TSUKUBA J. MATH.
Vol. 16 No. 1 (1992), 169–172

# A REMARK ON FOLIATIONS ON A COMPLEX PROJECTIVE SPACE WITH COMPLEX LEAVES

Dedicated to Professor Hisao Nakagawa on his sixtieth birthday

By

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## Introduction.

Let $\mathcal{F}$ be a foliation on a Riemannian manifold $M$. The distribution on $M$ which is defined to be orthogonal to $\mathcal{F}$ is said to be normal to $\mathcal{F}$ and denoted by $\mathcal{F}^{\perp}$.

Nakagawa and Takagi [8] showed that any harmonic foliation on a compact Riemannian manifold of non-negative constant sectional curvature is totally geodesic if the normal distribution is minnimal. And succesively the present author [2] proved a complex version of their result, that is, the above result holds also on a complex projective space with a Fubini-Study metric. However, recently, Li [4] pointed out a serious mistake in the proof of the result of Nakagawa and Takagi, and so of the author's. Therefore those results are now open yet.

On the other hand, Li [4] have studied a harmonic foliation on the sphere along the method of Nakagawa and Takagi, and obtained some interesting results.

The purpose of this paper is to give a complex analogue of the Li's results. Let $\boldsymbol{P}_{n+p}(\boldsymbol{C})$ be the complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c$. Let $\mathcal{F}$ be a complex foliation on $\boldsymbol{P}_{n+p}(\boldsymbol{C})$ with $q$-complex codimention and $h$ the second fundamental tensor of $\mathcal{F}$. Then we shall prove the following;

THEOREM. *If the normal distribution $\mathcal{F}^{\perp}$ is minimal, we have*

$$\int_{\boldsymbol{P}_{n+p}(\boldsymbol{C})} S\left\{\left(2-\frac{1}{2p}\right)S-\frac{n+2}{2}c\right\}*1\geqq 0,$$

*where $S$ denotes the square of the length of $h$ and $*1$ the volume element of $\boldsymbol{P}_{n+p}(\boldsymbol{C})$.*

COROLLARY. *Under the condition of the above theorem,*

Received March 25, 1991.

(1) *if* $S<\frac{n+2}{4-1/p}c$, *then* $\mathcal{F}$ *is totally geodesic,*

(2) *if* $\mathcal{F}$ *is not totally geodesic and if* $S\leqq\frac{n+2}{4-1/p}c$, *then* $S=\frac{n+2}{4-1/p}c$.

## 1. Outline of the proof.

We use the following convention on the range of indecies;

$$A, B, C, \cdots=1, \cdots, 2(n+p),$$

$$i, j, k, \cdots=1, \cdots, 2n,$$

$$\alpha, \beta, \gamma, \cdots=2n+1, \cdots, 2(n+p).$$

Let $\{e_A\}$ be an locally defined orthonormal frame field on $\boldsymbol{P}_{n+p}(\boldsymbol{C})$ such that each $e_i$ is always tangent to $\mathcal{F}$. Then the component $R_{ABCD}$ of the curvature tensor of $\boldsymbol{P}_{n+p}(\boldsymbol{C})$ is given by

$$R_{ABCD}=\frac{c}{4}(\delta_{AD}\delta_{BC}-\delta_{AC}\delta_{BD}+J_{AD}J_{BC}-J_{AC}J_{BD}-2J_{AB}J_{CD}), \tag{1.1}$$

where $J=(J_{AB})$ denotes the complex structure. If we denote by $h_{BC}^A$ the components of $h$, each $h_{\alpha\beta}^i$ vanishes. Since all leaves of $\mathcal{F}$ are complex, we obtain

$$\sum h_{kj}^\alpha J_{ki}=\sum h_{ik}^\alpha J_{kj}=\sum h_{ij}^\beta J_{\alpha\beta}. \tag{1.2}$$

Note that the first equalities imply $\sum h_{ii}^\alpha=0$, that is, all leaves of $\mathcal{F}$ are minimal.

We consider a globally defined vector field $v=\sum v_A e_A$ on $\boldsymbol{P}_{n+p}(\boldsymbol{C})$ defined by

$$v_k=\sum h_{ij}^\alpha h_{ijk}^\alpha, \qquad v_\alpha=0,$$

and calculate its divergencc $\delta v$. Since, by using (1.1) and (1.2), culculation of $\delta v$ is carried out in a similar fashion to that in [4], we write down the result directry;

$$\delta v=\sum h_{ijk}^\alpha h_{ijk}^\alpha+\frac{n+2}{2}cS-\sum N(H^\alpha H^\beta-H^\beta H^\alpha)-\sum(Tr\, H^\alpha H^\beta)^2. \tag{1.3}$$

For the notation $H^\alpha$ and $N$, see [1], [4]. By an estimation

$$\sum N(H^\alpha H^\beta-H^\beta H^\alpha)+\sum(Tr\, H^\alpha H^\beta)^2\leqq\frac{4p-1}{2p}S^2, \tag{1.p}$$

we obtain from (1.3)

$$\begin{aligned}\delta v &\geqq \sum h^{\alpha}_{ijk} h^{\alpha}_{ijk} + \frac{n+2}{2} cS - \frac{4p-1}{2p} S^2 \\ &= \sum h^{\alpha}_{ijk} h^{\alpha}_{ijk} - S\left\{\left(2-\frac{1}{2p}\right)S - \frac{n+2}{2}c\right\},\end{aligned}$$

that is,

$$S\left\{\left(2-\frac{1}{2p}\right)S - \frac{n+2}{2}c\right\} \geqq -\delta v + \sum h^{\alpha}_{ijk} h^{\alpha}_{ijk}. \tag{1.5}$$

Thus integrating the both side of (1.5), we have

$$\int_{P_{n+p}(C)} S\left\{\left(2-\frac{1}{2p}\right)S - \frac{n+2}{2}c\right\} *1 \geqq \int_{P_{n+p}(C)} \sum h^{\alpha}_{ijk} h^{\alpha}_{ijk} *1 \geqq 0. \qquad \text{(q.e.d.)}$$

REMARK. Consider the case where $c=1$, and assume $S=\frac{n+2}{4-1/p}$. Then the minimality of $\mathcal{F}^{\perp}$ implies that $n=p=1$. This is obtained by a similar augument in Chern, do-Carmo and Kobayashi [1] or Ogiue [5].

Moreover if the metric is bundle-like, this cannot be occur (cf. Theorem 3. [4].)

ADDED IN PROOF $\left(\text{Non-existence of the case } S=\frac{n+2}{4-1/p}c\right)$

As is mentioned in the above remark, if $S=\frac{n+2}{4-1/p}c$, then both $n$ and $p$ must equal to 1. However this is impossible because $\boldsymbol{P}_2(\boldsymbol{C})$ can not admit even a plane field ([3]). For the interesting results about the existence of plane fields on 4-manifolds, see [6], [7].

## References


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