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ON TYPE NUMBER OF REAL HYPERSURFACES IN $P_n(C)$

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Introduction.

Let $P_n(\boldsymbol{C})$ denote an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $4c$. Real hypersurfaces in $P_n(\boldsymbol{C})$ have been studied by many differential geometers (See [2], [3], [4], [5] and [6]).

In particular, as for a problem with respect to the typec number t, i. e., the rank of the second fundamental form of real hypersurfaces M in $P_n(\boldsymbol{C})$, R. Takagi showed in [6] that there is a point p on M such that $t(p)\geqq 2$, and M. Kimura and S. Maeda [4] gave an example of real hypersurfaces in $P_n(\boldsymbol{C})$ satisfying $t=2$, which is non-complete. In this paper we shall prove

THEOREM 1. *Let M be a complete real hypersurface in $P_n(\boldsymbol{C})$ $(n\geqq 3)$. Then there exists a point p on M such that $t(p)\geqq 3$.*

REMAPK. It is known that a certain geodesic hypersphere in $P_2(\boldsymbol{C})$ has a property $t=2$ (cf. [2], [7]). Thus the assumption $n\geqq 3$ in Theorem 1 can not be removed.

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1. Preliminaries.

Let M be a real hypersurface in $P_n(\boldsymbol{C})$ $(n\geqq 2)$. Let $\{e_1, \cdots, e_{2n}\}$ be a local field of orthonormal frame in $P_n(\boldsymbol{C})$ such that, restricted to M, $e_1, \cdots, e_{2n-1}$ are tangent to M. Denote its dual frame field by $\theta_1, \cdots, \theta_{2n}$. We use the following convention on the range of indices unless otherwise stated; $A, B, \cdots, =1, \cdots, 2n$ and $i, j, \cdots, =1, \cdots, 2n-1$.

The connection forms θ_{AB} are defined as the 1-forms satisfying

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$$d\theta_A=-\sum\theta_{AB}\wedge\theta_B, \qquad \theta_{AB}+\theta_{BA}=0. \tag{1.1}$$

Restrict the forms under consideration to M. Then, we set $\theta_{2n}=0$ and the forms $\theta_{2n,i}$ can be written as

$$\phi_i\equiv\theta_{2n,i}=\sum h_{ij}\theta_j, \qquad h_{ij}=h_{ji}. \tag{1.2}$$

The quadratic form $\sum h_{ij}\theta_i\otimes\theta_j$ is called the *second fundamental form* of M with direction of e_{2n}. The curvature forms Θ_{ij} of M are defined by

$$\Theta_{ij}=d\theta_{ij}+\sum\theta_{ik}\wedge\theta_{kj}. \tag{1.3}$$

We denote by J the complex structure of $P_n(\boldsymbol{C})$, and put

$$Je_i=\sum J_{ji}e_j+f_ie_{2n}.$$

Then the almost contact structure (J_{ij}, f_k) satisfies

$$\begin{cases} \sum J_{ik}J_{kj}=f_if_j-\delta_{ij}, & \sum f_jJ_{ji}=0 \\ \sum f_i^2=1, & J_{ij}+J_{ji}=0. \end{cases} \tag{1.4}$$

$$\begin{cases} dJ_{ij}=\sum(J_{ik}\theta_{kj}-J_{jk}\theta_{ki})-f_i\phi_j+f_j\phi_i, \\ df_i=\sum(f_j\theta_{ji}-J_{ji}\phi_j). \end{cases} \tag{1.5}$$

The equations of Gauss and Codazzi are given by

$$\Theta_{ij}=\phi_i\wedge\phi_j+c\theta_i\wedge\theta_j+c\sum(J_{ik}J_{jl}+J_{ij}J_{kl})\theta_k\wedge\theta_l, \tag{1.6}$$

$$d\phi_i=-\sum\phi_j\wedge\theta_{ji}+c\sum(f_iJ_{jk}+f_jJ_{ik})\theta_j\wedge\theta_k, \tag{1.7}$$

respectively.

2. Lemmas.

Let M be a real hypersurface in $P_n(\boldsymbol{C})$. We choose an arbitary point p in M, and use the following convention on the range of indices; $a, b, \cdots, =1, \cdots, t(p)$ and $r, s, \cdots, =t(p)+1, \cdots, 2n-1$. Then we can take a field $\{e_1, \cdots, e_{2n}\}$ of orthonormal frame on a neighborhood of p in such a way that the 1-forms ϕ_i can be written as

$$\begin{cases} \phi_a=\sum h_{ba}\theta_b, & h_{ab}=h_{ba}, \\ \phi_r=0, \end{cases} \tag{2.1}$$

at p. We call such a field $\{e_1, \cdots, e_{2n}\}$ to be associated with a point p.

Under this notation we have

LEMMA 2.1. *Assume that $J_{rs}(p)=0$ at a point p on M. Then $t(p)\geqq n-1$. Furthermore, the equality holds if and only if $f_a=0$ and $J_{ab}=0$ at p.*

PROOF. By (1.4) we have

$$\sum_b J_{ab}^2+\sum_r J_{ar}^2+f_a^2=1, \tag{2.2}$$

$$\sum_a J_{ra}^2+f_r^2=1. \tag{2.3}$$

Summing up (2.2) on a, and (2.3) on r, we have

$$\sum_{a,b} J_{ab}^2+\sum_{a,r} J_{ar}^2+\sum_a f_a^2=t(p), \tag{2.4}$$

$$\sum_{a,r} J_{ar}^2+\sum_r f_r^2=2n-1-t(p). \tag{2.5}$$

Substituting (2.5) into (2.4) and making use of $\sum_a f_a^2+\sum_r f_r^2=1$, we have

$$\sum_{a,b} J_{ab}^2+2\sum_a f_a^2=2(t(p)-(n-1))\geqq 0,$$

and so our assertion follows.

This concludes the proof.

Now we consider a point p where the type number t attains the maximal value, say T. Then there is a neighborhood U of p, on which the function t is constant and the equation (2.1) holds.

Put $\theta_{ar}=\sum A_{arb}\theta_b+\sum B_{ars}\theta_s$. Then, taking the exterior derivative of $\phi_r=0$ and using (1.7), we have

$$\sum h_{ab}\theta_b\wedge\theta_{ar}-c\sum(f_r J_{ij}+f_i J_{rj})\theta_i\wedge\theta_j=0,$$

from which we have

$$\sum(h_{ac}A_{crb}-h_{bc}A_{cra})-cf_a J_{rb}+cf_b J_{ra}-2cf_r J_{ab}=0, \tag{2.6}$$

$$\sum h_{ab}B_{brs}-cf_a J_{rs}+cf_s J_{ra}-2cf_r J_{as}=0, \tag{2.7}$$

$$f_s J_{rt}-f_t J_{rs}+2f_r J_{st}=0. \tag{2.8}$$

It is easy to see that (2.8) is reduced to

$$f_r J_{st}=0. \tag{2.9}$$

Similarly, taking the exterior derivative of $\phi_a=\sum h_{ba}\theta_b$ and using the equation (1.7) of Codazzi, we have

$$\begin{aligned}\sum_b\{&dh_{ab}-\sum_c(h_{ac}\theta_{cb}+h_{cb}\theta_{ca}-\sum_r h_{ac}A_{crb}\theta_r-cf_b J_{ac}\theta_c\\&+cf_c J_{ab}\theta_c-2cf_a J_{bc}\theta_c)+c\sum_r(f_b J_{ar}\theta_r-f_r J_{ab}\theta_r+2f_a J_{br}\theta_r)\}\wedge\theta_b=0.\end{aligned}$$

Therefore, we can put

$$\begin{aligned}&dh_{ab}-\sum_c(h_{ac}\theta_{cb}+h_{cb}\theta_{ca}-\sum_r h_{ac}A_{crb}\theta_r-cf_b J_{ac}\theta_c+cf_c J_{ab}\theta_c\\&\quad-2cf_a J_{bc}\theta_c)+c\sum_r(f_b J_{ar}\theta_r-f_r J_{ab}\theta_r+2f_a J_{br}\theta_r)=\sum C_{abc}\theta_c,\end{aligned} \tag{2.10}$$

where $C_{abc}=C_{acb}=C_{bac}$.

Under such a situation we have

LEMMA 2.2. *If* $J_{rs}=0$ *on* U, *then* $T\geqq n$ *on* U.

PROOF. If $T<n$, then by Lemma 2.1 we have $T=n-1$, and $f_a=0$ on U. For a suitable choice of a field $\{e_r\}$ of orthonormal frames, if necessary, we may set $f_{2n-1}=1$ and $f_r=0$ for $r=n, \cdots, 2n-2$. Then from (1.5) we have

$$0=df_r=-\sum J_{ar}\phi_a .$$

But, since rank $J=2n-2$, we have $\det(J_{ar})\neq 0 (a=1, \cdots, n-1, r=n, \cdots, 2n-2)$. Thus the above equation implies $\phi_a=0$, which contradicts the fact that $\det(h_{ab})\neq 0$.

This concludes the proof.

In the remainder of this section we restrict the forms under consideration to the following open set V_T defined by

$$V_T=\{p\in M \mid J_{rs}(p)\neq 0, t(p)=T\},$$

where $J_{rs}(p)\neq 0$ means "$J_{rs}(p)\neq 0$ for some $r, s=T+1, \cdots, 2n-1$". First from (2.9) we have $f_r=0$. Thus we may set $f_1=1$, and $f_a=0$ for $a\geqq 2$. Hence we have

$$J_{1a}=0, \qquad J_{1r}=0. \tag{2.11}$$

Furthermore, $df_a=0$ and $df_r=0$ give

$$\theta_{1a}=\sum J_{ba}\phi_b, \tag{2.12}$$

$$A_{1ra}=\sum h_{ab}J_{br}, \tag{2.13}$$

$$B_{1rs}=0. \tag{2.14}$$

The equation (2.7) amounts to

$$\sum h_{ab}B_{brs}=cf_aJ_{rs}. \tag{2.15}$$

LEMMA 2.3. $\det(h_{ab})=0 (a, b=2, \cdots, T)$ *on* V_T.

PROOF. Here indices a, b run from 2 to T. If $\det(h_{ab})\neq 0$, then by (2.15) we have $B_{ars}=0$, which together with (2.14) gives $J_{rs}=0$. A contradiction to the fact $J_{rs}(p)\neq 0$ on V_T.

This concludes the proof.

3. Proof of Theorem 1.

We keep the notation in section 2. If $J_{rs}=0$ on a nonempty open set, then

Lemma 2.2 proves Theorem 1. Therefore, we have only to consider the case where the open set V_T defined in section 2 is not empty.

Assume $T=2$. Then we shall derive a contradiction. First by Lemma 2.3 we have $h_{22}=0$.

Now we put $F=h_{12}$. Then from (2.13) we have

$$A_{1r1}=FJ_{2r}, \qquad A_{1r2}=0. \tag{3.1}$$

Put $a=1$ and $b=2$ in (2.6) to get

$$FA_{2r2}-FA_{1r1}-cJ_{r2}=0. \tag{3.2}$$

Then (3.1) and (3.2) give

$$FA_{2r2}=(F^2-c)J_{2r}. \tag{3.3}$$

According to (2.12), we have

$$\theta_{12}=0.$$

Put $a=1$ and $b=2$ in (2.10). Then, together with (3.3) we find

$$dF+(F^2+c)\sum J_{2r}\theta_r=\sum D_{12a}\theta_a, \tag{3.4}$$

where we have put $D_{12a}=C_{12a}-2cJ_{2a}$.

Let p be any point of V_2 and let $\alpha: I\to V_2$ be the maximal integral curve of the unit dual vector field $\sum J_{2r}\theta_r$ on V_2 such that $\alpha(0)=p$, $\sum_r J_{2r}\theta_r(\alpha'(t))=1$ and $D_{12a}\theta_a(\alpha'(t))=0$, where $(0\in)I$ denotes an open interval of R.

Assume that there exists $\sup I$, say t_0. Since M is complete, we have a point $p_0=\lim_{t\to t_0}\alpha(t)$ on M. We assert $F(p_0)=\lim_{t\to t_0}F(\alpha(t))=0$. In order to prove this assertion, it suffices to show that $t(p_0)\leqq 1$, because $\det(h_{ij})=-F^2$ by $h_{22}=0$ at p_0. For this we assume $t(p_0)=2$. Thus we can consider that our frame field $\{e_i\}$ is defined also on the neighborhood of p_0. Since $p_0\in\overline{V}_2$, $J_{rs}(p_0)=0$ for any $r, s\geqq 3$. Then by Lemma 2.1 we have $t(p_0)\geqq n-1\geqq 3$ for $n\geqq 4$, which is a contradiction. For a case where $n=3$ also by using Lemma 2.1 we get $J_{ab}=0$ and $f_a=0$ at p_0 for all $a, b=1, 2$. This also contradicts to the fact that $f_1=1$ at p_0, which proves our assertion.

Now we shall show that $\inf I=-\infty$. Indeed, if there exists $t_1=\inf I$, then we find $\lim_{t\to t_1}F(\alpha(t))=0$ by an argument similar to the above. Thus there is a real number t' such that $t_1<t'<t_0$, $dF=0$ at $\alpha(t')$. Then (3.4) gives $J_{2r}=0$. From this together with $J_{1r}=0$ in (2.11) it follows that $\operatorname{rank}J\leqq 2n-3$, which makes a contradiction to the fact that $\operatorname{rank}J=2n-2$.

Now the function $|F|$ defined on the interval $(-\infty, t_0)$ satisfies by (3.4)

$$\frac{d|F|}{dt}=F^2+c \quad \text{or} \quad \frac{1}{F^2+c}\frac{d|F|}{dt}=1.$$

Solving the above differential equation, we have

$$F(\alpha(t))=\sqrt{c}\tan\sqrt{c}(t-t_0),$$

which is a contradiction, because $F(\alpha(t))$ is defined on $(-\infty, t_0)$ but the right hand side can not be defined at the points such that $\sqrt{c}(t-t_0)=(2k+1)\pi/2$, where k is an integer.

For a case where $\sup I=\infty$, we can take a point $\alpha(t_0)\in V_2$ such that $F(\alpha(t_0))=F_0\neq 0$ for $t_0<\infty$. This case also contains the situation such that $F(\alpha(t))$ is defined on $(-\infty, \infty)$ and $\alpha(t)$ is contained in V_2. Using the similar method to the above, we also get

$$F(\alpha(t))=\sqrt{c}\tan\sqrt{c}(t-t_0+s_0) \quad \text{on} \quad (-\infty, t_0]$$

where we have put $\sqrt{c}s_0=\tan^{-1}(F_0/\sqrt{c})$. This also makes a contradiction as in the above case.

It completes the proof of Theorem 1.

References

[1] J. Berndt, Real hypersurfaces with constant principal curvature in complex hyperbolic space, J. reine angew, Math. **395** (1989), 132-141.

[2] T.E. Cecil and P.J. Ryan, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. **269** (1982), 481-499.

[3] U-H. Ki, H. Nakagawa and Y.J. Suh, Real hypersurfaces with harmornic Weyl tensor of a complex space form, Hiroshima Math. J. **20** (1990), 93-102.

[4] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. **202** (1989), 299-311.

[5] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc., **212** (1975), 355-364.

[6] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math., **10** (1975), 495-506.

[7] R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures, J. Math. Soc. Japan, **27** (1975), 43-53.

[8] R. Takagi, Real hypersurfaces in a complex projective space with constant principal curvatures II, J. Math. Soc. Japan, **27** (1975), 507-516.

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