=1$ ) is Gorenstein injective. By if  $\text{pd } L < \infty$  then  $\text{pd } L \leq n$ . Hence  $\text{Ext}^1(L, G) = \text{Ext}^{n+1}(L, N) = 0$ . So by Prop. 1.11,  $G$  is Gorenstein injective.

2) $\Rightarrow$ 3). Given the partial injective resolution  $0 \rightarrow N \rightarrow E^0 \rightarrow \dots \rightarrow E^{n-1}$  let  $G = \text{Coker}(E^{n-2} \rightarrow E^{n-1})$  (or  $\text{Coker}(N \rightarrow E^0)$  if  $n=1$ ). We argue that  $G$  is Gorenstein injective.

Let  $\text{pd } L < \infty$ . Then since we have a Gorenstein injective resolution  $0 \rightarrow N \rightarrow G^0 \rightarrow \dots \rightarrow G^n \rightarrow 0$  and since  $\text{Ext}^i(L, G') = 0$  when  $G'$  is Gorenstein injective and  $i \geq 1$ , we see that  $\text{Ext}^{n+1}(L, N) = 0$ . Hence  $\text{pd } L \leq n$ . But then an argument as in 1) $\Rightarrow$ 2) gives  $\text{Ext}^1(L, G) = 0$ .

3) $\Rightarrow$ 1) by Enochs-Jenda ([9], Theorem 2.4).

1) $\Rightarrow$ 4) and 4) $\Rightarrow$ 5) are by arguments dual to those for 1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 3), 5) $\Rightarrow$ 1) by ([9], Theorem 2.4).

### 4. Gext.

If  $R$  is Gorenstein then the right derived functors of  $\text{Hom}(-, -)$  computed using either a resolution of  $M$  by finitely generated Gorenstein projective

modules or one of  $N$  by Gorenstein injective modules will be denoted  $G \text{ ext}^i(M, N)$ .

Then it is easy to check the following properties of  $G \text{ ext}$ :

- a)  $G \text{ ext}^0(-, -) \cong \text{Hom}(-, -)$
- b)  $G \text{ ext}^i(C, -) = 0$  if  $i \geq 1$  and  $C$  is a finitely generated Gorenstein projective module.
- c)  $G \text{ ext}^i(-, G) = 0$  for  $i \geq 1$  and  $G$  a Gorenstein injective module.
- d) For any short exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of finitely generated modules left exact by  $\text{Hom}(C, -)$  when  $C$  is finitely generated and Gorenstein projective gives rise to a long exact sequence

$$\cdots \rightarrow G \text{ ext}^i(M'', -) \rightarrow G \text{ ext}^i(M, -) \rightarrow G \text{ ext}^i(M', -) \rightarrow G \text{ ext}^{i+1}(M'', -) \rightarrow \cdots$$

(We note that the condition above is exactly what is needed to construct a commutative diagram

$$\begin{array}{ccccccc} & & \ddots & & \ddots & & \ddots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_0 & \longrightarrow & C_0 & \longrightarrow & C_0'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

satisfying the obvious conditions).

- e) Any short exact sequence  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  left exact by  $\text{Hom}(-, G)$  whenever  $G$  is Gorenstein injective gives rise to a long exact sequence
 
$$\cdots \rightarrow \text{Ext}^i(-, N') \rightarrow \text{Ext}^i(-, N) \rightarrow \text{Ext}^i(-, N'') \rightarrow \text{Ext}^{i+1}(-, N') \rightarrow \cdots$$
- f) Then are natural transformations

$$G \text{ ext}^i(-, -) \longrightarrow \text{Ext}^i(-, -)$$

which are also natural in the exact sequences as in d) and e).

**PROPOSITION 4.1.** *If  $R$  is  $n$ -Gorenstein then for a left  $R$ -module  $L$  the following are equivalent:*

- 1)  $\text{proj. dim } L < \infty$  (and so  $\leq n$ )
- 2)  $G \text{ ext}^i(L, -) \rightarrow \text{Ext}^i(L, -)$  is an isomorphism for all  $i \geq 0$
- 3)  $\text{Ext}^i(L, G) = 0$  for all Gorenstein injective modules  $G$ .



PROOF. 1) $\Rightarrow$ 3) is part of Corollary 4.4 of [8].

3) $\Rightarrow$ 2). 3) implies  $\text{Ext}^i(L, G)=0$  for all Gorenstein injective modules  $G$  and  $i \geq 1$ , again by Corollary 4.4 of [8]. Furthermore  $G \text{ ext}^i(L, G)=0$  for all such  $G$  and  $i \geq 1$ . So for  $i \geq 0$  we get a commutative diagram

$$\begin{array}{ccccccc} G \text{ ext}^i(L, G) & \longrightarrow & G \text{ ext}^i(L, H) & \longrightarrow & G \text{ ext}^{i+1}(L, N) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{Ext}^i(L, G) & \longrightarrow & \text{Ext}^i(L, H) & \longrightarrow & \text{Ext}^{i+1}(L, N) & \longrightarrow & 0 \end{array}$$

For  $i=0$ , the two first vertical maps are isomorphism since  $\text{Ext}^0(-, -) = G \text{ ext}^0(-, -) = \text{Hom}(-, -)$ . Hence we get  $G \text{ ext}^1(L, N) \rightarrow \text{Ext}^1(L, N)$  is an isomorphism. Then induction on  $i$  gives the desired result.

2) $\Rightarrow$ 1). Since  $R$  is  $n$ -Gorenstein,  $G\text{-id } N \leq n$  for all modules  $N$  by Theorem 3.2 above. So  $G \text{ ext}^i(L, N)=0$  for  $i \geq n+1$  and all  $N$ . Hence  $\text{Ext}^{n+1}(L, N)=0$  for all  $N$  and so  $\text{pd } L \leq n$ .

REMARK 4.2. If  $R$  is  $n$ -Gorenstein, for all left  $R$ -modules  $M$  and  $N$   $G \text{ ext}^1(M, N) \rightarrow \text{Ext}^1(M, N)$  is an injection.

PROOF. If  $N \rightarrow G$  is a Gorenstein injective envelope of  $N$  let  $0 \rightarrow N \rightarrow G \rightarrow H \rightarrow 0$  be the associated exact sequence. Then we get a commutative diagram

$$\begin{array}{ccccccc} \text{Hom}(M, G) & \longrightarrow & \text{Hom}(M, H) & \longrightarrow & G \text{ ext}^1(M, N) & \longrightarrow & 0 \\ \parallel & & \parallel & & \downarrow & & \\ \text{Hom}(M, G) & \longrightarrow & \text{Hom}(M, H) & \longrightarrow & \text{Ext}^1(M, N) & & \end{array}$$

with exact rows. The result then follows.

If we consider the elements of  $\text{Ext}^1(M, N)$  as classes of short exact sequences  $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$  we get that

COROLLARY 4.3. *The short exact sequence  $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$  corresponds to an element of  $G \text{ ext}^1(M, N) \subset \text{Ext}^1(M, N)$  is and only if for every Gorenstein injective module  $G$ ,  $\text{Hom}(L, G) \rightarrow \text{Hom}(N, G) \rightarrow 0$  is exact.*

PROOF. Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & K \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & \bar{K} \longrightarrow 0 \end{array}$$

be a commutative diagram with exact rows and  $N \rightarrow G$  a Gorenstein injective preenvelope of  $N$  and  $E$  an injective module. Given a map  $M \rightarrow K$ , using a

pull-back diagram we get the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & K \longrightarrow 0 \end{array}$$

and so get the element corresponding to  $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$  of  $G \text{ ext}^1(M, N)$ . Then if  $G'$  is Gorenstein injective, any map  $N \rightarrow G'$  gives rise to a map  $G \rightarrow G'$  (since  $N \rightarrow G$  is a preenvelope) and so we get the desired map  $L \rightarrow G'$ .

Now given a map  $M \rightarrow \bar{K}$  form the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & \bar{K} \longrightarrow 0 \end{array}$$

with exact rows. If  $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$  has the desired property, then the map  $N \rightarrow G$  gives rise to a map  $L \rightarrow G$  and so to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & K \longrightarrow 0 \end{array}$$

This shows that  $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$  corresponds to an element of  $G \text{ ext}^1(M, N)$  and completes the proof.

If  $M$  is taken finitely generated in the previous result, a dual argument gives that the elements of  $G \text{ ext}^1(M, N)$  correspond to the sequences  $0 \rightarrow N \rightarrow L \rightarrow M \rightarrow 0$  such that  $\text{Hom}(C, L) \rightarrow \text{Hom}(C, M) \rightarrow 0$  is exact when  $C$  is a finitely generated Gorenstein projective module. vpp

## 5. $G \text{ tor}$ .

LEMMA 5.1. *If  $R$  is  $n$ -Gorenstein then for a left  $R$ -module  $C$  the following are equivalent :*

- 1)  $C$  is Gorenstein projective
- 2)  $\text{Ext}^i(C, L) = 0$  for all  $L$  with  $\text{id } L < \infty$  and all  $i \geq 1$
- 3)  $\text{Ext}^i(C, L) = 0$  for all  $L$  with  $\text{id } L < \infty$
- 4)  $\text{Tor}_i(L, C) = 0$  for all right  $R$ -modules  $L$  with  $\text{id } L < \infty$  and  $i \geq 1$
- 5)  $\text{Tor}_1(L, C) = 0$  for all right  $R$ -modules  $L$  with  $\text{pd } L < \infty$
- 6)  $\text{Hom}_s(C, Q/Z)$  is a Gorenstein injective right  $R$ -module.

PROOF. 1)  $\Rightarrow$  2) is by 2)  $\Rightarrow$  3) is trivial.

3) $\Rightarrow$ 1). By [2], there is an exact sequence  $0 \rightarrow L \rightarrow \bar{C} \rightarrow C \rightarrow 0$  with  $\bar{C}$  a finitely generated Gorenstein projective module and  $\text{id } L < \infty$ . By 3), this splits and so  $C$  is Gorenstein projective.

2) $\Rightarrow$ 4). Letting  $M^+ = \text{Hom}_Z(M, Q/Z)$  for any left or right  $R$ -module, we have

$$\text{Tor}_i(L, C)^+ = \text{Ext}^i(C, L^+)$$

If  $\text{pd } L < \infty$  then  $\text{id } L^+ < \infty$ , so  $\text{Ext}^i(C, L^+) = 0$  for  $i \geq 1$  and  $\text{pd } L < \infty$ . Hence  $\text{Tor}_i(L, C) = 0$  for such  $i$  and  $L$ .

4) $\Rightarrow$ 5) is trivial.

5) $\Rightarrow$ 6) and 6) $\Rightarrow$ 3).  $\text{Tor}_1(L, C) = 0$  if and only if  $\text{Ext}^1(L, C^+) = \text{Tor}_1(L, C)^+ = 0$  so using Proposition 1.11 we get our claims.

As a result we get

**THEOREM 5.2.** *If  $R$  is  $n$ -Gorenstein then  $-\otimes-$  is left balanced on  $\mathcal{F}\mathcal{G} \times \mathcal{F}\mathcal{G}$  (finitely generated right  $R$ -modules for the first  $\mathcal{F}\mathcal{G}$  and left for the second) by  $\mathcal{F}\mathcal{G} \text{ Gor Proj} \times \mathcal{F}\mathcal{G} \text{ Gor Proj}$  (again left and right).*

**PROOF.** Let  $\dots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0$  be a Gorenstein projective resolution of a finitely generated right  $R$ -module and let  $D$  be a finitely generated Gorenstein projective left  $R$ -module. Then

$$\dots \rightarrow C_1 \otimes D \rightarrow C_0 \otimes D \rightarrow M \otimes D \rightarrow 0$$

is exact if and only if

$$0 \rightarrow (M \otimes D)^+ \rightarrow (C_0 \otimes D)^+ \rightarrow (C_1 \otimes D)^+ \rightarrow \dots$$

i. e. if and only if

$$0 \rightarrow \text{Hom}(M, D^+) \rightarrow \text{Hom}(C_0, D^+) \rightarrow \text{Hom}(C_1, D^+) \rightarrow \dots$$

is exact. But  $D^+$  is Gorenstein injective by the Lemma 5.1. Hence this sequence is exact by Theorem 2.1.

So we can now compute left derived functors of  $-\otimes-$  computed using Gorenstein projective resolutions of finitely generated modules (see [6]).

These derived functors will be defined denoted  $G \text{ tor}_i(M, N)$ .

Then it is easy to check the following:

- a)  $G \text{ tor}_0(-, -) \cong -\otimes-$
- b)  $G \text{ tor}_i(C, -) = 0$  if  $i \geq 1$  and  $C$  is a finitely generated Gorenstein projective right  $R$ -module.
- c)  $G \text{ tor}_i(-, D) = 0$  if  $i \geq 1$  and  $D$  is a finitely generated Gorenstein projective left  $R$ -module

- d) for any exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of finitely generated right  $R$ -modules which remains exact when  $-\otimes D$  is applied to it with finitely generated Gorenstein projective, there is a long exact sequence
- $$\cdots \rightarrow G \operatorname{tor}_{i+1}(M'', -) \rightarrow G \operatorname{tor}_i(M', -) \rightarrow G' \operatorname{tor}_i(M, -) \rightarrow G \operatorname{tor}_i(M'', -) \rightarrow \cdots$$
- e) same as d) but for an exact sequence of left  $R$ -modules  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$
- f) there are natural transformations

$$\operatorname{Tor}_i(-, -) \longrightarrow G \operatorname{tor}_i(-, -)$$

for each  $i \geq 0$  and these natural transformations commute with the connecting homomorphisms associated with short exact sequences as in d) and e). If  $i=0$  the natural transformations is an isomorphism.

**PROPOSITION 5.3.** *If  $R$  is  $n$ -Gorenstein then for a finitely generated right  $R$ -module  $L$  the following are equivalent :*

- 1)  $\operatorname{pd} L < \infty$  (so  $\operatorname{pd} L \leq n$ )
- 2)  $\operatorname{Tor}_i(L, -) \rightarrow G \operatorname{tor}_i(L, -)$  is an isomorphism for all  $i \geq 0$
- 3)  $\operatorname{Tor}_1(L, D) = 0$  for all finitely generated Gorenstein projective modules  $D$
- 4)  $\operatorname{Tor}_i(L, D) = 0$  for all finitely generated Gorenstein projective modules  $D$  and all  $i \geq 1$ .

**PROOF.** 1) $\Rightarrow$ 4).  $\operatorname{Tor}_i(L, D)^+ = \operatorname{Ext}^i(L, D^+)$ . By Lemma 5.1 6),  $D^+$  is Gorenstein injective. Then Lemma 1.3 says  $\operatorname{Ext}^i(L, D^+) = 0$  for  $i \geq 1$  and so  $\operatorname{Tor}_i(L, D) = 0$  for  $i \geq 1$ .

4) $\Rightarrow$ 3) is trivial.

2) $\Rightarrow$ 1). By Theorem 3.2  $G\text{-pd} L \leq n$  so  $G \operatorname{tor}_i(L, N) = 0$  for  $i \geq n+1$ . Therefore  $\operatorname{Tor}_i(L, N) = 0$  for all  $i \geq n+1$  and all  $n$ . Hence  $\operatorname{ph} L \leq n$ .

4) $\Rightarrow$ 2). This is standard. For example, it is straightforward modification of ([4], Proposition 4.4).

3) $\Rightarrow$ 4). By the definition of a Gorenstein projective module, there is an exact sequence  $0 \rightarrow K \rightarrow P \rightarrow D \rightarrow 0$  with  $P$  finitely generated and projective and  $K$  Gorenstein projective. So  $\operatorname{Tor}_2(L, D) = \operatorname{Tor}_1(K, D) = 0$ . Similarly we get  $\operatorname{Tor}_i(L, D) = 0$  for all  $i \geq 1$ .

### References

- [1] M. Auslander and M. Bridger, Stable Module Theory, A.M.S. Memoirs, No. 94, Amer. Math. Soc., Providence, R.I., 1969.
- [2] M. Auslander and R. Buchweitz, The homological theory of maximal Cohen-Macaulay approximations, Soc. Math. de France, Mémoire 38 (1989), 5-37.

- [ 3 ] K. Brown, *Cohomology of Groups*, Springer-Verlag, New York, 1982.
- [ 4 ] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, Princeton, N. J., 1956.
- [ 5 ] S. Eilenberg and J.C. Moore, *Foundations of Relative Homological Algebra*, A.M.S. Memoirs, No. 55, Amer. Math. Soc., Providence, R.I., 1965.
- [ 6 ] E. Enochs and O. Jenda, *Balanced functors applied to modules*, *J. Algebra* **92** (2), 1985, 303-310.
- [ 7 ] E. Enochs, *Injective and flat covers, envelopes and resolvents*, *Israel J. Math.* **39** (3), 1981, 189-209.
- [ 8 ] E. Enochs and O. Jenda, *Gorenstein injective and projective modules* (to appear in *Math. Zeit.*).
- [ 9 ] E. Enochs and O. Jenda, *Resolvents and dimensions of modules*, *Arch. Math.* vol. **56**, 1991, 528-532.
- [10] Y. Iwanaga, *On rings with finite self-injective dimension II*, *Tsukuba J. Math.* **4**, 1980, no. 1, 107-113.

Edgar E. Enochs  
Department of Mathematics  
University of Kentucky  
Lexington, KY 40506-0027  
U. S. A.

Overtoun M.G. Jenda  
Department of Algebra, Combinatorics and Analysis  
Auburn University  
Auburn, Alabama 36849-5307  
U. S. A.