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FIBREWISE CONVERGENCE AND EXPONENTIAL LAWS

By

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Abstract. We show that the category $\mathbf{Conv}_B$ of convergence spaces over B is a convenient category for any $B \in \mathbf{Conv}$. It is shown that without any condition on spaces the category $\mathbf{Conv}_B$ and the category $\mathbf{Conv}_B^B$ of sectioned convergence spaces over B hold various exponential laws in a natural way. In $\mathbf{Conv}_B$, we can construct exponential object in terms of function spaces. Our fibrewise mapping space structure generalizes the fibrewise compact-open topology in some case.

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1. Introduction.

The fibrewise viewpoint is standard in the theory of fibre bundles. It also has an important role to play in homotopy theory. Fibrewise topology, as a natural generalization of topology, has emerged recently as a subject in its own right with a rich potential for research. I.M. James has been promoting the fibrewise viewpoint systematically in topology [13-19].

In homotopy theory, the category **Top** of topological spaces is not a very good one to work in for many problems. **Top** is not cartesian closed. So is not the category $\mathbf{Top}_B$ of topological spaces and maps over a fixed space B. So, many attempts have been made to find a suitable category, allowing a convenient category of fibred spaces. A convenient category means that it contains

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all the spaces of real interest, that it have all limits and colimits, and that it be cartesian closed. So far, compactly generated spaces and quasi-topological spaces have been main objectives. However, in a structural point of view, it has not been completely successful to find a convenient category of fibred spaces. P.I. Booth [4] obtained many interesting exponential laws for quasi-topological spaces. However, quasi-topological spaces do not form a category, but a quasi-category, which is illegitimate and hence not a suitable replacement for **Top** (cf. [12]).

In this paper, we will introduce a new approach to fibrewise topology using the notion of convergence [3, 9] and develop a theory of fibrewise convergence, mainly focusing on the adjoints of the fibre product and the fibre smash product, respectively. In 1986, Adámek and Herrlich [1] showed that a topological category $\boldsymbol{A}$ is a quasitopos (=final epi-sinks in $\boldsymbol{A}$ are preserved by pullbacks) if and only if, for each $B\in\boldsymbol{A}$, the comma category $\boldsymbol{A}_B$ is cartesian closed. Thus, to find a convenient category of fibred spaces, we must first choose a quasitopos. It is well-known that the category **Conv** of convergence spaces forms a quasitopos (cf, [1, 21]) and it is very useful category in various respects, containing the category **Top** as a bireflective subcategory (cf. [3, 21]). So, we work with the category of convergence spaces. We will show that the category $\mathbf{Conv}_B$ of convergence spaces over B is a convenient category for every $B\in\mathbf{Conv}$. In fact, it turns out that without any restriction on spaces the category $\mathbf{Conv}_B$ and the category $\mathbf{Conv}_B^B$ of sectioned convergence spaces over B hold various exponential laws including the exponential laws for fibred section spaces and fibred relative lifting spaces and homotopy versions of all exponential laws mentioned above. We note that an exponential object in $\mathbf{Conv}_B$ can be constructed in terms of function spaces even though a constant map in $\mathbf{Conv}_B$ is not a morphism (cf. 27.18, [2]). Our fibrewise mapping space structure generalizes the fibrewise compact-open topology in some cases. Using those exponential laws, we can obtain naturally improved versions of many interesting properties concerned by many researchers [4-8, 14, 18, 20, 22-24]. The terminology and notation of [2, 13] will be used throughout.

2. Convergence spaces over a base.

For a set X, we denote by $\mathcal{F}(X)$ the set of all filters on X and $\mathcal{P}(\mathcal{F}(X))$ the power set of $\mathcal{F}(X)$. A *convergence space* [3] is a pair (X, c) of a set X and a function $c: X\to\mathcal{P}(\mathcal{F}(X))$, called a *convergence structure*, subject to the following axioms: for each $x\in X$,

(1) $\dot{x}\in c(x)$, where $\dot{x}$ is the filter generated by $\{x\}$.

(2) if $\mathcal{F}\in c(x)$ and $\mathcal{F}\subseteq\mathcal{G}$, then $\mathcal{G}\in c(x)$,

(3) if $\mathcal{F}$, $\mathcal{G}\in c(x)$, then $\mathcal{F}\cap\mathcal{G}\in c(x)$.

The filters in $c(x)$ are said to be *convergent to* x. We usually write $\mathcal{F}\to x$ instead of $\mathcal{F}\in c(x)$. By a *continuous map* $f: X\to Y$ between convergence spaces is meant a function $f: X\to Y$ such that $f(\mathcal{F})\to f(x)$ in Y whenever $\mathcal{F}\to x$ in X. The category **Conv** is formed by all convergence spaces and all continuous maps between them.

Let X be a topological spaces. By assigning to each $x\in X$ $c(x)$=the set of all filters on X, convergent to x, we obtain a convergence structure. Hence any topology can be interpreted as a convergence structure. Let (X, c) be a convergence space. A subset U of X is said to be open if it belongs to every filter which converges to a point of U. The collection τ_c of all open subsets of X forms a topology on X. Note that (X, τ_c) is the topological reflection of (X, c).

Given a space $B\in$**Conv**, an object (X, p) of the comma category $\mathbf{Conv}_B$ is called a *convergence space over* B and p the projection. As usual, (X, p) is also simply denoted by X. A morphism $f:(X, p)\to(Y, q)$ in $\mathbf{Conv}_B$ is called a *continuous map over* B. For topological space B, each $((X, c), p)\in\mathbf{Conv}_B$ has the topological reflection $((X, \tau_c), p)$. Hence $\mathbf{Top}_B$ is a bireflective subcategory of $\mathbf{Conv}_B$.

It is easy to see the following facts; Initial (resp. final) structures in **Conv** determine initial (resp. final) structures in $\mathbf{Conv}_B$ over $\mathbf{Set}_B$. The limit (resp. colimit) in **Conv** of a natural source (resp. sink) in $\mathbf{Conv}_B$ is the limit (resp. colimit) in $\mathbf{Conv}_B$. Therefore, $\mathbf{Conv}_B$ has initial structures over $\mathbf{Set}_B$ and, hence, limits and colimits. Moreover, as does in **Conv** final epi-sinks in $\mathbf{Conv}_B$ are preserved by pullbacks and hence finite products of quotient maps are quotient in $\mathbf{Conv}_B$. From now on, B means any convergence space.

Note that, for (X, p), $(Y, q)\in\mathbf{Conv}_B$, the pull-back $X\times_B Y$ of p and q is the product of X and Y in $\mathbf{Conv}_B$. Since $\mathbf{Conv}_B$ is cartesian closed, the functor $X\times_B\text{-}$. $\mathbf{Conv}_B\to\mathbf{Conv}_B$ has a right adjoint -^X, an exponential functor. An exponential object Y^X is not necessarily a function space. However, in $\mathbf{Conv}_B$, we can construct exponential object in terms of function spaces.

For (X, p), $(Y, q)\in\mathbf{Conv}_B$, consider the set

$$map_B(X, Y)=\bigcup_{b\in B} map(X_b, Y_b)$$

with the natural projection (pq), where $map(X_b, Y_b)$ denotes the set of continuous maps of X_b into Y_b and we define a convergence structure c on $map_B(X, Y)$ as follows; For a filter $\mathcal{F}$ on $map_B(X, Y)$ and $f\in map(X_b, Y_b)$, $\mathcal{F}\in c(f)$ if and only if

(1) for each $x\in X_b$, $(\mathcal{F}\cap\dot{f})(\mathcal{A}\cap\dot{x})\to f(x)$ in Y whenever $\mathcal{A}\to x$ in X, where $F(A)=\bigcup_{b\in B}F_b(A_b)$ for $F\in\mathcal{F}\cap\dot{f}$ and $A\in\mathcal{A}\cap\dot{x}$,

(2) $(pq)(\mathcal{F})\to(pq)(f)$ in B.

By a routine work, we can show that c is a convergence structure over B. We note that if B is a singleton space, then c is the continuous convergence structure on $map(X, Y)$.

THEOREM 2.1. *For any convergence space X over B, $map_B(X, \_)$ is a right adjoint of $X\times_B\_$. Therefore the exponential object Y^X is isomorphic to $map_B(X, Y)$ in* $\mathbf{Conv}_B$.

PROOF. Consider the evaluation map $ev: X\times_B map_B(X, Y)\to Y$ defined by $ev(x, f)=f(x)$. Then ev is a map over B. For the continuity of ev, let $\mathcal{U}\to(x, f)$ in $X\times_B map_B(X, Y)$ with $(x, f)\in X_b\times map(X_b, Y_b)$. Then there exist filters $\mathcal{A}$ on X and $\mathcal{F}$ on $map_B(X, Y)$ such that $\mathcal{A}\to x$ in X and $\mathcal{F}\to f$ in $map_B(X, Y)$ and $\mathcal{A}\times_B\mathcal{F}\subseteq\mathcal{U}$, where $\mathcal{A}\times_B\mathcal{F}$ is the filter generated by $\{A\times_B F\mid A\in\mathcal{A}, F\in\mathcal{F}\}$. Note that $ev(A\times_B F)=F(A)$. Hence $(\mathcal{F}\cap\dot{f})(\mathcal{A}\cap\dot{x})\subseteq ev(\mathcal{A}\times_B\mathcal{F})\subseteq ev(\mathcal{U})$. Therefore ev is continuous. In fact, ev is a co-universal map for Y with respect to the functor $X\times_B\_$. Let $(Z, r)\in\mathbf{Conv}_B$ and $f: X\times_B Z\to Y$ a continuous map over B. Define $\bar{f}: Z\to map_B(X, Y)$ by $\bar{f}(r)(x)=f(x, z)$. (If $X_b=\emptyset$, then $\bar{f}(z)$ is the empty map $\emptyset_b: X_b\to Y_b$.) Then $\bar{f}$ is a map over B. Let $\mathcal{H}\to z$ in Z with $z\in Z_b$ and $\mathcal{A}\to x$ in X with $x\in X_b$. Then

$$f((\mathcal{A}\cap\dot{x})\times_B(\mathcal{H}\cap\dot{z}))\subseteq(\bar{f}(\mathcal{H})\cap\dot{\bar{f}}(z))(\mathcal{A}\cap\dot{x})$$

and $(pq)\circ\bar{f}=r$. Hence $\bar{f}(\mathcal{H})\to\bar{f}(z)$ in $map_B(X, Y)$. Thus $\bar{f}$ is continuous. Clearly, $ev\circ(1_X\times_B\bar{f})=f$ and such a map $\bar{f}$ is unique.

Since $\mathbf{Conv}_B$ is cartesian closed, we have the following exponential law as a corollary.

THEOREM 2.2. *For $X, Y, Z\in\mathbf{Conv}_B$,*

$$\Psi: map_B(X\times_B Y, Z)\longrightarrow map_B(X, map_B(Y, Z))$$

is an isomorphism in $\mathbf{Conv}_B$, where $\Psi(f)(x)(y)=f(x, y)$.

For $X, Y\in\mathbf{Conv}_B$, we denote by $Map_B(X, Y)$ the convergence space of all continuous maps $X\to Y$ over B, equipped with a subspace structure of $map(X, Y)$ in $\mathbf{Conv}$.

LEMMA 2.3. *For (X, p), $(X, q)\in\mathbf{Conv}_B$, consider $X\times B$ and $X\times Y$ as objects* *in* $\mathbf{Conv}_B$ *with projections* π_2 *and* $q\circ\pi_2$ *respectively. Then* $\alpha : (X\times B)\times_B Y\to X\times Y$ *is an isomorphism in* $\mathbf{Conv}_B$, *where* $\alpha((x, b), y)=(x, y)$.

PROOF. It is immediate from the property of products in **Conv**.

PROPOSITION 2.4. *For* $X, Y\in\mathbf{Conv}_B$,

$$\sigma : Map_B(X, Y) \longrightarrow Map_B(B, map_B(X, Y))$$

is an isomorphism in $\mathbf{Conv}_B$,*where* $\sigma(f)(b)=f_b : X_b\to Y_b$, *the restriction of* f *on* X_b.

PROOF. Using the cartesian closedness of **Conv**, Theorem 2.1. and Lemma 2.3., the following commutative diagram;

$$\begin{tikzcd}
X\times_B map_B(X, Y) \arrow[r, "ev"] & Y \\
X\times_B(B\times Map_B((X, Y)) \arrow[u, "1\times_B ev"'] \arrow[r, phantom, "\cong"] & X\times Map_B(X, Y) \arrow[u, "ev"']
\end{tikzcd}$$

Since $X\times\_$ is a left adjoint of $map(X, \_)$, we have a continuous map $\widehat{ev} : Map_B(X, Y)\to map(B, map_B(X, Y))$ such that $ev\circ(1_X\times\widehat{ev})=\overline{ev}$. In fact, σ is the corestriction of $\widehat{ev}$. Consider the following diagram;

$$\begin{tikzcd}
X\times map(X, Y) \arrow[rr, "ev"] & & Y \\
X\times Map_B(B, map_B((X, Y)) \arrow[u] \arrow[d, phantom, "\wr"] & & \\
X\times_B(B\times Map_B(B, map_B(X, Y))) \arrow[rr, "1_X\times_B ev"] & & X\times_B map_B(X, Y) \arrow[uu, "ev"']
\end{tikzcd}$$

Again, by the exponential law in **Conv**, we have a continuous map $\beta : Map_B(B, map_B(X, Y))\to map(X, Y)$ such that $ev\circ(1_X\times_B ev)=ev\circ\beta$. In fact, σ^{-1} is the corestriction of β.

REMARK 2.5. The space $Map_B(B, map_B(X, Y))$ is the space of sections of $map_B(X, Y)$. So, usually it is denoted by $sec_B\, map_B(X, Y)$. This proposition shows that exponential objects in $\mathbf{Conv}_B$ may not be hom-objects in that category.

By combining Theorem 2.2. and Proposition 2.4., we have another exponential law.

THEOREM 2.6. *For* $X, Y, Z\in\mathbf{Conv}_B$,

$$\Phi : Map_B(X \times_B Y,\ Z) \longrightarrow Map_B(X,\ map_B(Y,\ Z))$$

is an isomorphism in $\mathbf{COnv}_B$, *where* $\Phi(f)(x)(y)=f(x,\ y)$.

PROOF. $Map_B(X \times_B Y,\ Z) \cong Map_B(B,\ map_B(X \times_B Y,\ Z))$
$\cong Map_B(B,\ map_B(X,\ map_B(Y,\ Z))) \cong Map_B(X,\ map_B(Y,\ Z))$.

REMARK 2.7. Since, in $\mathbf{Conv}_B$, $B \times_B X \cong X$ in a natural way, Theorem 2.6. implies Proposition 2.4. Therefore given the isomorphism in Theorem 2.2., Proposition 2.4. and Theorem 2.6. are equivalent.

Mapping Spaces

We collect some interesting properties of mapping spaces in $\mathbf{Conv}_B$.

1. Since $\mathbf{Conv}_B$ is cartesian closed, we can show the followings (cf. [2]); (a) $X \times_B -$ preserves final epi-sinks, (b) $map_B(X,\ -)$ preserves initial sources and (c) $map_B(-,\ X)$ carries final epi-sinks to initial sources. In particular. $X \times_B (\coprod_B Y_i) \cong \coprod_B (X \times_B Y_i)$, $map_B(X, \prod_B Y_i) \cong \prod_B map_B(X,\ Y_i)$ and $map_B(\coprod_B Y_i,\ X) \cong \prod_B map_B(Y_i,\ X)$.

2. Given $(X,\ p)$, $(Y,\ q) \in \mathbf{Conv}_B$, if q is quotient or $Map_B(X, Y) \neq \emptyset$, then $(pq): map_B(X,\ Y) \to B$ is quotient (cf. Theorem 5.1. in [4]). If we take $Y=B$, then this shows $map_B(X,\ B) \cong B$. Let α be the compositon $map_B(B,\ X) \xrightarrow{((1_B p),1)} B \times_B map_B(B,\ X) \xrightarrow{ev} X$. Then α is bijective and the adjoint of $\pi_2 : B \times_B X \to X$ is α^{-1}. Hence $map_B(B,\ X) \cong X$.

3. For each $B' \in \mathbf{Conv}$ and a continuous map $\xi : B' \to B$, a functor $\xi : \mathbf{Conv}_B \to \mathbf{Conv}_{B'}$ is defined, where $\xi^* X = X \times_B B'$ and $\xi^*(f) = f \times_B 1_{B'}$. In fact, ξ^* has a left adjoint functor ξ_*, defined by $\xi_*(X,\ p)=(X,\ \xi \circ p)$ and $\xi_*(f)=f$, and hence preserves products. By Theorem 1.1. and modification of proof of Proposition 6.9. in [14], we can show that the natural map $\xi_\# : map_{B'}(\xi^* X, \xi^* Y) \to \xi^* map_B(X, Y)$ is an isomorphism in $\mathbf{Conv}_{B'}$.

4. For $(Z,\ r) \in \mathbf{Conv}_B$ and a non-empty space F, define $\mathcal{O}_F(Z)$ to be the subspace of $map(F,\ Z)$ of maps $f : F \to Z$ such that $r \circ f$ is constant. Then, $\mathcal{O}_F(Z) \in \mathbf{Conv}_B$ with the projection $q_F(r)(f) = rf(x)$ and we have an isomorphism in $\mathbf{Conv}_B$ $\alpha : \mathcal{O}_F(Z) \to map_B(F \times B,\ Z)$, where $\alpha(f)(x,\ b) = f(x)$, using Lemma 2.3. and exponential laws in $\mathbf{Conv}$ and $\mathbf{Conv}_B$. Hence $map_B(F \times B,\ Z)$ is embedded in $map(F,\ Z)$ (cf. Proposition 3.1. [7]).

5. Using the similar argument in Theorem 6.1. of [4], we can show the following; Given $(X,\ p)$, $(Y,\ q) \in \mathbf{Conv}_B$, if p and q are Hurewicz (resp. Dold) fibrations, then so is (pq).

6. Let B be a discrete topological space, X a locally compact Hausdorff topological space over B and Y a topological space over B. Then $map_B(X,\ Y)$

carries the fibrewise compact-open topology: Suppose $\mathcal{F}\to f$ in $map_B(X, Y)$ with $f\in map(X_b, Y_b)$. Since B is discrete, $(pq)(\mathcal{F})\to(pq)(f)=b$ in B implies $map(X_b, Y_b)\in\mathcal{F}$. Let (K, V) be a fibrewise compact-open neighborhood of f. Then, for each $x\in K_b$, $V\in(\mathcal{F}\cap\dot{f})(\mathcal{N}_x)$, where $\mathcal{N}_x$ is the neighborhood filter at x in X. Since K_b is compact, there exist $x_1, \cdots, x_n\in X_b$, $U_{x_i}\in\mathcal{N}_{x_i}$ and $F_{x_i}\in\mathcal{F}\cap\dot{f}$ such that $F_{x_i}(U_i)\subseteq V$ for each $i=1, \cdots, u$. Let $F=F_{x_1}\cap\cdots\cap F_{x_n}\cap map(X_b, Y_b)$. Then $F\in\mathcal{F}\cap\dot{f}$ and$F\subseteq(K, V)$. Hence $\mathcal{F}\to f$ with respect to the fibrewise compact-open topology. Conversely, let $\mathcal{N}_f$ be the neighborhood filter at f with respect to the fibrewise compact-open topology, where $f\in map(X_b, Y_b)$. Let V be an open neighborhood of $f(x)$ in Y. Since X is locally compact over B, there is a compact neighborhood K of x in X_b such that $f(X)\subseteq V_b$. Note that K is compact over B, since B is T_1. In fact, $(K, V)\in\mathcal{N}_f$ and $(K, V)(K)\subseteq V$. Hence $\mathcal{N}_f\to f$ in $map_B(X, Y)$. In general, $map_B(X, Y)$ does not carry the fibrewise compact-open topology. For example, let $X=Y=B=\{0, 1\}$, the Sierpinski space with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$ and the identity map as its projection. Consider the filter $\mathcal{F}=\{\{\underline{0}, \underline{1}\}\}$, where $\underline{0}: \{0\}\to\{0\}$ and $\underline{1}: \{1\}\to\{1\}$. Then $\mathcal{F}\to\underline{1}$ in $map_B(X, Y)$, but $\mathcal{F}\not\to\underline{1}$ with respect to the fibrewise compact-open topology. Note that $\{\underline{1}\}=(\{1\}, Y)$ is the fibrewise compact-open neighborhood of $\underline{1}$.

3. Sectioned space over a base.

A *sectioned space over* B is a triple consisting of a convergence space X and continuous maps

$$B \xrightarrow{s} X \xrightarrow{p} B$$

such that $ps=1_B$. Usually X alone is a sufficient notation. The map p is called a projection and the map s the section. Let X, Y be sectioned space over B, with projections p, q and sections s, t, respectively. By a *map of sectioned space over* B, we mean a continuous map $f: X\to Y$ of convergence spaces such that $qf=p$ and $fs=t$. The category $\mathbf{Conv}_B^B$ is formed by all sectioned spaces over B and all maps between them. By a similar argument in $\mathbf{Conv}_B$, the category $\mathbf{Top}_B^B$ is shown to be a bireflective subcategory of $\mathbf{Conv}_B^B$. Note that products of sectioned spaces in $\mathbf{Conv}_B$ serve as products in $\mathbf{Conv}_B^B$.

Let X, Y be sectioned spaces over B, with projections p, q and sections s, t, respectively. Consider the convergence space

$$X\wedge_B Y=\bigcup_{b\in B}\{(X_b\times Y_b)/((s(b)\times Y_b)\cup(X_b\times t(b))\}$$

equipped with the quotient structure with respect to the natural map $\phi: X\times_B Y\to X\wedge_B Y$. Then the triple $(\phi\circ(s, t), X\wedge_B Y, p\wedge q)$ is a sectioned space over B, called the smash product of X and Y, where the map $p\wedge q$ is induced by the projection for $X\times_B Y$. We note that the smash product is not the product in the category $\mathbf{Conv}_B^B$. We denote by $map_B^B(X, Y)$ the subspace of $map_B(X, Y)$ of pointed maps, where the base points in the fibres are determined by sections. The space $map_B^B(X, Y)$ is a sectioned space over B: In fact, the projection is induced by (pq) and the section is induced by the adjoint of $X\times_B B\xrightarrow{\pi_2} B\xrightarrow{t} Y$. For any $X\in\mathbf{Conv}_B^B$, $map_B^B(B, X)\cong B$ via its projection. Denote $\dot{I}=\{0\}\amalg\{1\}$. Then $map_B^B(B\times\dot{I}, X)\cong X$, since $map_B(\_, X)$ carries coproducts into products.

THEOREM 3.1. *For any sectioned space X over B, $map_B^B(X, \_)$ is a right adjoint of $X\wedge_B\_$.*

PROOF. Let $Y\in\mathbf{Conv}_B^B$. Consider the map $e: X\wedge_B map_B^B(X, Y)\to Y$ defined by $e([x, f])=f(x)$, where $[x, f]=\phi(x, f)$. Then $e\circ\phi=ev$ implies that e is a morphism in $\mathbf{Conv}_B^B$. In fact, e is a co-universal map for Y with respect to the functor $X\wedge_B\_$. Given $Z\in\mathbf{Conv}_B^B$ and a morphism $f: X\wedge_B Z\to Y$ in $\mathbf{Conv}_B^B$, define $\bar{f}: Z\to map_B^B(X, Y)$ by $\bar{f}(z)(x)=f([x, z])$. Then, using Theorem 2.1., it is easy to see that $\bar{f}$ is a unique morphism in $\mathbf{Conv}_B^B$ such that $e\circ(1_X\wedge_B\bar{f})=f$, since $map_B^B(X, Y)$ is a subspace of $map_B(X, Y)$.

THEOREM 3.2. *For $X, Y, Z\in\mathbf{Conv}_B^B$,*

$$\phi: map_B^B(X\times_B Y, Z)\longrightarrow map_B^B(X, map_B^B(Y, Z))$$

is an isomorphism in $\mathbf{Conv}_B^B$, where $\phi(f)(x)(y)=f([x, y])$.

PROOF. Clearly, ϕ is bijective. Using Theorem 3.1. and a parallel method in Theorem 2.2., we can show that ϕ is an isomorphism in $\mathbf{Conv}_B^B$. We note that the smash product $\wedge_B$ is commutative and associative.

REMARK 3.3. If B is a singleton space $*$, then this theorem gives an exponential law of pointed convergence spaces. This type of exponential law plays a central role on duality in homotopy theory (cf. [10, 11, 23]).

For $X, Y\in\mathbf{Conv}_B^B$, we denote by $Map_B^B(X, Y)$ the convergence space of all morphisms $X\to Y$ in $\mathbf{Conv}_B^B$, equipped with a subspace structure of $map(X, Y)$. Clearly, $Map_B^B(X, Y)$ is a subspace of $Map_B(X, Y)$ in $\mathbf{Conv}_B$.

PROPOSITION 3.4. *For $X, Y\in\mathbf{Conv}_B^B$,*

$$\sigma: Map_B^B(X, Y)\longrightarrow Map_B^B(B, map_B^B(X, Y))$$

is an isomorphism in $\mathbf{Conv}_B^B$, where $\sigma(f)(b)=f_b: X_b\to Y_b$, the restriction of f on X_b.

PROOF. Clearly, σ is bijective. We note that $Map_B^B(B, map_B^B(X, Y)) = Map_B(B, map_B^B(X, Y))$ and our σ is the restriction of σ in Proposition 2.4. Since $map(B, \_)$ preserves initial sources, the result follows immediately.

By combining Theorem 3.2. and Proposition 3.4., we have another exponential law;

THEOREM 3.5. *For* $X, Y, Z \in \mathbf{Conv}_B^B$,

$$\varphi : Map_B^B(X \wedge_B Y, Z) \longrightarrow Map_B^B(X, map_B^B(Y, Z))$$

is an isomorphism in $\mathbf{Conv}_B^B$, *where* $\varphi(f)(x)(y) = f(x, y)$.

REMARK 3.6. Using our exponential laws and modifying the proof in [7], we can obtain in our context the exponential laws for fibred section space, and fibred relative lifting spaces and homotopy versions of all exponential laws mentioned above without any restriction on spaces.

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