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SHARP CHARACTERS OF FINITE GROUPS HAVING PRESCRIBED VALUES

By

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Let χ be a generalized character of a finite group G with $L=\{\chi(g)|g\in G, g\neq 1\}$. Cameron and Kiyota [2] called that the pair (G, χ) is L-sharp if $|G| = \prod_{\alpha\in L}(\chi(1)-\alpha)$, and posed the problem of determining all the L-sharp pairs (G, χ) for various sets L of complex numbers. In [2] and Cameron, Kiyota and Kataoka [3], L-sharp pairs (G, χ) for several sets L are characterized or partially settled. In this paper, we consider the cases $L=\{l, l+1, l+2, l+3\}$ with $l\in \boldsymbol{Z}$, and $L=\{0\}\cup L'$ where L' is a family of algebraic conjugates. The results are as follows.

THEOREM 1. *Let G be a finite group and χ be a faithful character of degree n of G. Suppose that (G, χ) is $\{l, l+1, l+2, l+3\}$-sharp with $l\in \boldsymbol{Z}$, and normalized. Then*

(1) *$l=-2$ or -1, and χ is irreducible;*

(2) *G is isomorphic to one of the following groups:*

$$\mathrm{SL}(2, 3)\ (n=2 \text{ and } l=-2);$$

$$S_5\ (n=4 \text{ and } l=-1);$$

$$A_6\ (n=5 \text{ and } l=-1);$$

$$M_{11}\ (n=10 \text{ and } l=-1).$$

By inspection of character tables, it is easily verified that the above four groups have sharp characters of type $\{l, l+1, l+2, l+3\}$ with $l=-2$ or -1. We note that the case $l=-1$ was proved by [2].

THEOREM 2. *Let G be a finite group and χ be a faithful irreducible character of G. Suppose that (G, χ) is L-sharp with $L=\{0\}\cup L'$ where L' is a family of algebraic conjugates and $|L'|\geqq 2$. Then G is dihedral of twice odd prime order, and χ is an irreducible character of degree 2.*

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In Theorem 2, the pair (G, χ) is normalized since χ is irreducible. When χ is a (possibly reducible) character of G and (G, χ) is normalized, Cameron and Kiyota [2] proved that the theorem 2 is true under either of the following hypotheses:

(1) n is coprime to $f_{L'}(n)$;

(2) $|L'|=2$.

1. Some preliminrry results.

For a given finite set L of complex numbers, let $f_L(x)$ denote the monic polynomial of least degree having L as its set of roots, that is,

$$f_L(x)=\prod_{\alpha\in L}(x-\alpha).$$

Let G be a finite group and χ be a generalized character of G with $\chi(1)=n$. Let $L=\{\chi(g)|g\in G, g\neq 1\}$. Then we may say that the pair (G, χ) is of *type* L. If (G, χ) is of type L, then it is known by Blichfeldt [1] that $f_L(n)$ is a rational integer and $|G|$ divides $f_L(n)$. We say that the pair (G, χ) is *L-sharp* if (G, χ) is of type L and $|G|=f_L(n)$. Thus χ is faithful whenever (G, χ) is L-sharp. We note that the L-sharpness of (G, χ) is equivalent to the condition $f_L(n)=\rho_G$, where ρ_G is the regular character of G.

Adding a multiple of the principal character 1_G to χ adds the same quantity to n and to each element of L, and so does not affect the sharpness of (G, χ). Accordingly, we say that (G, χ) is *normalized* if $(\chi, 1_G)=0$.

Throughout this section, let G be a finite group and let χ be a faithful generalized character of G. The first four lemmas appear in the work [2] of Cameron and Kiyota. We will make use of these results later.

LEMMA 1.1 (Proposition 1.3 in [2]). *Let (G, χ) be L-sharp and normalized, where $L\subseteqq \boldsymbol{R}$.*

(1) *If $|L|=2$, say $L=\{l_1, l_2\}$, then $(\chi, \chi)_G=1-l_1l_2$.*

(2) *If $|L|>2$ and $\min(L)$, $\max(L)\in \boldsymbol{Z}$, then $(\chi, \chi)_G\leqq -\min(L)\cdot\max(L)$.*

LEMMA 1.2 (Corollary 1.4 in [2]). *Let χ be a faithful character of G. With the hypotheses of Lemma 1.1,*

(1) *If $|L|=2$, then $\min(L)<0\leqq\max(L)$;*

(2) *If $|L|>2$ and $\max(L)$, $\min(L)\in \boldsymbol{Z}$, then $\min(L)<0<\max(L)$.*

LEMMA 1.3 (Proposition 1.6 in [2]). *Let F be a monic polynomial with integer coefficients and degree d, and L a finite subset of complex numbers such* *that each element of $F(L)$ is the image under F of exactly d elements of L. If (G, χ) is L-sharp, then $(G, F(\chi))$ is $F(L)$-sharp.*

LEMMA 1.4 (Proposition 2.4 in [2]). *Let χ be a faithful character of G. Let (G, χ) be $\{0, l\}$-sharp with $l \neq -1$ and normalized. Then*

(1) *$-l$ is a prime power;*

(2) *$|G|$ is bounded by a function of l;*

(3) *If $-l=p$ is prime, then $G=P \rtimes Z_{p-1}$, where P is a non-abelian group of order p^3.*

Next we introduce the result [3] for classification of $\{-1, 1\}$-sharp pairs and two Theorems concerning L which contains a family of algebraic conjugates.

THEOREM 1.5 (Main Theorem in [3]). *Let χ be a faithful character of degree n of G. If (G, χ) is $\{-1, 1\}$-sharp, then G is isomorphic to one of the following twelve groups:*

D_8 *and* Q_8 $(n=3)$;

S_4 *and* SL(2, 3) $(n=5)$;

GL(2, 3) *and the binary octahedral group* $(n=7)$;

S_5 *and* SL(2, 5) $(n=11)$;

PSL(2, 7) $(n=13)$;

A_6 $(n=19)$;

the double cover $\hat{A}_7$ *of* A_7 $(n=71)$;

M_{11} $(n=89)$.

THEOREM 1.6 (Theorem 4.1 in [2]). *Let χ be a faithful character of G and L a family of algebraic conjugates and $|L|>1$. If (G, χ) is L-sharp and normalized, then G is cyclic of odd prime order, and χ is either a linear character of G, or the sum of two complex conjugate linear characters of G.*

THEOREM 1.7 (Theorem 7.3 in [2]). *Let χ be a faithful character of G and $L=\{0\} \cup L'$, where L' is a family of algebraic conjugates. Suppose either that n is coprime to $f_{L'}(n)$ or that $|L'|=2$. If (G, χ) is L-sharp and normalized, then G is dihedral of twice odd prime order, and χ is an irreducible character of degree 2.*

2. Proof of Therem 1.

From now on, let G be a finite group and χ a faithful character of degree n of G. We construct new sharp pairs from old ones.

PROPOSITION 2.1. *Let l_1 and l_2 be integers with $l_1<0<l_2$ and $l_1+l_2\neq 0$. Let $(\chi, \chi)_G=m$, and let $\varphi=\chi^2-(l_1+l_2)\chi-m1_G$. Suppose that (G, χ) is $\{0, l_1, l_2, l_1+l_2\}$-sharp. Then*

(1) *(G, φ) is $\{-m, -m-l_1l_2\}$-sharp;*

(2) *(G, φ) is normalized and $(\varphi, \varphi)=1-m(m+l_1l_2)$ if (G, χ) is.*

PROOF. (1) Let $L=\{0, l_1, l_2, l_1+l_2\}$ and $F(x)=x^2-(l_1+l_2)x-m$. Then (G, φ) is clearly of type $F(L)=\{-m, -m-l_1l_2\}$, and

$$\begin{aligned} f_L(n)&=n(n-l_1)(n-l_2)(n-l_1-l_2)\\ &=(F(n)+m)(F(n)+m+l_1l_2)\\ &=f_{F(L)}(\varphi(1)). \end{aligned}$$

This identity shows that (G, φ) is $F(L)$-sharp.

(2) If (G, χ) is normalized, then we have, by orthogonality relation,

$$(\varphi, 1_G)=(\chi^2, 1_G)-m=0 .$$

Thus (G, φ) is normalized. Also it follows from (1) that

$$\rho_G=\varphi^2+(2m+l_1l_2)\varphi+m(m+l_1l_2)1_G .$$

Hence we have

$$(\varphi, \varphi)=(\varphi^2, 1_G)=1-m(m+l_1l_2),$$

and the proof is complete.

In the proof of Proposition 2.1, we notice that φ is a generalized character not necessarily character. However, φ is faithful as χ is so.

COROLLARY 2.2. *Let $(\chi, \chi)_G=m$, and let $\varphi=\chi^2+\chi-m1_G$. If (G, χ) is $\{-2, -1, 0, 1\}$-sharp and normalized, then*

(1) *χ is irreducible, and (G, φ) is $\{-1, 1\}$-sharp and normalized;*

(2) *φ is a character.*

PROOF. Under the same notation as in Proposition 2.1, we put $l_1=-2$ and $l_2=1$. Then it follows from Lemma 1.1 (2) that $(\chi, \chi)=m\leqq 2$. Hence m must be equal to 1 or 2. However, if $m=2$, then by Proposition 2.1, (G, φ) is

$\{-2, 0\}$-sharp and φ is an irreducible character of G. Hence it follows from Lemma 1.4 that G is a non-abelian group of order 8. In particular, we have $\varphi(1)=4$. This is impossible since the groups of order 8 have no irreducible character of degree 4. Thus m must be equal to 1. Therefore χ is irreducible and (G, φ) is $\{-1, 1\}$-sharp. Also we then have $(\varphi, \varphi)=2$.

So, if φ is not a character, it is the difference of two irreducible characters. But χ^2 is the sum of its symmetric and alternating parts, and the symmetric part contains the principal character 1_G. This is impossible as $\varphi=\chi^2+\chi-1_G$. Hence the proof is complete.

COROLLARY 2.3. *Let* $(\chi, \chi)_G=m$, *and let* $\varphi=\chi^2-\chi-m1_G$. *If* (G, χ) *is* $\{-1, 0, 1, 2\}$*-sharp and normalized, then*

(1) χ *is irreducible, and* (G, φ) *is* $\{-1, 1\}$*-sharp and normalized*;

(2) φ *is a character.*

PROOF. The result follows from the similar argument as Corollary 2.2.

Now we are ready to prove the theorem 1 stated in the introduction.

PROOF OF THEOREM 1. It follows from Lemma 1.1 and Lemma 1.2 that $l(l+3)<0$. Hence we have $l=-2$ or -1. Now let $(\chi, \chi)_G=m$ and let $\varphi=\chi^2-(2l+3)\chi-m1_G$ with $l=-2$ or -1. Then, by Corollary 2.2 and 2.3, (G, φ) is $\{-1, 1\}$-sharp. So we can quote the classification theorem 1.5 of sharp pairs of type $\{-1, 1\}$. If $l=-2$, then since 3, 7 and 13 are not of the form n^2+n-1, G is isomorphic to one of the following groups:

$$S_4 \text{ and } SL(2, 3)\ (n=2);$$

$$S_5 \text{ and } SL(2, 5)\ (n=3);$$

$$A_6\ (n=4);$$

$$\text{the double cover } \hat{A}_7 \text{ of } A_7\ (n=8);$$

$$M_{11}\ (n=9).$$

Since the irreducible character of degree 2 of S_4 is not faithful and the irreducible character of degree 3 of SL(2, 5) is not rational, G is not S_4 and SL(2, 5). Moreover, the other four groups except the SL(2, 3) have no irreducible characters of given degree n by inspection of character tables, and so the result follows. (Of course, the irreducible character of degree 2 of SL(2, 3) satisfies the assumption.)

For the case $l=-1$, the similar argument as $l=-2$ gives the result.

3. Proof of Theorem 2.

Throughout this section, let χ be a faithful irreducible character of degree n of a finite group G, and let $L=\{0\}\cup L'$, where L' is a family of algebraic conjugates with $|L'|=t$. We also set

$$a=|\{x\in G \mid \chi(x)=0\}|$$

$$b=|\{x\in G \mid \chi(x)=\alpha\}|$$

for $\alpha\in L'$, and

$$-s=\sum_{\alpha\in L}\alpha .$$

Suppose that (G, χ) is L-sharp and normalized. Since (G, χ) is of type L, the elements of L' occur equally often, each b times, as values of χ, and so

$$|G|=1+a+bt . \tag{3.1}$$

Moreover, since (G, χ) is normalized, $(\chi, 1_G)=0$ implies

$$n-bs=0 , \tag{3.2}$$

and so s must be a positive integer.

PROPOSITION 3.1. *Under the above notation, if (G, χ) is L-sharp, then the followings hold.*

(1) $|G|=nf_{L'}(n)$ *where* $f_{L'}(n)=\prod_{\alpha\in L}(n-\alpha)$.

(2) *There is a non-identity p-element g of G, for some prime p, such that* $\chi(g)\neq 0$.

(3) *For the same prime p as in* (2), $f_{L'}(n)$ *is a power of p.*

PROOF. Statement (1) follows from definition.

(2) If not, then the restriction of χ to every Sylow subgroup P of G is a multiple of the regular character of P, whence $|P|$ devides n, and so $|G|$ divides n. This is impossible and so (2) holds.

(3) Let g be an element of order p^d of G such that $\chi(g)\neq 0$. Since $\chi(g)$ is a sum of p^dth roots of unity, L' is contained in the field $\boldsymbol{Q}(e^{2\pi i/p^d})$. If $(p, m)=1$, it is well known from Galois theory that $\boldsymbol{Q}(e^{2\pi i/p^d})\cap\boldsymbol{Q}(e^{2\pi i/m})=\boldsymbol{Q}$. Therefore p is a unique prime such that $L'\subseteq\boldsymbol{Q}(e^{2\pi i/p^d})$, since $L'\not\subseteq\boldsymbol{Q}$. Thus if Q is a Sylow q-subgroup of G, for any prime q different from p, then the restriction of χ to Q is a multiple of the regular character of Q, whence $|Q|$ divides n. Thus the p'-part of the order of G divides n, and so statement (1)

implies that $f_{L'}(n)$ is a power of p, and the proof is complete.

Since the Galois group of $\boldsymbol{Q}(e^{2\pi i/p^d})$ over $\boldsymbol{Q}$ acts transitively on L', G has t distinct Galois conjugates, say $\chi=\chi_1, \chi_2, \cdots, \chi_t$, of χ. Now we set $\varphi=\chi_1+\chi_2+\cdots+\chi_t$. Clearly, φ is a faithful character of G with $(\varphi, 1_G)=0$, and the pair (G, φ) is of type $\{0, -s\}$.

PROPOSITION 3.2. *Let φ be as above. Under the same notation as in Proposition* 3.1, *if (G, χ) is L-sharp, then*

(1) $f_{L'}(n)=s(1+bt)$;

(2) *b is the p'-part of the order of G.*

PROOF. (1) Using (3.2), the inner product of φ with χ gives

$$1=(\varphi, \chi)=\frac{1}{|G|}(n^2t+bs^2)=\frac{ns(1+bt)}{|G|}.$$

Thus $f_{L'}(n)=s(1+bt)$.

(2) If follows from Proposition 3.1 and statement (1) that $s(1+bt)$ is a power of p. In particular, b is relatively prime to p and therefore $|G|=bs^2(1+bt)$ means b is the p'-part of the order of G as desired.

PROPOSITION 3.3. *Under the same notation as in Proposition* 3.1, *if (G, χ) is L-sharp, then the following hold.*

(1) *$N=\{g\in G \mid \chi(g)\neq 0\}$ is the unique minimal normal subgroup of G.*

(2) *For any $\alpha\in L'$, $C_\alpha=\{g\in G \mid \chi(g)=\alpha\}$ is a single conjugacy class of G. In particular, N is an elementary abelian p-subgroup of G.*

PROOF. (1) Set $\Theta=\mathrm{Irr}(G)-\{$all irreducible constituents of $\varphi\}$. Then, for any $\theta\in\Theta$, we have

$$(\theta, \varphi)=\frac{1}{|G|}\sum_{g\in G}\theta(g)\overline{\varphi(g)}$$

$$=\frac{1}{|G|}\{nt\theta(1)-\sum_{g\in N-\{1\}}s\theta(g)\},$$

whence by (3.2),

$$\sum_{g\in N-\{1\}}\theta(g)=bt\theta(1).$$

Thus we obtain $\theta(g)=\theta(1)$ for any element g of N, and so $N\subseteq\bigcap_{\theta\in\Theta}\mathrm{Ker}\,\theta$. Let g be a non-identity element of N. If there exists a non-identity element h of $\bigcap_{\theta\in\Theta}\mathrm{Ker}\,\theta$ that is not contained in N, then the second orthogonality relation applied to the conjugacy classes containing g and h yields

$$0=\sum_{\theta\in\Theta}\theta(g)\theta(h)=\sum_{\theta\in\Theta}\theta(1)^2 .$$

a contradiction. Thus $N=\bigcap_{\theta\in\Theta}\operatorname{Ker}\theta$, and so N is a normal subgroup of G.

Let M be any proper normal subgroup of G, and put Ψ be the set of irreducible characters ψ of G with kernel containing M. As χ is faithful, χ does not contained in Ψ. Thus we have $N\subseteqq\operatorname{Ker}\psi$ for every $\psi\in\Psi$, and so $M=\bigcap_{\psi\in\Psi}\operatorname{Ker}\psi\supseteqq N$. Hence N is the unique minimal normal subgroup of G.

(2) Let g, h be any elements of C_α and let θ be any irreducible character of G. Then we have $\theta(g)=\theta(h)=\theta(1)$, and so C_α is a single conjugacy class of G.

Clearly N is a p-group as $|N|=1+bt$ is a power of p. Since, for any $\beta\in L'$, each element of C_β is a power of an element of C_α, every element of $N-\{1\}$ is of order p. In particular, N is an elementary abelian p-subgroup. This completes the proof of Proposition 3.3.

PROOF OF THEOREM 2. By Theorem 1.7, we may assume that $t\geqq 3$. Let $N=\{g\in G\,|\,\chi(g)\neq 0\}$. By Proposition 3.3, N is an elementary abelian normal p-subgroup of G. Hence we have, by Clifford's Theorem,

$$\chi_N=s\sum_{i=1}^{b}\lambda_i$$

for some linear character λ_i of N. Hence we have, by Proposition 3.1 and 3.2,

$$s|N|=f_{L'}(n)=s^t\prod_{\alpha\in L'}(b-\alpha/s).$$

Also, clearly, the pair $(N, \sum_{i=1}^{b}\lambda_i)$ is of type $\{\alpha/s\,|\,\alpha\in L'\}$. This yields that $\prod_{\alpha\in L'}(b-\alpha/s)$ is divisible by $|N|$, and so we have $s=1$ as $t\geqq 3$. In particular, the pair (N, χ_N) is of type L'. Hence it follows from Theorem 1.6 that N must be cyclic of order p and $n=2$. Thus G is dihedral of order $2p$ and χ is an irreducible character of degree 2. This completes the proof of Theorem 2.

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