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A COMBINATORIAL PROOF FOR ARTIN'S PRESENTATION OF THE BRAID GROUP B_n AND SOME CYCLIC ANALOGUE

By

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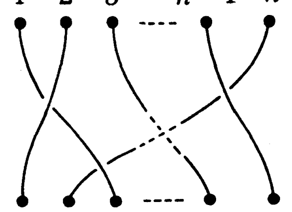
1. Artin's presentation.

For each $n\geqq 1$, let S_n be the symmetric group on n letters $\{1, 2, \cdots, n\}$, and B_n the geometric braid group with n strings.



There is a natural homomorphism, called χ_n, of B_n onto S_n. As usual, S_{n-1} and B_{n-1} are regarded as subgroups of S_n and B_n respectively, and then the restriction of χ_n to B_{n-1} coinsides with χ_{n-1}. Put $B_n^0=\chi_n^{-1}(S_{n-1})$. Then B_{n-1} is a subgroup of B_n^0.

Let $\tilde{B}_n$ be the group presented by the generators:

$$\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$$

and the defining relations:

$$\begin{cases} \sigma_i\sigma_j\sigma_i=\sigma_j\sigma_i\sigma_j & \text{if } |i-j|=1\,; \\ \sigma_i\sigma_j=\sigma_j\sigma_i & \text{if } |i-j|\neq 0, 1\,. \end{cases}$$

Put

$$\tau_i=\sigma_{n-1}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\cdots\sigma_{n-1} \qquad \text{for } 1\leqq i\leqq n-2\,,$$

$$\tau_{n-1}=\sigma_{n-1}^2\,.$$

Let $\tilde{B}_n^0$ be the subgroup of $\tilde{B}_n$ generated by $\sigma_1, \cdots, \sigma_{n-2}, \tau_1, \cdots, \tau_{n-1}$. Then there is a natural homomorphism of $\tilde{B}_{n-1}$ into $\tilde{B}_n^0$.

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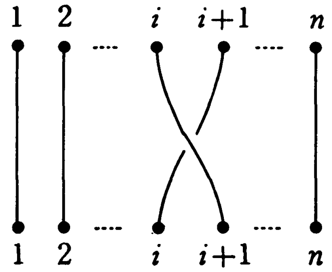
Taking σ_i to the i-th fundamental braid:



for $1 \leqq i \leqq n-1$, we obtain a homomorphism, called ϕ_n, of $\tilde{B}_n$ onto B_n. Then the following result is well-known.

ARTIN'S THEOREM. *ϕ_n is an isomorphism.*

PROOF. We proceed by induction on n. The result is trivial if $n=1, 2$. Suppose $n \geqq 3$, and that ϕ_{n-1} is an isomorphism. Forgetting the n-th string, we obtain a homomorphism, called θ, of B_n^0 onto B_{n-1}. Hence, $B_n^0 = B_{n-1} \ltimes \operatorname{Ker}\theta$, and $\operatorname{Ker}\theta \cong F_{n-1}$, where F_{n-1} is the free group of rank $n-1$. This fact implies that $\tilde{B}_n^0$ is isomorphic to B_n^0 under ϕ_n. Let $\rho = \sigma_1\sigma_2 \cdots \sigma_{n-1}$, and put

$$\tilde{X} = \tilde{B}_n^0 \cup \tilde{B}_n^0 \rho \cup \cdots \cup \tilde{B}_n^0 \rho^{n-1} .$$

Then $\tilde{B}_n = \langle \tilde{B}_n^0, \rho \rangle$, and $\tilde{X}$ is a subgroup since

$$\rho\sigma_i = \sigma_{i+1}\rho \qquad (1 \leqq i \leqq n-2),$$

$$\rho\sigma_{n-2} = \sigma_{n-2}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\rho^2 , \qquad \rho^2\sigma_{n-2} = \sigma_1\sigma_2 \cdots \sigma_{n-2}\tau_{n-1}\rho ,$$

$$\rho\tau_i = \sigma_1\sigma_2 \cdots \sigma_{i-1}\sigma_i^2\sigma_{i-1}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\rho \qquad (1 \leqq i \leqq n-2),$$

$$\rho\tau_{n-1} = \tau_1\rho ,$$

$$\rho^n = (\sigma_1\sigma_2 \cdots \sigma_{n-2})^{n-1}\tau_{n-1}\tau_{n-2} \cdots \tau_2\tau_1 ,$$

Therefore, $\tilde{X} = \tilde{B}_n$, and the group index $[\tilde{B}_n : \tilde{B}_n^0]$ is at most n, which implies $[\tilde{B}_n : \tilde{B}_n^0] = n$. Hence, ϕ_n is an isomorphism. ■

2. Some cyclic analogue.

Here we consider the braid group $B_{n+1} = \langle \sigma_1, \sigma_2, \cdots, \sigma_n \rangle$ with $n \geqq 3$ and a certain subgroup. Put

$$\delta = \sigma_n^{-2}\sigma_1\sigma_2 \cdots \sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\sigma_n^2 ,$$

$$\pi = \sigma_1\sigma_2 \cdots \sigma_{n-1}\sigma_n^2$$

and set $C_{n+1}^0 = \langle \sigma_1, \cdots, \sigma_{n-1}, \delta \rangle \subset B_{n+1}$. Then $B_{n+1}^0 = \langle C_{n+1}^0, \pi \rangle$. Let C_{n+1}^* be the group presented by the generators:

$$\beta_1, \beta_2, \cdots, \beta_n$$

and the defining relations:

$$\begin{cases} \beta_i\beta_j\beta_i=\beta_j\beta_i\beta_j & \text{if } |i-j|=1,\ n-1 \\ \beta_i\beta_j=\beta_j\beta_i & \text{if } |i-j|\neq 0,\ 1,\ n-1 \end{cases}$$

and Z the infinite cyclic group generated by ζ. We construct the semi-direct product, called $B_{n+1}^*=Z\ltimes C_{n+1}^*$ of Z and C_{n+1}^* with $\zeta\beta_i\zeta^{-1}=\beta_{i+1}$ $(1\leqq i\leqq n-1)$ and $\zeta\beta_n\zeta^{-1}=\beta_1$. Then there is a homomorphism ψ_1 of B_{n+1}^* onto B_{n+1}^0 with

$$\psi_1:\begin{cases} \beta_i \longmapsto \sigma_i & (1\leqq i\leqq n-1); \\ \beta_n \longmapsto \delta; & \\ \zeta \longmapsto \pi. & \end{cases}$$

On the other hand, there is a homomorphism ψ_2 of B_{n+1}^0 onto B_{n+1}^* with

$$\psi_2:\begin{cases} \sigma_i \longmapsto \beta_i & (1\leqq i\leqq n-1); \\ \tau_i \longmapsto \gamma_i & (1\leqq i\leqq n), \end{cases}$$

where $B_{n+1}^0=\langle\sigma_1, \cdots, \sigma_{n-1}, \tau_1, \cdots, \tau_n\rangle\cong B_n\ltimes F_n$ and

$$\gamma_i=\beta_{i-1}^{-1}\cdots\beta_2^{-1}\beta_1^{-1}\zeta\beta_{n-1}^{-1}\cdots\beta_{i+1}^{-1}\beta_i^{-1}.$$

Then one can see both $\psi_1\psi_2=id.$ and $\psi_2\psi_1=id.$ Hence we obtain the following.

THEOREM. *$B_{n+1}^0\cong B_{n+1}^*$ and $C_{n+1}^0\cong C_{n+1}^*$.*

Therefore, the group C_{n+1}^0 may be called a braid covering of the affine Weyl group $W_a(S_n)$ associated with S_n. We can describe this fact more precisely as follows. Let f_n be the canonical gradation homomorphism of $F_n\cong\langle\tau_1, \cdots, \tau_n\rangle$ onto $\boldsymbol{Z}$, and put $E_n=\operatorname{Ker} f_n$. Then $C_{n+1}^0\cong B_n\ltimes E_n$ and E_n is the normal subgroup of F_n generated by

$$\tau_1\tau_2^{-1}, \tau_2\tau_3^{-1}, \cdots, \tau_{n-1}\tau_n^{-1}.$$

Hence, we obtain a homomorphism ν_{n+1} of $C_{n+1}^0\cong B_n\ltimes E_n$ onto

$$S_n\ltimes E_n/[F_n, F_n]\cong S_n\ltimes\boldsymbol{Z}^{n-1}\cong W_a(S_n).$$

The (C_{n+1}^0, ν_{n+1}) gives the above braid covering of $W_a(S_n)$. Put $Q_{n+1}=\operatorname{Ker}\nu_{n+1}$. Then $Q_{n+1}\cong P_n\ltimes[F_n, F_n]$, where P_n is the kernel of χ_n and called the pure braid group with n strings.

We refer to [1], [2] for braid groups, and [3] for affine Weyl groups.

References

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