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VERY AMPLE INVERTIBLE SHEAVES OF NEW TYPE ON ABELIAN VARIETIES

By

Ryuji SASAKI

Introduction.

In 1919 Comessatti [1] proved the following theorem, which we learned by Lange's paper [2].

THEOREM (Comessatti). *Let* Jac(C) *denote the Jacobian variety of a smooth projective curve* C *of genus* 2. *If an ample divisor* D *on* Jac(C) *satisfies* $(D^2)=2$ *and* $(C\cdot D)=n$ *for* $n\geqq 3$, *then the divisor* $C+D$ *is very ample.*

The aim of the present paper is to generalize this theorem. Our result is

THEOREM. *Let* A *be an abelian variety defined over an algebraically closed field of any characteristic. Let* L *and* M *be ample invertible sheaves on* A *with* $h^0(A, L)=h^0(A, M)=1$. *Let* D *and* E *be positive divisors such that* $L=\mathcal{O}_A(D)$ *and* $M=\mathcal{O}_A(E)$. *Assume that any component of* D *is not algebraically equivalent to a component of* E. *Then* $L\otimes M$ *is very ample.*

We prove the theorem in §1. In §2 we show that the Commessatti's theorem is a special case of ours. In the last §3 we discuss projective embeddings of abelian varieties with real multiplication.

At first I set up unnecessary assumption in the theorem. I could find the above theorem as a result of the referee's pertinent suggestion. Here I thank the referee for his kind advice.

1. Proof of theorem.

We shall use the following notation. For details we refer to [4]. Let A be an abelian variety of dimension g defined over an algebraically closed field k of arbitrary characteristic and let $\hat{A}=\mathrm{Pic}^0(A)$ denote its dual variety. The translation $x\to x+a$ by a point a of A is denoted by T_a. We denote by P the

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Poincaré invertible sheaf on the product $A\times\hat{A}$ and by P_α the restriction $P|_{A\times\{\alpha\}}$. For an invertible sheaf L on A, the homomorphism $a\rightarrow T_a{}^*(L)\otimes L^{-1}$ of A to $\hat{A}$ is denoted by φ_L and its kernel by $K(L)$. When L is ample, we have $P_{\varphi_L(a)}\cong T_a{}^*(L)\otimes L^{-1}$. The Riemann-Roch theorem asserts $\deg\varphi_L=\chi(L)^2$ and $\chi(L)=(L^g)/g!$ where $\chi(L)$ is the Euler-Poincaré characteristic of L and (L^g) is the g-fold self-intersection number of L. If L is ample and $h^0(A, L)=1$, then φ_L is an isomorphism and $(L^g)=g!$.

Now we shall prove the theorem. Let

$$\Phi=\Phi_{|L\otimes M|}: A\longrightarrow \boldsymbol{P}(\Gamma(A, L\otimes M))$$

be the rational map associated with the complete linear system $|L\otimes M|$. What we should do is to establish the following statements:

(1.1) Given $a, b\in A$ with $a\neq b$, there is a divisor $F\in|L\otimes M|$ such that $a\in \mathrm{Supp}(F)$ and $b\notin \mathrm{Supp}(F)$.

(1.2) Given any tangent t to A at a, there is a dividor $F\in|L\otimes M|$ such that $a\in\mathrm{Supp}(F)$ and t is not tangential to F.

In the following we shall use the same letter for a divisor and its support. Let

$$D=\sum_{i=1}^{r}D_i \quad\text{and}\quad E=\sum_{j=1}^{s}E_j$$

be decompositions into irreducible components and $\mathcal{O}_A(D_i)=L_i$, $\mathcal{O}_A(E_j)=M_j$. Since $h^0(A, L)=1$, it follows that L_i and $L_{i'}$ are not algebraically equivalent for $i\neq i'$. We denote by A_i the quotient of A by the connected component $K(L_i)^0$ of $K(L)$ containing the origin 0. Then there is an ample invertible sheaf $\bar{L}_i$ on A_i such that $h^0(A_i, \bar{L}_i)=1$ and $\pi^*(\bar{L}_i)\cong L_i$, where π is the canonical surjection. Moreover we have

$$A\cong A_1\times\cdots\times A_r \quad\text{and}\quad L\cong p_1{}^*(\bar{L}_1)\otimes\cdots\otimes p_r{}^*(\bar{L}_r),$$

where $p_i: A_1\times\cdots\times A_r\rightarrow A_i$ is the i-th projection; cf. [7], Lem. 1.6. The same results hold for M: there is an ample invertible sheaf $\bar{M}_j$ on $B_j=A/K(M_j)^0$ such that $h^0(B_j, \bar{M}_j)=1$ and we have

$$A\cong B_1\times\cdots\times B_s \quad\text{and}\quad M\cong p_1{}^*(\bar{M}_1)\otimes\cdots\otimes p_s{}^*(\bar{M}_s).$$

Now we shall prove (1.1). Let $\psi=-\varphi_M{}^{-1}\circ\varphi_L$, then we have

$$T_{\psi(a)}{}^*(M)\cong M\otimes P_{\varphi_M(\psi(a))}\cong M\otimes P_{-\varphi_L(a)}\cong M\otimes L\otimes T_a{}^*(L)^{-1}.$$

Hence we have

(1.3) $$T_a^*(D)+T_{\psi(a)}^*(M)\in|L\otimes M| \qquad \text{for all} \quad a\in A.$$

Let a and b be points in A. Suppose that, for any $F\in|L\otimes M|$, $a\in F$ implies $b\in F$. For every i, if $p\in T_a^*(D_i)$ then $a\in T_p^*(D_i)\subset T_p^*(D)+T_{\psi(p)}(E)$. This last divisor is a member in $|L\otimes M|$ by (1.3); hence it contains b. If $b\in T_p^*(D)$, then $p\in T_b^*(D)$. If $b\in T_{\psi(p)}^*(E)$, then $\psi(p)\in T_b^*(E)$, i.e., $p\in\psi^*(T_b^*(E))$. Thus we have

$$T_a^*(D_i)\subset T_b^*(D)\cup\psi^*(T_b^*(E)).$$

Since D_i is irreducible, we have

(1.4) $$T_a^*(D_i)=T_b^*(D_{i'}) \qquad \text{for some } i'$$

or

(1.5) $$T_a^*(D_i)=\psi^*(T_b^*(E_j)) \qquad \text{for some } j.$$

Suppose (1.5) holds. Since $T_b\circ\psi=\psi\circ T_{\psi^{-1}(b)}$, we have

$$T_a^*(D_i)=\psi^*(T_b^*(E_j))=T_{\psi^{-1}(b)}^*(\psi^*(E_j)).$$

This implies that $\varphi_L(D_i)$ is algebraically equivalent to $\varphi_M(E_j)$. Therefore $K(\varphi_L(L_i))^0=K(\varphi_M(M_j))^0$ and there are ample inveritible sheaves $(\bar{L}_i)^\wedge$ and $(\bar{M}_j)^\wedge$ such that $h^0(X, (\bar{L}_i)^\wedge)=h^0(X, (\bar{M}_j)^\wedge)=1$ and $\pi^*((\bar{L}_i)^\wedge)\cong\varphi_L(L_i)$, $\pi^*((\bar{M}_j)^\wedge)\cong\varphi_M(M_j)$, where $X=\hat{A}/K(\varphi_L(L_i))^0$ and $\pi: \hat{A}\to X$ is the canonical surjection. Then $(\bar{L}_i)^\wedge$ and $(\bar{M}_j)^\wedge$ are algebraically equivalent. Moreover X is isomorphic to both of the dual abelian varieties of A_i and B_j; hence $A_i\cong B_j$, and $(\bar{L}_i)^\wedge\cong\varphi_{L_i}(\bar{L}_i)$, $(\bar{M}_j)^\wedge\cong\varphi_{\bar{M}_j}(\bar{M}_j)$. We identify A_i with $\hat{\hat{A}}_i$ via the canonical isomorphism induced by the Poincaré invertible sheaf P; cf. [4] § 13. Then $\varphi_{L_i}^{-1}=\varphi_{(L_i)^\wedge}$ and $\varphi_{\bar{M}_j}^{-1}=\varphi_{(\bar{M}_i)^\wedge}$. Since $\varphi_{(L_i)^\wedge}=\varphi_{(\bar{M}_j)^\wedge}$, $\varphi_{L_i}=\varphi_{\bar{M}_j}$; hence $\bar{L}_i$ is algebraically equivalent to $\bar{M}_j$. It follows that L_i is algebraically equivalcnt to M_j. This contradicts to the assumption. Thus we see that (1.5) does not occur.

If (1.4) holds, then D_i is algebraically equivalent to $D_{i'}$; hence $i=i'$ and $T^*_{a-b}(D_i)=D_i$. Therefore $T^*_{a-b}(D)=(D)$ and $a-b\in K(L)=\{0\}$, so we have $a=b$. This completes the proof of (1.1).

Now we shall show (1.2). We shall prove this only for $a=0$, since the general case follows by applying the result to translates of L and M. Suppose (1.2) is not true (with $a=0$). Then there is a non-zero tangent vector to the origin such that, for any member $F\in|L\otimes M|$ containing 0, $\langle t, df\rangle=0$ where f is a local equation of F. If $p\in D$ then $0\in T_p^*(D)+T_{\psi(p)}^*(E)$. This is a member of $|L\otimes M|$, so t is tangent to it. $0\in T_{\psi(p)}^*(E)$ means $p\in\psi^*(E)$. Since any component D_i does not equal to a component of $\psi^*(E)$ (cf. the proof of (1.1)), t is tangent to $T_p^*(D)$ at 0 for general $p\in D$. V be the invariant vector field defined by t. Then V_p is tangent to D for all i and general $p\in D_i$. It follows that V is tangent to D. This is equivalent to the property:

(1.6) For any open subset $U\subset A$ and any local equation f of D_L on U,

$$V(f)=h\cdot f \quad \text{for some} \quad h\in\mathcal{O}_A(U).$$

Let $\Lambda=\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$. We regard t as a Λ-valued point of A. Then the translation T_t on $A\times\Lambda$ induced by t is given by $(a, s)\to(a+t(s), s)$. Let L_Λ denote the pull-back of L via the projection $A\times\Lambda\to A$. Then we have $T_t^*L_\Lambda\cong L_\Lambda$ by (1.6). This means that t is a Λ-valued point of $K(L)=\{0\}$. Therefore t must be 0. This is a contradiction. Thus we have proved the theorem.

2. Proof of Comessatti's theorem.

In this section we shall show that Comessatti's theorem is a special case of the theorem proved in the previous section.

LEMMA. *Let L_0 and L_1 be ample invertible sheaves on a g-dimensional abelian variety A with $h^0(A, L_0)=h^0(A, L_1)=1$. Then the following statements are equivalent:*

(2.1) $\qquad L_0$ *is algebraically equivalent to* L_1.

(2.2) $\qquad (L_0{}^i\cdot L_1{}^{g-i})=g!$ *for* $i=0, 1, \cdots, g$.

PROOF. Let $P(n)=P_{L_0,L_0\otimes L_1^{-1}}(n)=\chi(L_0{}^n\otimes L_1{}^{-1})$. Then we have

$$P(n)=\frac{1}{g!}\left\{\sum_{i=0}^{g}(-1)^i\binom{g}{i}(L_0{}^{g-i}\cdot L_1{}^i)(n+1)^{g-i}\right\}. \tag{2.3}$$

(2.1) is equivalent to $K(L_0\otimes L_1^{-1})=A$, and it is also equivalent to $P(n)=n^g$; cf. [5] App. By (2.3), it is equivalent to (2.2). Q.E.D.

COROLLARY. *Let L_0 and L_1 be ample invertible sheaves on abelian surface A with $h^0(A, L_0)=h^0(A, L_1)=1$. Then we have the following:*

(2.4) $\qquad (L_0\cdot L_1)\geqq 2$;

(2.5) $\qquad (L_0\cdot L_1)=2$ *if and only if L_0 is algebraically equivalent to L_1.*

PROOF. (2.4) Since $(L_0^2)>0$, $(L_0\cdot L_1)^2\geqq(L_0^2)(L_1^2)=4$; hence $(L_0\cdot L_1)\geqq 2$. (2.5) follows the lemma. Q.E.D.

THEOREM (Comessatti). *Let $L\cong\mathcal{O}_A(C)$ and M be ample invertible sheaves on an abelian surface A with $h^0(A, L)=h^0(A, M)=1$, where C is an irreducible curve* *on A. If $(L\cdot M)\geqq 3$, then $L\otimes M$ is very ample.*

PROOF. Combining our theorem and (2.5), we get the resul. Q. E. D.

3. Application.

Let K be a totally real algebraic number field of degree g and $\mathfrak{o}_K$ the ring of integers of K. Let $\{\sigma_1, \sigma_2, \cdots, \sigma_g\}$ be the set of embeddings of K into the field $\boldsymbol{R}$ of real numbers. Let $\Phi: K\to \boldsymbol{M}_g(\boldsymbol{C})$ denote the representation of K over the field of complex numbers defined by

$$\Phi(a)=\begin{pmatrix} \sigma_1(a) & & 0 \\ & \ddots & \\ 0 & & \sigma_g(a)\end{pmatrix} \qquad (a\in K).$$

Then there are a simple abelian variety A over $\boldsymbol{C}$ of dimension g, an ample invertible sheaf L on A with $h^0(A, L)=1$ and a ring homomorphism $\theta: K\to \mathrm{End}_{\boldsymbol{Q}}(A)$ such that

(3.1) $$\theta(\mathfrak{o}_K)\subset \mathrm{End}(A);$$

(3.2) $r_a\circ\theta$ is equivalent to Φ where r_a is the analytic representation of $\mathrm{End}_{\boldsymbol{Q}}(A)$ with respect to some basis for the universal covering space of A,

(3.3) $\rho\cdot\theta=\theta$ where $\rho: \mathrm{End}_{\boldsymbol{Q}}(A)\to \mathrm{End}_{\boldsymbol{Q}}(A)$ is the Rosati involution defined by L, i. e., $\rho(f)=\varphi_L{}^{-1}\cdot\hat{f}\cdot\varphi_L$.

For detais we refer to [8].

We regard $\mathfrak{o}_K$ as a subring of $\mathrm{End}(A)$ via θ. Let $\varepsilon\in K$ be a unit of infinite order. Then we have

PROPOSITION. (1) *$L\otimes\varepsilon^*(L)$ is very ample.*

(2) $$h^0(L\otimes\varepsilon^*(L))=\sum_{i=0}^{g} s_i(\sigma_1(\varepsilon^2), \cdots, \sigma_g(\varepsilon^2))$$

where s_i is the i-th fundamental symmetric polynomial and $s_0=1$.

PROOF. (1) There is a positive divisor D on A such that $L\cong\mathcal{O}_A(D)$. Then D is irreducible. Otherwise A is isomorphic to a product $B\times C$ of abelian varieties of smaller dimension; cf. [7], Lem. 1.6. This contradicts to the fact that A is simple. If L is algebraically equivalent to $\varepsilon^*(L)$, then ε is an automorphism of the polarized abelian variety (A, L). Therefore the order of ε is finite; cf. [4] §20 The. 5. This is a contradiction. By the theorem we see that $L\otimes\varepsilon^*(L)$ is very ample.

(2) By the Riemann-Roch theorem, we have

$$(3.4)\qquad h^0(A,\ L\otimes\varepsilon^*(L))=\chi(L\otimes\varepsilon^*(L))$$

$$=\frac{1}{g!}\left\{\sum_{i=0}^{g}\binom{g}{i}(L^{g-i}\cdot\varepsilon^*(L)^i)\right\}$$

and

$$(3.5)\qquad \chi(L^n\otimes\varepsilon^*(L)^{-1})=\frac{1}{g!}\left\{\sum_{i=0}^{g}(-1)^i\binom{g}{i}(L^{g-i}\cdot\varepsilon^*(L)^i)n^{g-i}\right\}.$$

On the other hand (3.5) is equal to the characteristic polynomial $P(n)$ of the endomorphism; cf. [2] Lem. 2.3:

$$\varphi_L{}^{-1}\cdot\varphi_{\varepsilon^*L}=\varphi_L{}^{-1}\cdot\hat{\varepsilon}\cdot\varphi_L\cdot\varepsilon=\varphi_L{}^{-1}\cdot\varphi_L\cdot\varepsilon\cdot\varepsilon=\varepsilon^2$$

Here we used (3.3). By (3.2), we have

$$P(n)=\prod_{i=1}^{g}(n-\sigma_i(\varepsilon^2)).$$

Comparing (3.4) and (3.5), we get (2). Q. E. D.

EXAMPLE (Lange [2]). Let $K=\boldsymbol{Q}(\sqrt{5})$ and $\varepsilon=1+\sqrt{5}/2$. Let a triplet (A, L, θ) be as above. Then $L\otimes\varepsilon^*L$ is very ample and

$$h^0(A,\ L\otimes\varepsilon^*L)=1+tr(\varepsilon^2)+Nm(\varepsilon^2)=5.$$

EXAMPLE. Let $K=\boldsymbol{Q}(\varepsilon)$, where ε is a roof of $X^3-2X^2-X+1=0$. Then K is totally real and ε is a unit of infinite order. Let a triplet (A, L, θ) be as above. Then $L\otimes\varepsilon^*L$ is very ample and

$$h^0(A,\ L\otimes\varepsilon^*L)=1+tr(\varepsilon^2)+\{\sigma_2(\varepsilon^2)\sigma_3(\varepsilon^2)+\sigma_3(\varepsilon^2)\sigma_1(\varepsilon^2)$$
$$+\sigma_1(\varepsilon^2)\sigma_2(\varepsilon^2)\}+Nm(\varepsilon^2)$$
$$=1+6+5+1=13.$$

In conclnsion we raise a qustion:

What is the smallest dimension $d(g)+1$ of the space of the global sections of very ample invertible sheaves on abelian varieties of dimension g?

It is well-known that $d(2)=4$. Is $d(3)$ equal to 12?

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Department of Mathematics,
College of Science and Technology,
Nihon University,
Chiyoda-ku, Tokyo, Japan