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WEAKLY NORMAL FILTERS AND LARGE CARDLINALS

By

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0. Introduction.

In this paper, κ denotes an uncountable regular cardinal and λ a cardinal $\geqq\kappa$. For any such pair, $P_\kappa\lambda$ is the set $\{x\subset\lambda: |x|<\kappa\}$.

An "ideal on $P_\kappa\lambda$" is always a "proper, nonprincipal, κ-complete, fine ideal on $P_\kappa\lambda$" unless specified. (An ideal I is fine if for all $\alpha<\lambda$, $\{x\in P_\kappa\lambda: \alpha\notin x\}\in I$.) For any ideal I, $I^+=P(P_\kappa\lambda)-I$ and I^* is the filter dual to I.

DEFINITON. An ideal I as well as I^* are said to be *weakly normal* iff for every regressive function $f: P_\kappa\lambda\to\lambda$,

$$(\exists\gamma<\lambda)(\{x\in P_\kappa\lambda: f(x)<\gamma\}\in I^*).$$

The above definition is a translation of Kanamori's "weak normality" for filters on κ in [**5**]. There is another weak normality presented by Mignone [**10**]. It is known that our notion is Mignone's weak normality plus some saturation property and every $cf\lambda$-saturated normal ideal on $P_\kappa\lambda$ has our weak normality.

κ is said to be λ-compact if there is a fine ultrafilter on $P_\kappa\lambda$. If κ is λ-compact, $P_\kappa\lambda$ carries many fine ultrafilters. Moreover every fine ultrafilter has a weakly normal fine ultrafilter which is Rudin-Keisler ordering below it. So, it may be a natural question whether κ is large if a weakly normal filter on $P_\kappa\lambda$ exists.

In §1, we consider a case where one can say κ is large.

Kunen-Paris [**7**] and Kunen [**6**] consider the possibility of $S(\kappa, \eta)$ holding for various κ, η where

$S(\kappa, \eta)\equiv$There is a κ-complete η-saturated ideal on κ. §2 is devoted to an application of their methods to weakly normal ideals on $P_\kappa\lambda$.

Much of our notation is standard, and Jech [**4**] or [**7**] should be consulted.

§1. κ may be strong compact.

For a reader's convenience, we give a proof of a lemma which appears

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in [**2**].

LEMMA 1.1. *If I is weakly normal, then there is no disjoint family of $cf\,\lambda$-many sets in I^+.*

PROOF. Suppose that $\{A_\alpha : \alpha < cf\lambda\} \subset I^+$ is a pairwise disjoint family. Let $\{\lambda_\alpha : \alpha < cf\lambda\}$ be a cofinal increasing sequence in λ. We can assume that $A_\alpha \subset \widehat{\{\lambda_\alpha\}}$ for all $\alpha < cf\lambda$. Define a regressive function $f : P_\kappa\lambda \to \lambda$ so that $f'' A_\alpha = \{\lambda_\alpha\}$. Then $B = \{x \in P_\kappa\lambda : f(x) \leqq \gamma\} \in I^*$ for some $\gamma < \lambda$. Now pick a $\lambda_\alpha > \gamma$. By the definition of f, $A_\alpha \subset f^{-1}(\{\lambda_\alpha\})$ and $f^{-1}(\{\lambda_\alpha\}) \cap B = 0$. This contradicts to $A_\alpha \in I^+$. $\square$

If $P_\kappa\lambda$ carries a weakly normal ideal I and $cf\lambda \leqq \kappa$, I is κ-saturated. So, κ is measurable in some inner model. In general, we can say at least κ is not small.

LEMMA 1.2. (See Matsubara [**9**]) *Let $\kappa = \delta^+$ and I an ideal on $P_\kappa\lambda$. If $\gamma \leqq \lambda$, $cf\gamma > \delta$ and $X \in I^*$, then X can be decomposed into γ-many disjoint I-positive subsets.*

THEOREM 1.3. *If $P_\kappa\lambda$ carries a weakly normal ideal, then κ is weakly inaccessible.*

PROOF. There is no $cf\lambda$-many disjoint sets in I^+. Hence κ is weakly inaccessible by 1.2 if $cf\lambda \geqq \kappa$. When $cf\lambda < \kappa$, I is $cf\lambda$-saturated, and there is no λ-saturated ideal on $P_\kappa\lambda$ if κ is a successor cardinal. $\square$

If $cf\lambda$ is small, κ becomes very large. In fact we have a direct analogue of Proposition 3.8 in DiPrisco and Marek [**3**].

THEOREM 1.4. *Let $2^{<cf\lambda} < \kappa$. If there is a weakly normal ideal on $P_\kappa\lambda$, then κ is λ-compact.*

SKETCH OF PROOF. Assuming that every set $X \in I^+$ can be partitioned into two disjoint sets in I^+, we construct a tree $T \subset I^+$ with fewer than κ-many branches such that $P_\kappa\lambda$ is the union of the intersections of each branches which are all sets in I.

So, there is a set $X \in I^*$ which is not a union of two disjoint sets in I^+. Then $I|X = \{Y \subset P_\kappa\lambda : Y \cap X \in I\}$ is a prime ideal and $(I|X)^*$ is a κ-complete fine ultrafilter on $P_\kappa\lambda$. $\square$

COROLLARY 1.5. *Suppose that $cf\lambda = \omega$ and $P_\kappa\lambda$ carries a weakly normal ideal.*

Then κ is λ-compact.

Combining weakly compactness, the method of 1.4 yields following.

THFOREM 1.6. *If κ is weakly compact, $P_\kappa\lambda$ carries a κ-saturated ideal and $cf\lambda=\kappa$, then κ is λ-compact.*

PROOF. Assume that every $X\in I^+$ is a disjoint union of two sets in I^+. A tree T with αth level T_α is defined as follows.

$$T_0=\{P_\kappa\lambda\} .$$

For $X\in T_\alpha$, the immediate successors of X are two sets in I^+ such that X is their disjoint union.

For α limit, $T_\alpha=\{\cap S:\ S$ is an α-branch$\}\cap I^+$.

Since I is κ-saturated, $|T_\alpha|<\kappa$ for each $\alpha<\kappa$. Also every $T_\alpha\neq 0$ since κ is inaccessible and I is κ-complete.

By the tree property, there is a branch B through T. Let $B=\{A_\alpha : \alpha<\kappa\}$ and $A_\alpha=A_{\alpha+1}\cup A'_{\alpha+1}$. Then $\{A'_{\alpha+1} : \alpha<\kappa\}$ is an almost disjoint family which contradicts the κ-saturation of I. □

We have shown that κ is λ-compact if there is a weakly normal ideal on $P_\kappa\lambda$ and one of (a) and (b) is satisfied.

(a) $2^{<cf\lambda}<\kappa$

(b) κ is weakly compact and $cf\lambda=\kappa$.

§2. κ may not be very large.

Next we examine the other situation and present some consistency results such that κ is not λ-compact although $P_\kappa\lambda$ bears weakly normal ideals. Theorem 1.4 and 1.6 impose limitations on κ and λ.

We need several lemmas.

LEMMA 2.1. *Let I be weakly normal. If $f: X\to\lambda$ is a regressive function and $X\in I^+$, then $\{x\in X:\ f(x)\leqq\gamma\}\in I^+$ for some $\gamma<\lambda$.*

PROOF. We extend f to $g: P_\kappa\lambda\to\lambda$ which is also regressive and $g|X=f$. Then $Y=\{x\in P_\kappa\lambda:\ g(x)\leqq\gamma\}\in I^*$ for some $\gamma<\lambda$ and $Z=X\cap Y\in I^+$. $g|Z=f|Z$. □

This lemma says that our weak normality is stronger than Mignone's virsion of weak normality.

LEMMA 2.2. *$\{x: cf(sup x)<\kappa\}\in I^+$ for every weakly normal ideal.*

PROOF. We only have to show that $\{x: \sup x\in x\}\in I$. Then we have $\{x: x$ is cofinal in $\sup x\}\in I^*$ and the lemma is proved.

Suppose that $\{x: \sup x\in x\}\in I^+$. By the previous lemma, there is a $\gamma<\lambda$ such that $\{x: \sup x\leqq\gamma\}\leqq I^+$. Now $\{x: x\subset\gamma+1\}\in I^+$ contradicting that I is fine. □

LEMMA 2.3. *Suppose that I is weakly normal and $\alpha<min(cf\lambda, \kappa)$. Then $\{x: cf(sup\, x)>\alpha\}\in I^*$.*

PROOF. Assume that $\{x: cf(\sup x)\leqq\alpha\}\in I^+$. Since $\alpha<\kappa$ and I is κ-complete, there is an $\alpha_0\leqq\alpha$ such that $A=\{x: cf(\sup x)=\alpha_0\}\in I^+$. $I|A$ is also weakly normal and $A\in(I|A)^*$.

Let $\{\beta_\xi^x|\xi<\alpha_0\}$ be a cofinal increasing sequence in $\sup x$ for each $x\in A$. $f_\xi: A\to\lambda$ can be defined so that $\beta_\xi^x<f_\xi(x)\in x$. Now, for every $\xi<\alpha_0$ we have $\gamma_\xi<\lambda$ such that $\{x\in A: f_\xi(x)\leqq\gamma_\xi\}\in(I|A)^*$. Since $\alpha_0\leqq\alpha<cf\lambda$, $\gamma=\sup\{\gamma_\xi: \xi<\alpha_0\}<\lambda$. Then $A_\xi=\{x\in A: f_\xi(x)\leqq\gamma\}\in(I|A)^*$ for each $\xi<\alpha_0$ and $B=\cap\{A_\xi: \xi<\alpha_0\}\in(I|A)^*$. But $\sup x\leqq\gamma$ for every $x\in B$, which is a contradiction. □

LEMMA 2.4. *Let $\delta>\omega$. Suppose that I is a weakly normal ideal such that $\{x\in P_\kappa\lambda: cf(\sup x)\geqq\delta\}\in I^*$ and P is a δ-c.c. forcing notion in V. Then I generates a weakly normal ideal J on $P_\kappa\lambda$ in the generic extension $V[G]$.*

PROOF. In $V[G]$, J is defined by $X\in J^*$ iff $Y\subset X$ for some $Y\in I^*$. By our assumption and 2.2, $\delta<\kappa$. Hence J is an ideal on $P_\kappa\lambda$ extending I.

Let f be a regressive function on $P_\kappa\lambda$ in $V[G]$ and $\underset{\sim}{f}$ its name always denoting a regressive function on $P_\kappa\lambda$. In V, for each $x\in P_\kappa\lambda$, let $A_x=\{\alpha\in x: p\Vdash\underset{\sim}{f}(x)=\alpha$ for some $p\in P\}$.

Since P satisfies the δ chain condition, $|A_x|<\delta$. Hence $\{x: \sup A_x<\sup x\}\in I^*$ because $\{x: cf(\sup x)\geqq\delta\}\in I^*$. Using the weak normality, we have a $\gamma<\lambda$ such that $B=\{x: \sup A_x\leqq\gamma\}\in I^*$. For each $x\in B$, $1\Vdash\underset{\sim}{f}(x)\leqq\gamma$. Clearly every condition forces "$B\subset\{x: \underset{\sim}{f}(x)\leqq\gamma\}$" and B is a member of J^*. □

We only had to require $\{x: cf(\sup x)\geqq\delta\}\in I^+$, since $I|X$ is also weakly normal for any $X\in I^+$.

We can now show that the inaccessibility of κ is necessary in Theorem 1.4.

THEOREM 2.5. *It is consistent that there is a weakly normal ideal on $P_\kappa\lambda$ with $\omega<cf\lambda<\kappa$ and κ is not inaccessible.*

PROOF. Assume that κ is inaccessible, $\omega<cf\lambda<\kappa$, and I is a weakly normal ideal on $P_\kappa\lambda$ in the ground model. We blow up 2^ω to κ by c.c.c. forcing. All the cardinals and cofinalities are preserved and I generates a weakly normal ideal in the generic extension by 2.3 and 2.4. □

Two problems remain: Is it posiible that $P_\kappa\lambda$ carries a weakly normal ideal and κ is inaccessible (weakly compact) and nevertheless κ is not λ-compact? Is it consistent that κ is inaccessible, not weakly compact and there is a weakly ideal on $P_\kappa\lambda$?

For this purpose We state $P_\kappa\lambda$ generalization of what Kunen and Paris [7] did for saturated ideals on κ that is measurable in the ground model.

Let G be Q-generic over V and U is a $V-\kappa$-complete V-weakly normal filter on $P(P_\kappa\lambda)\cap V$ which lies in $V[G]$ i.e., We have in $V[G]$

(1) $U\subset P(P_\kappa\lambda)\cap V$ and $0\notin U$

(2) $X\in U\wedge X\subset Y\in P(P_\kappa\lambda)\cap V\to Y\in U$

(3) $\{X_\alpha:\ \alpha<\delta\}\in V\cap P(U)\wedge\delta<\kappa\to\cap\{X_\alpha:\ \alpha<\delta\}\in U$

(4) For every $\alpha<\lambda$, $\{x\in P_\kappa\lambda\cap V:\ \alpha\in x\}\in U$

(5) For each $f\in V$ which is regressive on $P_\kappa\lambda\cap V$, there is a $\gamma<\lambda$ such that $\{x\in P_\kappa\lambda\cap V:\ f(x)\leqq\gamma\}\in U$.

In addition, assume that Q satisfies the $cf\lambda$-c.c. and $\underset{\sim}{U}$ is a name such that $[\![\underset{\sim}{U}$ is a $V-\kappa$-complete V-weakly normal filter on $P(P_\kappa\lambda)\cap V]\!]^{B(Q)}=1$. In V, let $F=\{X\subset P_\kappa\lambda:\ [X\in\underset{\sim}{U}]^{B(Q)}=1\}$.

LEMMA 2.6. *F is a weakly normal filter on $P_\kappa\lambda$.*

PROOF. F is clearly a filter on $P_\kappa\lambda$. For its weak normality, let $f: P_\kappa\lambda\to\lambda$ regressive. Then, $[\![\{x\in P_\kappa\lambda:\ f(x)\leqq\gamma\}\in U$ for some $\gamma<\lambda]\!]^{B(Q)}=1$.

For $p\in Q$, $p(\gamma)$ is the ordinal such that p forces "$p(\gamma)$ is the least ordinal γ so that $\{x\in P_\kappa\lambda:\ f(x)\leqq\gamma\}\in\underset{\sim}{U}$".

Set $A=\{\gamma<\lambda:\ \gamma=p(\gamma)$ for some $p\}$.

Since Q is a $cf\lambda$-c.c. notion, p and p' are incompatible if $p(\gamma)\neq p'(\gamma)$, $|A|<cf\lambda$. So, $\beta=\sup A<\lambda$ and $[\![\{x:\ f(x)\leqq\beta\}\in\underset{\sim}{U}]\!]^{B(Q)}=1$. Hence $\{x\in P_\kappa\lambda:\ f(x)\leqq\beta\}\in F$. □

The situation of lemma is familiar in the large cardinal theory.

Suppose that $cf\lambda\geqq\kappa$, $R=P*Q$, G is P-generic over V and H is Q-generic over $V[G]$, $[\![\underset{\sim}{Q}$ is $cf\lambda$-c.c.$]\!]^{B(Q)}=1$, and there exists a $V[G]-\kappa$-complete $V[G]$-weakly normal filter on $P(P_\kappa\lambda)\cap V[G]$ in $V[G][H]$. Then we can find a weakly normal filter on $P_\kappa\lambda$ in $V[G]$.

Assume that $V\models GCH+\kappa$ is supercompact, and R is a usual backward Easton forcing adding α^{++} subsets to every inaccessible $\alpha\leqq\kappa$. Let G' be R-generic over V. Then $V[G]\models\kappa$ is supercompact+there is a normal ultrafilter on $P_\kappa\lambda$. $R=P*Q$ where $[\![Q$ adds κ^{++} subsets to $\kappa]\!]^{B(P)}=1$ and $V[G]=V[G_1][H]$, G_1 is P-generic and H is Q-generic over $V[G]$. Since $V[G_1]\models 2^\kappa=\kappa^+ + 2^\alpha=\alpha^{++}$ for any inaccessible $\alpha<\kappa$, κ is a inaccessible cardinal that is not measurable in $V[G_1]$. Moreover Q satisfies κ^+-c.c. in $V[G_1]$. Hence we get a weakly normal filter on $P_\kappa\lambda$ in $V[G_1]$ if $cf\lambda\geqq\kappa^+$.

However, it is not clear whether κ is weakly compact or not in this construction. So, we follow Kunen's argument in [**6**].

Again we start from the universe V_0 where κ is supercompact. Without loss of generality, supercompactness of κ is indestructible under κdirected closed forcing (Ref. [**8**]). Let $V_1=V_0[G]$, G is V_0-generic over P adding a Cohen subset of κ. Since P is κ-directed closed, κ is supercompact in V_1. $V_1=V_0[H][K]$, H is Q-generic over V_0, K is R-generic over $V_1[H]$, where Q is a notion of forcing which adds a κ-Suslin tree T, and R is T itself.

$$V_1\models\exists U: V_0[H]\text{-weakly normal filter on } P(P_\kappa\lambda)\cap V_0[H]$$

and

$$V_0[H]\models Q \text{ is the } \kappa\text{-c.c.}+cf\lambda\geqq\kappa$$

We have a weakly normal filter in $V_0[H]$ and κ is inaccessible but not weakly compact since there is a κ-Suslin tree. We have proved;

THEOREM 2.7. *Con* ($\exists\kappa$(*κ is supercompact*)) *implies Con* ($\exists\kappa$(*κ is not weakly compact*+*κ is inaccessible*+*$P_\kappa\lambda$ carries a weakly normal filter*+$cf\lambda\geqq\kappa$)).

Next we try the case where κ is weakly compact but not measurable. It is impossible that $cf\lambda\leqq\kappa$ by Theorem 1.4, 1.6. The following lemma is standard.

LEMMA 2.8. *Let $j: V\to M$ be a λ-supercompact embedding with the critical point κ, $P\subset V_\kappa$, $j(P)=P\oplus Q$, G be $j(P)$-generic. If U is defined as;*

$$p\Vdash_{j(P)}\underset{\sim}{X}\in\underset{\sim}{U} \text{ iff for some } P\text{-term } \underset{\sim}{X}' \quad p\Vdash_P\underset{\sim}{X}=\underset{\sim}{X}'\subset P_\kappa\lambda$$

and

$$M\models(p\Vdash_{j(P)}j''\lambda\in j(X')),$$

then $V[G]\models U$ is a $V[G_1]$-κ-complete $V[G_1]$-normal ultrafilter on $P_\kappa\lambda\cap V[G_1]$, where G_1 is V-generic on P. (*Note that $j(p)=p$ for all $p\in P$.*)

PROOF. U is trivially a proper fine filter. Let $\tau\in V[G_1]$ and $\tau(\alpha)\in U$ for all $\alpha<\delta<\kappa$. For some $p\in G$, $p\Vdash\forall\alpha<\delta(\tau'(\alpha)=\tau(\alpha)\in\underset{\sim}{U})$. So, p forces in M that

$j''\lambda\in j(\tau'(\alpha))$. ($\tau'$ is a P-term that that every condition forces that $\tau'=\bigcap_{\alpha<\delta}\tau(\alpha)$.) Then $j(P)\Vdash j(\tau')=\bigcap_{\alpha<\delta}j(\tau(\alpha))$. Hence $p\Vdash j''\lambda\in j(\tau')$ in M, which says that U is $V[G_1]$-κ-complete.

To show that U is ultra, pick a P-terms $\underline{X}$ and $\underline{X}^c$ which always denotes X and $P_\kappa\lambda-X$ respectively for given $X\in P_\kappa\lambda\cap V[G_1]$. If X is not in U, then we can find a $p\in G$ such that $p\Vdash\neg j''\lambda\in j(\underline{X})$ since $\{p: p$ decides "$j''\lambda\in j(\underline{X})$"$\}$ is dense in $j(P)$. So, $p\Vdash j''\lambda\in j(\underline{X}^c)$ and $P_\kappa\lambda-X\in U$.

For the normality of U, let $f: P_\kappa\lambda\to\lambda$ be a regressive function in $V[G_1]$ and $\underline{f}$ be its P-name. Then $j(P)\Vdash j(\underline{f})(j''\lambda)\in j''\lambda$. $V[G]\supset M[G]$ and $j(f)(j''\lambda)=j(\alpha)$ for some $\alpha<\lambda$ in $M[G]$. So, "$j(\underline{f})(j''\lambda)=j(\alpha)$" is forced by some $p\in G$. In $V[G_1]$, let $X=\{x\in P_\kappa\lambda: f(x)=\alpha\}$ and $\underline{X}$ be its P-name. $j(P)\Vdash j(\underline{X})=\{x\in P_{j(\kappa)}j(\lambda): j(\underline{f})(x)=j(\alpha)\}$. Hence p forces "$j''\lambda\in j(\underline{X})$" and $X\in U$. □

THEOREM 2.9. *If the existence of a supercompact cardinal is consistent, it is also consistent that κ is weakly compact, not measurable, and there is a weakly normal filter on $P_\kappa\lambda$ with $cf\lambda\geqq\kappa^+$.*

PROOF. We follow the proof of Theorem 4.4 in [7]. Let P be a forcing notion adding one generic subset to each inaccessible cardinal less than κ. $j: V\to M$ is an elementary embedding with the critical point κ and M is closed under 2^λ-sequences. Note that $cf\lambda\geqq\kappa^+$. We can furthere assume that $2^\kappa\geqq\lambda$.

$j(P)=P\oplus Q\oplus R$, where Q adds a subset to κ and R treats inaccessibles between κ and $j(\kappa)$ in M. So, Q satisfies the κ^+-c.c. and R is λ^+-closed. Let G_1, G_2, G_3 be V-generic over P, Q, R respectively. By the lemma, we have a $V[G_1]$-κ-complete $V[G_1]$-normal ultrafilter on $P_\kappa\lambda\cap V[G_1]$ in $V[G_1\times G_2\times G_3]$.

$P(P_\kappa\lambda)\cap V[G_1\times G_3]=P(P_\kappa\lambda)\cap V[G_1]$ since R is λ^+-closed and $\lambda=(\lambda^{<\kappa})^V=(\lambda^{<\kappa})^{V[G_1]}=(\lambda^{<\kappa})^{V[G_1\times G_3]}$.

Now, U is a $V[G_1\times G_3]$-κ-complete $V[G_1\times G_3]$-normal ultrafilter on $P(P_\kappa\lambda)\cap V[G_1\times G_3]$ in $V[G_1\times G_2\times G_3]$.

Since Q is κ^+-c.c., we can find a weakly normal filter on $P_\kappa\lambda$ in $V[G_1\times G_2]$ using Lemma 2.6. It is known that κ is weakly compact but not measurable in $V[G_1\times G_3]$. □

Recent work of Abe and Matsubara show that weakly normal filters on $P_\kappa\lambda$ exist if there is a precipitous ideal I with no pairwis disjoint family of $cf\lambda$-many sets in I^+.

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