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RIGIDITY OF COMPACT IDEAL BOUNDARIES OF MANIFOLDS JOINED BY HAUSDORFF APPROXIMATIONS

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

By

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§0. Introduction

The concept of ideal boundary of Hadamard manifolds was introduced by Eberlein and O'Neill [3] in 1973, which had marked a milestone in the study of the geometry of noncompact Riemannian manifolds. Since then, it has been utilized in various fields of research on Hadamard manifolds.

Subsequently, the concept of ideal boundary was also defined on other classes of Riemannian manifolds. One is done by Kasue [6] on asymptotically nonnegatively curved manifolds, and the other by Shioya [9][10] on complete open surfaces admitting total curvature.

Based on their works, it is an interesting problem to study to what extent the structure of manifolds is determined by information about their ideal boundaries. In our previous papers, we characterized in [8] the rigidity of product manifolds by information on ideal boundary, and in [1] the Euclidean factor of a Hadamard manifold in terms of the polar points on ideal boundary.

Recently, Kubo [7] proved that given two connected complete oriented and noncompact Riemannian 2-manifolds with finite total curvature, if there is a Hausdorff approximation between them, then their ideal boundaries are isometric. This means that if ideal boundaries are not isometric, then there is no Hausdorff approximation between their underlying open surfaces.

In this paper, for other two classes of Riemannian manifolds mentioned above, we shall study the same rigidity problem on ideal boundaries by a different method from Kubo's, and prove the following theorems.

THEOREM A. *Let M and N be Hadamard manifolds with ideal boundaries $M(\infty)$ and $N(\infty)$ respectively, which are assumed to be compact with respect to*

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the Tits-topology. If there exists a Hausdorff approximation from M to N, then $(M(\infty), \mathrm{Td})$ *is isometric to* $(N(\infty), \mathrm{Td})$.

THEREM B. *Let M and N be manifolds of asymptotically nonnegative curvature with ideal boundaries $M(\infty)$ and $N(\infty)$ respectively. If there exists a Hausdorff approximation from M to N, then* $(M(\infty), \mathrm{Td})$ *is isometric to* $(N(\infty), \mathrm{Td})$.

It should be noted that the ideal boundaries treated in both Kubo's theorem and our Theorems A, B are compact with respect to the Tits-topology. The result seems to remain true even in the case when ideal bonndaries are non-compact, but we shall need another approach to prove it.

§ 1. Definitions

We shall first summarize the definitions concerning ideal boundary for Hadamard manifolds and for manifolds of asymptotically nonnegative curvature. For details, we refer to [2] for the former and to [6] for the latter case. In what follows, geodesics are assumed to be parametrized by arc length unless otherwise stated.

Let M be a Hadamard manifold, which is a simply connected complete Riemannian manifold of nonpositive curvature. Two geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to M$ are said to be *asymptotic* if the distance function $t \to d_M(\gamma_1(t), \gamma_2(t))$ is bounded from above for all $t \geqq 0$. Then we define the *ideal boundary* $M(\infty)$ of M as the set of all asymptotic classes of geodesic rays in M.

The *Tits metric* on this boundary is defined in the following manner. For given points $x \in M$ and $z \in M(\infty)$, we have a unique geodesic ray γ_{xz} in M emanating from x, whose asymptotic class $\gamma_{xz}(\infty)$ is z. Then the angle $\angle(z, w)$ on $M(\infty)$ is defined by

$$\angle(z, w) = \sup_{x \in M} \angle_x(z, w),$$

where $\angle_x(z, w) = \angle(\gamma'_{xz}(0), \gamma'_{xw}(0))$. The *Tits metric* $\mathrm{Td}(\cdot, \cdot)$ is the interior metric $\angle_i$ induced from this angle.

Now we assume that M is a manifold of asymptotically nonnegative curvature, that is, the sectional curvature K_M of M satisfies $K_M \geqq -k \circ r_0$, where r_0 is the distance function from a fixed point $o \in M$, called the base point of M, and $k(t)$ is a nonnegative monotone nonincreasing function on $[0, \infty)$ such that the integral $\int_0^\infty t \cdot k(t) dt$ is finite.

Let p be an arbitrarily fixed point of M. For sufficiently large t, the metric sphere $S_t(p)$ around p of radius t is a Lipschitz hypersurface of M consisting of k connected components, where k is the number of the ends of M. On $S_t(p)$, we introduce the interior metric, denoted by $d_{p,t}$, induced from the metric d_M restricted on $S_t(p)$.

Two rays σ and γ are called *equivalent* if $\lim_{t\to\infty} d_M(\sigma(t), \gamma(t))/t=0$, and then denoted by $\sigma\sim\gamma$. The *ideal boundary* $M(\infty)$ of M is, by definition, the set of all equivalence classes of $\sim$. We write $\sigma(\infty)$ for the equivalence class of σ. For any fixed point $p\in M$, we define the *Tits metric* Td on $M(\infty)$ by

$$(*) \qquad \mathrm{Td}(\sigma(\infty), \gamma(\infty)) := \lim_{t\to\infty} \frac{d_{p,t}(\sigma\cap S_t(p), \gamma\cap S_t(p))}{t}.$$

Then Td is well-defined on $M(\infty)$ and is independent of the choice of p. (cf. Proposition 2.1 in [6])

REMARK. The equivalence relation $\sim$ is a natural extension of the asymptotic rlation. We can see that on Hadamard manifolds the equivalence relation $\sim$ coincides with the asymptotic relation. Moreover, the equation $(*)$ defining the metric Td for asymptotically nonnegatively curved manifolds is also valid for the metric Td for Hadamard manifolds.

Next, following [4], we shall recall Hausdorff convergence. The definition in [4] is slightly different from the original one introduced by Gromov in [5]. However this is more tractable in our discussion.

Let $\mathfrak{MST}$ denote the set of all isometry classes of metric spaces. For any isometry class $X\in\mathfrak{MST}$, we denote a representative metric space of X also by the same symbol X. For $X, Y\in\mathfrak{MST}$, a (not necessary continuous) map $\phi: X\to Y$ is said to be a Δ-*Hausdorff approximation* if ϕ satisfies the following conditions:

(1) The Δ-neighborhood $B_\Delta(\phi(X))=\{x\in Y \mid d(x, \phi(X))<\Delta\}$ of $\phi(X)$ in Y is equal to Y.

(2) For any points $x, y\in X$, we have

$$|d_X(x, y)-d_Y(\phi(x), \phi(y))|<\Delta.$$

The *Hausdorff distance* $d_H(X, Y)$ between X and Y is defined to be the infimum of the positive numbers Δ such that there exist Δ-Hausdorff approximations from X to Y and from Y to X.

We should note that d_H is not a metric, but it satisfies that for $X, Y, Z\in \mathfrak{MST}$

$$d_H(X, Z) \leqq 2\{d_H(X, Y) + d_H(Y, Z)\}.$$

Hence d_H defines a uniform structure on $\mathfrak{MS T}$.

Now let $\mathfrak{CMST}$ denote the set of all isometric classes of *compact* metric spaces. Then, with respect to the uniform topology defined by d_H, the following holds.

THEOREM 1.1. (Theorem 1.5 in [4], also cf. Proposition 3.6 in [5]) $\mathfrak{CMST}$ *is Hausdorff and complete.*

The Hausdorffness means that the uniform structure on $\mathfrak{CMST}$ is metrizable, and that we may treat d_H as if it is a distance function.

On the other hand, in the noncompact case, we need to study the category of pointed and locally compact metric spaces.

We denote by $\mathfrak{MST}_0$ the set of all isometry classes of pointed metric spaces (X, p) with a base point $p \in X$ such that the closure $\bar{B}_R(p, X)$ of R-neighborhood of p in X is compact for every $R>0$. Let $(X, p), (Y, q) \in \mathfrak{MST}_0$ and $\phi: (X, p) \to (Y, q)$ be a pointed map, namely $\phi(p) = q$. We say that ϕ is a Δ*-pointed Hausdorff approximation* if $\phi(\bar{B}_{1/\Delta}(p, X)) \subset \bar{B}_{1/\Delta}(q, Y)$ and if the restriction of ϕ on $\bar{B}_{1/\Delta}(p, X)$ into $\bar{B}_{1/\Delta}(q, Y)$ is a Δ-Hausdorff approximation. Then the *pointed Hausdorff distance* $d_{p,H}((X, p), (Y, q))$ is also defined to be the infimum of the numbers Δ such that there exist Δ-pointed Hausdorff approximations from (X, p) to (Y, q) and from (Y, q) to (X, p).

It should be noted that $\mathfrak{MST}_0$ is also Hausdorff and complete, but the limit space depends on the choice of base points.

§2. The case of Hadamard manifolds

Let M be a Hadamard manifold and d_M the distance function on M. If the ideal boundary of M is compact (with respect to the Tits-topology), then there exists the tangent cone of M at infinity, that is, the pointed Hausdorff limit of pointed spaces $((M, (1/t)d_M), p)$ exists for $t \to \infty$, and is isometric to the cone of $M(\infty)$. We shall first prove this fact and make use of it in the proof of Theorem A.

We recall the definition of the cone $(\mathfrak{C}(M(\infty)), o)$ of $M(\infty)$ with vertex o. For a pair of points $(s, w), (t, z) \in [0, \infty) \times M(\infty)$, we set

$$\delta((s, w), (t, z)) := \sqrt{s^2 + t^2 - 2st\cos(\widetilde{\mathrm{Td}}(w, z))},$$

where $\widetilde{\mathrm{Td}}(w, z) := \min\{\pi, \mathrm{Td}(w, z)\}$. Using the function δ, we can define an equivalence relation as follows:

$$(s, w)\sim(t, z) \Longleftrightarrow \delta((s, w), (t, z))=0.$$

Then it is immediate that δ gives rise to a distance function on the quotient space $\{[0, \infty)\times M(\infty)\}/\sim$. This metric space $(\{[0, \infty)\times M(\infty)\}/\sim, \delta)$ is called the cone of $M(\infty)$ and is denoted by $\mathfrak{C}(M(\infty))$. We mean by the vertex of $\mathfrak{C}(M(\infty))$ the equivalence class $[(0, z)](z\in M(\infty))$. Then the following holds:

PROPOSITION 2.1. *If the ideal boundary is compact (with respect to the Tits-topology), then for any fixed point p on M the sequence of pointed metric spaces $((M, d_t^M), p)$ converges to the cone $(\mathfrak{C}(M(\infty)), o)$ of $M(\infty)$ with vertex o in the sense of pointed Hausdorff distance:*

$$\lim_{t\to\infty}((M, d_t^M), p)=(\mathfrak{C}(M(\infty)), o),$$

where $d_t^M=(1/t)d_M$.

PROOF. Let R be an arbitrary large number. Let $\bar{B}_R^t(p)$ denote the closed geodesic ball around p with radius R in $M_t:=(M, d_t^M)$. Then we can identify $\bar{B}_R^t(p)$ with the closed disk $\bar{B}_R=\{v\in T_pM \mid \|v\|\leqq R\}$ in T_pM, and $\bar{B}_R$ with the closed ball $\bar{B}_R(o)$ in $\mathfrak{C}(M(\infty))$ as follows:

$$T_pM\supset\bar{B}_R\ni v \longleftrightarrow \gamma_v(t)\in\bar{B}_R^t(p)\subset M_t,$$

$$T_pM\supset\bar{B}_R\ni v \longleftrightarrow [(\|v\|, \gamma_{v/\|v\|}(\infty))]\in\bar{B}_R(o)\subset\mathfrak{C}(M(\infty)).$$

The metric on $\bar{B}_R$ induced from $(\bar{B}_R^t(p), d_t^M)$ or $(\bar{B}_R(o), \delta)$ through this identification is also denoted by the same symbol d_t^M or δ, respectively.

It is known in [2] that the sequence $\{d_t^M\}$ converges to the metric δ. We remark that the sequence $\{d_t^M\}$ restricted on $\bar{B}_R$ converges uniformly to the metric δ on $\bar{B}_R$, where $\bar{B}_R$ is equipped with the standard metric.

In fact, since $\bar{B}_R$ is homeomorphic to $(\bar{B}_R^t(p), d_t^M)$, d_t^M is a continuous function on $\bar{B}_R\times\bar{B}_R$. On the other hand, it is proved in Proposition 2.1 of [8] that $(M(\infty), \mathrm{Td})$ is compact if and only if the unit tangent sphere is homeomorphic to $(M(\infty), \mathrm{Td})$. Therefore $\bar{B}_R$ is homeomorphic to $\bar{B}_R(o)$. Hence δ is also continuous on $\bar{B}_R\times\bar{B}_R$. Since the sequence $\{d_t^M\}$ of monotone non-decreasing continuous functions converges to a continuous function δ on the compact set $\bar{B}_R\times\bar{B}_R$, the convergence is uniform.

This means that

$$\varepsilon_R(t):=\max_{(u,v)\in\bar{B}_R\times\bar{B}_R}|\delta(u, v)-d_t^M(u, v)|$$

converges to 0 as t tends to ∞.

Now for any $\varepsilon>0$, let $R=1/\varepsilon$. Then there is a number t_ε such that $\varepsilon_R(t)$

$<\varepsilon$ for all $t>t_\varepsilon$. Since the map

$$\Phi_t:(M_t,\ p)\longrightarrow(\mathfrak{C}(M(\infty)),\ o):\ \Phi_t(x)=[(d_t^M(p,\ x),\ \gamma_{px}(\infty))],$$

where γ_{px} denotes the geodesic emanating from p through x, is an ε-pointed Hausdorff approximation for $t>t_\varepsilon$, this completes the proof. ■

REMARK. We can see from the proof that when $(M(\infty),\ \mathrm{Td})$ is noncompact, the sequence $\{((M,\ d_t^M),\ p)\}$ of pointed metric spaces does not converge in the sense of pointed Hausdorff distance.

In fact, if the sequence converges in this sense, then the sequence $\{d_t^M\}$ of the continuous functions on $\bar{B}_R\times\bar{B}_R$ converges uniformly to δ. Hence δ is also continuous on $\bar{B}_R\times\bar{B}_R$. This means that $(M(\infty),\ \mathrm{Td})$ is homeomorphic to a standard sphere, and hence is compact.

PROOF OF THEOREM A. We assume that a Δ-Hausdorff approximation ϕ: $M\to N$ is given. Let p be any fixed point of M and $q:=\phi(p)\in N$.

From Proposition 2.1, there is a sequence of $\varepsilon(t)$-pointed Hausdorff approximation Φ_t: $((N,\ d_t^N),\ q)\to(\mathfrak{C}(N(\infty)),\ o)$ such that $\varepsilon(t)\to 0$ as $t\to\infty$. If we regard ϕ as a map from $(M,\ d_t^M)$ to $(N,\ d_t^N)$, then ϕ is a Δ/t-Hausdorff approximation and the composite $\Psi_t:=\Phi_t\circ\phi$: $((M,\ d_t^M),\ p)\to(\mathfrak{C}(N(\infty)),\ o)$ is a $((\Delta/t)+2\varepsilon(t))$-pointed Hausdorff approximation. Since $(\Delta/t)+2\varepsilon(t)\to 0$ as $t\to\infty$, it holds that

$$\lim_{t\to\infty}((M,\ d_t^M),\ p)=(\mathfrak{C}(N(\infty)),\ o).$$

On the other hand, the left side of this equality coincides with $(\mathfrak{C}(M(\infty)),\ o)$, and hence $(\mathfrak{C}(M(\infty),\ o)$ is isometric to $(\mathfrak{C}(N(\infty)),\ o)$. Since $(M(\infty),\ \mathrm{Td})$ is isometric to the metric sphere in $\mathfrak{C}(M(\infty))$ around a vertex o of radius 1 equipped with the interior metric induced from the restriction of δ, we can conclude that $(M(\infty),\ \mathrm{Td})$ is isometric to $(N(\infty),\ \mathrm{Td})$. ■

To conclude this section, we give an example of Hadamard manifolds whose ideal boundaries are isometric but no Hausdorff approximation exists between them.

EXAMPLE. Let M be a Hadamard 2-manifold equipped with a metric given as $ds^2=dr^2+f(r)^2d\theta^2$, where $(r,\ \theta)$ is a polar coordinate of M with origin o_M and f: $[0,\ \infty)\to[0,\ \infty)$ is a smooth function satisfying

$$\begin{cases} f(0)=0, & f'(0)=1, \quad f''(0)=0 \\ f''(0)\geqq 0 & (\text{for any } t\geqq 0) \\ f'(t)\equiv 2 & (\text{for } t\geqq 2). \end{cases}$$

Let N be a Hadamard 2-manifold with a metric $ds^2=dr^2+g(r)^2d\theta^2$, where g satisfies

$$\begin{cases} g(0)=0, & g'(0)=1, \quad g''(0)=0 \\ g''(0)\geqq 0 & (\text{for any } t\geqq 0) \\ g'(t)=2-\dfrac{1}{t} & (\text{for } t\geqq 2). \end{cases}$$

Then the difference of the girths of the geodesic spheres of radius t, with center o_M and o_N respectively, equals to $2\pi(f(t)-g(t))$. This is $2\pi(\log(t/2)+f(2)-g(2))$ for $t\geqq 2$, and goes to infinity as $t\to\infty$. Hence no Hausdorff approximation exists between them, but their ideal boundaries are isometric to a circle of girth 4π.

§ 3. The case of manifolds of asymptotically nonnegative curvature

For an asymptotically nonnegatively curved manifold, its ideal boundary is always compact and the counterpart of Proposition 2.1 is valid, which can be seen in the proof of Proposition 2.4 of [6] due to Kasue.

PROPOSITION 3.1. *Let M be a manifold of asymptotically nonnegative curvature and p be a base point of M. Then the sequence of pointed metric spaces $((M, d_t^M), p)$ converges to the cone $(\mathfrak{C}(M(\infty)), o)$ of $M(\infty)$ with vertex o in the sense of pointed Hausdorff distance:*

$$\lim_{t\to\infty}((M, d_t^M), p)=(\mathfrak{C}(M(\infty)), o),$$

where $d_t^M=(1/t)d_M$.

It should be noted that the Hausdorff limit in the proposition is independent of the choice of a base point $p\in M$.

Theorem B can be proved in a quite similar way to the case of Hadamard manifolds by applying Proposition 3.1. We also note that for two asymptotically nonnegatively curved manifolds, there exists no Hausdorff approximation between them in general, even if their ideal boundaries are isometric. Indeed, the example of the previous section gives also a counter example in this case.

§ 4. Appendix

Propositions 2.1 and 3.1 imply that if the ideal boundary is compact, then we may regard it as the Hausdorff limit of a sequence of geodesic spheres around arbitrary fixed point equipped with normalized metric. Namely we have the following

COROLLARY 4.1. *Let M be either a Hadamard manifold or a manifold of asymptotically nonnegative curvature and d_M be the distance function on M. If the ideal boundary is compact (with respect to the Tits-topolgy), then $(M(\infty), l)$ is obtained as the Hausdorff limit of normalized geodesic spheres around $p \in M$:*

$$\lim_{t\to\infty}(S_t(p),\ d_t)=(M(\infty),\ l),$$

where $l(z, w):=\lim_{t\to\infty}(d_M(\gamma_{pz}(t),\ \gamma_{pw}(t)))/t$ for any $z, w \in M(\infty)$.

If we consider a sequence of metric spaces $\{(S_t(p), (1/t)d_{p,t})\}$, then we obtain $(M(\infty), \mathrm{Td})$ as its Hausdorff limit. Naturally, Td is the interior metric of l.

From the viewpoint above, when we study the relation between structure of a manifold and that of its ideal boundary, we may deal with a sequence of geodesic spheres around any fixed point with normalized metric. Then we see that a Hausdorff approximation between metric spaces under consideration induces one between their geodesic spheres.

Indeed, for a Δ-Hausdorff a pproximation ϕ: $M \to N$ between metric spaces M and N, we construct a map $\tilde{\phi}$: $S_t^M(p) \to S_t^N(q)$ for an arbitrarily fixed point $p \in M$ and $q=\phi(p) \in N$ as follows:

$$\tilde{\phi}(x)=x' :=\gamma_{q\phi(x)} \cap S_t^N(q) \qquad \text{for} \quad x \in S_t^M(p),$$

where $\gamma_{q\phi(x)}$ is the ray emanating from q through $\phi(x)$. Then, applying the triangle inequality, the following lemma is obtained.

LEMMA 4.2. *$\tilde{\phi}$ is a 5Δ-Hausdorff approximation.*

Theorems A and B can be obtained also by using the compositions of $\tilde{\phi}$ and maps giving the Hausdorff convergence.

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