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NON-UNIQUE SOLUTIONS TO THE PLATEAU PROBLEM ON SYMMETRIC SPACES

By

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Abstract. In this paper, we will show that in each of the following compact symmetric spaces $SU(3)$, $SU(3)/SO(3)$, $SU(6)/Sp(3)$ and E_6/F_4, there are exactly 3 solutions to the minimal surface equations with a given prescribed boundary. The proof uses only elementary techniques in the calculus of variations.

I. Introduction

Symmetric spaces are among the most extensively studied objects in differential geometry. The inherent symmetry of a symmetric space makes it possible to solve certain geometric problems which otherwise would be intractable on a general manifold lacking such symmetries. One such problem is the search for solutions of the minimal surface equation and related questions on the uniqueness of solutions. The problem of uniqueness is a particularly difficult one, since there are no known necessary and sufficient conditions on a given boundary to guarantee that a solution surface is the only solution, not even in ordinary 3-space. It is therefore of immense value to have numerous examples of non-unique solutions (see [2] p. 109), especially in higher dimensions and non-Euclidean spaces.

The purpose of this paper is to exhibit a codimension-two boundary in compact symmetric spaces of rank two and the 3 minimizing surfaces which solves the Plateau problem with the given boundary. The technique used here is the so-called reduction of variables method [4] to reduce the problem to a corresponding ODE problem in a flat 2-dimensional space.

2. Symmetric Spaces and their Orbit Structures

The following symmetric spaces G/K will be considered:

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$$SU(3),\ SU(3)/SO(3),\ SU(6)/Sp(3) \text{ and } E_6/F_4 .$$

In each of the above cases, the group K acts on G/K via the adjoint action, and a fundamental domain for this action can be identified with a Weyl chamber W of the Cartan subspace of the Lie algebra of G. For the above symmetric spaces, W is simply a 2-dimensional equilateral triangle. The volume function which records the volume of each of the principal orbits is given by

$$v(x)=c(\sin d_1(x)\sin d_2(x)\sin d_3(x))^k$$

(see [3]) where c is a constant (which we will take to be 1), $d_i(x)$ are the distance functions of the point x to the edges of the triangle, and $k=1, 2, 4, 8$ depending on the symmetric space.

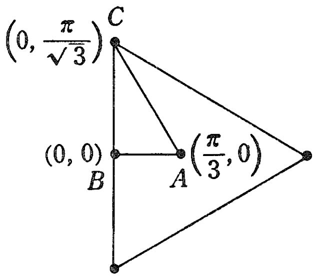

Fig. 1.

3. The Non-uniqueness Results

For our boundary, we select the principal orbit of the centroid of the above equilateral triangle W. We will show that an area-minimizing codimension-one surface having this principal orbit as boundary is one of the three orbits of the perpendicular drop from the centroid to a side of the triangle.

We first observe that the boundary is invariant under the adjoint K-action. Therefore, we need only consider the one-dimensional section of surfaces in W. The volume integral of a surface in G/K intersecting the section W at a curve $y=y(x)$, $x_0 \leqq x \leqq x_1$, is easily seen to be

$$\int_{x_0}^{x_1} v(x, y)\sqrt{1+y'^2}\, dx ,$$

where $v(x, y)=\sin x(\cos x+\cos\sqrt{3}y)^k$, and the corresponding Euler equation satisfied by the geodesics is,

$$(*)\ \sin x(\cos x+\cos\sqrt{3}y)y''=-k(1+y'^2)\{\sqrt{3}\sin x\sin\sqrt{3}y + y'(\cos 2x+\cos x\cos\sqrt{3}y)\}.$$

Notice that due to the symmetry of the volume function in W, it suffice to study the Euler equation on the smaller triangle ABC.

It is easily verified that in the extreme cases, the line segments AB and AC are geodesics. We now show that there are no other geodesic solutions starting from the point A and ending on the segment BC.

From the Euler equation, such a geodesic must intersect BC perpendicularly. For small x, let $y(x)$ denote such a geodesic, with $y(0)=b>0$, $y(\pi/3)=0$, and $y'(0)=0$. It is known that for $0<b<\pi/\sqrt{3}$, $y(x)$ is a graph for $0\leqq x<\pi/3$ [3]. We claim that such a geodesic cannot exist.

LEMMA 1. *$y'(x)$ is negative and decreasing for $0<x<\pi/3$.*

PROOF. The fact that $y'(x)$ is negative was already established in [4]. What is new here is that the function $y'(x)$ is decreasing. To prove this, we show that y'' is negative on the given interval interval $0<x<\pi/3$. Clearly, if $\cos 2x+\cos x\cos\sqrt{3}y\leqq 0$, then (*) shows that $y''<0$. Hence we need only consider the case where $\cos 2x+\cos y\cos\sqrt{3}y>0$. Suppose there is a point $x_1<\pi/3$ such that $y''(x_1)=0$ and $y''<0$ for $0<x<x_1$. We compute $y'''(x_1)$ and obtain

$$\frac{-\sin x_1(\cos x_1+\cos\sqrt{3}y_1)}{k(1+y'(x_1)^2)}y'''(x_1)=\{(1-y'(x_1)^2)\sqrt{3}\cos x_1\sin\sqrt{3}y_1$$
$$+y'(x_1)(2\sin x_1\cos\sqrt{3}y_1-2\sin 2x_1)\}.$$

From (*), we have

$$(**)\quad y'(x_1)=\frac{-\sqrt{3}\sin x_1\sin\sqrt{3}y_1}{\cos 2x_1+\cos x_1\cos\sqrt{3}y_1}.$$

Substituting this expression for $y'(x_1)$ into the right hand side of the y''' equation and multiplying the result by $(\cos 2x_1+\cos x_1\cos\sqrt{3}y_1)^2/(\sqrt{3}\sin\sqrt{3}y_1)$, we obtain:

$$\{(\cos 2x_1+\cos x_1\cos\sqrt{3}y_1)^2-3\sin^2 x_1\sin^2\sqrt{3}y_1\}\cos x_1$$
$$+2\sin x_1(\sin 2x_1-\sin x_1\cos\sqrt{3}y_1)(\cos 2x_1+\cos x_1\cos\sqrt{3}y_1).$$

Let us call the above expression $A(x_1, y_1)$. Claim: $\cos\sqrt{3}y_1>-\cos 3x_1$. This follows from the fact that in the triangle ABC, $y_1<-\sqrt{3}x_1+\pi/\sqrt{3}$. The strict inequality is due to the fact that if equality occurs, then (**) implies that $y'(x_1)=-\sqrt{3}$, and the geodesic is the line AC itself (Picard's thm.). Consequently, we also have $\sin\sqrt{3}y_1>\sin 3x_1$. We then expand the expression $A(x_1, y_1)$ and substitute the inequalities to arrive at

$$A(x_1, y_1)=\cos x_1\{(\cos 2x_1+\cos x_1 \cos\sqrt{3}y_1)^2-\sin^2 x_1 \sin^2\sqrt{3}y_1\}$$
$$+2\sin^2 x_1\{2\cos x_1 \cos 2x_1+\cos\sqrt{3}y_1-\cos x_1\}$$
$$>\cos x_1\{(\cos 2x_1-\cos x_1 \cos 3x_1)^2-\sin^2 x_1 \sin^2\sqrt{3}y_1\}$$
$$+2\sin^2 x_1\{2\cos x_1 \cos 2x_1-\cos 3x_1-\cos x_1\}>0\,.$$

Thus $y'''(x_1)<0$, which contradicts the non-negative slope of y'' at x_1. Lemma 1 is thus established.

With the help of the above lemma, we can now prove the main result of this paper.

THEOREM 2. *Let B be the codimension-two boundary surface on the above symmetric spaces G/K obtained as the principal orbit of the centroid of a Weyl chamber. Then there are exactly 3 volume-minimizing codimension-one surfaces in G/K having B as boundary.*

PROOF. Let $y(x)$ be a geodesic in W corresponding to a section of a smooth minimal surface in G/K with boundary B. Since $y(x)$ intersects the centroid, $y(\pi/3)=0$. We also know that $y(0)=b$, $0\leqq b<\pi/\sqrt{3}$, and $y'(0)=0$. Claim: $b=0$. Suppose not, i.e. $\pi/\sqrt{3}>b>0$ for some geodesic solution $y(x)$. Straightforward computations show that $y''(\pi/3)=y'''(\pi/3)=0$. Also, we have $|y'(\pi/3)|<\sqrt{3}$ (otherwise, Picard's thm. would imply $y(\pi)$ is the line AC). Hence a simple computation shows that

$$y''''(x/3)>0\,.$$

It follows that since $y(x)$ is real-analytic ([3] p. 587), its derivative y' must have a local maximum (coming from the left) at $x=\pi/3$, contradicting Lemma 1.

Thus, the only geodesic solutions on W of the above ODE (*) passing thru the centroid and meeting the sides of W are the 3 perpendicular bisectors from a vertex to the opposite sides of the triangle. By direct computation, AB is shorter than AC, as can be seen from the following table.

1	AB	AC
1	.875	1.125
2	.898	1.065
4	1.145	1.285
8	2.394	2.592

This proves that each of the 3 perpendicular drop from the centroid to a side of the triangle corresponds to a volume-minimizing solution to our original Plateau problem on G/K.

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References

[1] B. N. Cheng, Area-minimizing cone-type surfaces and coflat calibrations, Indiana Journal of Math., Vol. 37, No. 3 (1988), 1-29.

[2] S. Hildebrandt and A. Tromba, Mathematics and optimal from, Scientific American Book Inc., 1985.

[3] W. T. Hsiang and W. Y. Hsiang, On the existence of codimension-one minimal spheres in compact symmetric spaces of rank 2, II, J. of Diff. Geometry, 17 (1982), 583-594.

[4] W. Y. Hsiang and H. B. Lawson Jr., Minimal submanifolds of low cohomogeneity, J. of Diff. Geometry, **5** (1971), 1-38.

[5] R. S. Palais and C. L. Terng, A general theory of canonical form, AMS Trans. Vol. 300, No. 2 (1987), 771-789.

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