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# SPECIAL ALGEBRAIC PROPERTIES OF KÄHLER ALGEBRAS

By

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Homogeneous Kähler manifolds $M$ are frequently investigated via Kähler algebras $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$, where $\boldsymbol{g}$ denotes a Lie algebra of infinitesimal automorphisms of $M$ and $\boldsymbol{k}$ the isotropy subalgebra of some point in $M$. Moreover, $j$ corresponds to the complex structure tensor and $\rho$ to the Kähler form. In particular, Kähler algebras have been used intensively in the proof of the geometric Fundamental Conjecture for homogeneous Kähler manifolds: Every homogeneous Kähler manifold is a holomorphic fiber bundle over a homogeneous bounded domain in which the fiber is (with the induced Kähler metric) the product of a flat homogeneous Kähler manifold and a compact simply connected homogeneous Kähler manifold.

Two additional properties of Kähler algebras have proven to be particularly useful. One is that $\boldsymbol{g}$ or $\operatorname{ad}\boldsymbol{g}$ is an algebraic Lie algebra. The second one is the assumption that $\rho$ is the differential of a leftinvariant 1-form, $\rho=d\omega$. This is the case of "$j$-algebras". It has been investigated intensively by Gindikin, Piatetskii-Shapiro, Vinberg and others. The proof of the Fundamental Conjecture for homogenous Kähler manifolds is much shorter for $j$-algebras than for general Kähler algebras. This is due to some extent to the fact that one can embed a $j$-algebra into an algebraic $j$-algebra.

The purpose of this note is threefold. First we want to prove that for the Lie algebra $\boldsymbol{g}_M$ of all infinitesimal automorphisms of an arbitrary homogeneous Kähler manifold $M$, the Lie algebra $\operatorname{ad}\boldsymbol{g}_M$ is algebraic. Secondly, we decompose $\boldsymbol{g}_M$ into the orthogonal sum of $j$-invariant subalgebras. This decomposition will be of importance for a forthcoming publication in which we give a detailed description of $\boldsymbol{k}_M$ and the Kähler form $\rho$. The orthogonal decomposition in question has a simple geometric interpretation. It is essentially induced by a representation of the base domain (occuring in the Fundamental Conjecture) as a Siegel domain of type three.

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The first step in the orthogonal decomposition mentioned above is a generalization of [7; Theorem 2.5]. We show $\boldsymbol{g}_M=\boldsymbol{a}_M+\boldsymbol{h}_M$, where $\boldsymbol{a}_M$ is an abelian $j$-invariant ideal of $\boldsymbol{g}_M$ and $\boldsymbol{h}_M$ is a $j$-invariant subalgebra such that $\boldsymbol{k}_M\subset\boldsymbol{h}_M$ and $\boldsymbol{h}_M$ carries the structure of a $j$-algebra. More precisely, there exists some linear form $\omega$ on $\boldsymbol{h}_M$ such that $(\boldsymbol{h}_M, \boldsymbol{k}_M, j, d\omega)$ is a $j$-algebra. The third goal of this paper is to show that one can choose actually $\omega$ so that $\rho\,|\,\boldsymbol{h}_M=d\omega$ holds. This has several geometric consequences.

## § 1. Basic Definitions and Reductions.

**1.1.** Let $\boldsymbol{g}$ be a finite dimensional Lie algebra over $\boldsymbol{R}$, $\boldsymbol{k}$ a subalgebra of $\boldsymbol{g}$, $j$ a linear endomorphism of $\boldsymbol{g}$ and $\rho: \boldsymbol{g}\times\boldsymbol{g}\to\boldsymbol{R}$ a skew form. Then $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ is called a Kähler algebra if

$$j\boldsymbol{k}\subset\boldsymbol{k}, \quad j^2x=-x \pmod{\boldsymbol{k}} \quad \text{for all } x\in\boldsymbol{g}\,, \tag{1.1.1}$$

$$[k, jx]=j[k, x] \quad \text{for all } k\in\boldsymbol{k},\ x\in\boldsymbol{g}, \tag{1.1.2}$$

$$[jx, jy]=j[jx, y]+j[x, jy]+[x, y] \pmod{\boldsymbol{k}} \quad \text{for all } x, y\in\boldsymbol{g}, \tag{1.1.3}$$

$$\rho(\boldsymbol{k}, \boldsymbol{g})=0\,, \tag{1.1.4}$$

$$\rho(jx, jy)=\rho(x, y) \quad \text{for all } x, y\in\boldsymbol{g}\,, \tag{1.1.5}$$

$$\rho(jx, x)>0 \quad \text{for all } x\in\boldsymbol{g},\ x\notin\boldsymbol{k}\,, \tag{1.1.6}$$

$$\rho([x, y], z)+\rho([y, z], x)+\rho([z, x], y)=0 \quad \text{for all } x, y, z\in\boldsymbol{g}\,. \tag{1.1.7}$$

It was shown in [7; Proposition 1.1] that each Kähler algebra corresponds in a natural way to a transitive group of automorphisms of a Kähler manifold.

Since we are only interested in effectively acting groups, we *can—and will—assume*

$$\boldsymbol{k} \text{ does not contain any ideal of } \boldsymbol{g}\,. \tag{1.1.8}$$

Such Kähler algebras will be called *effective*.

Moreover, since the isotropy subgroup of the group of all automorphisms of a homogeneous Kähler manifold is compact, we *can—and will—assume*

$$\boldsymbol{k} \text{ is the Lie algebra of a compact Lie subgroup } K\subset G, \tag{1.1.9}$$

$$\text{where } \boldsymbol{g}=\text{Lie } G\,.$$

We will call such subalgebras briefly *compact*.

**1.2.** If $\rho(x, y)=\omega([x, y])$ for some linear form $\omega: \boldsymbol{g}\to\boldsymbol{R}$, then $(\boldsymbol{g}, \boldsymbol{k}, j, \omega)$ is called a *$j$-algebra*.

By abuse of language we also say $(\boldsymbol{g}, \boldsymbol{k}, j)$ is a $j$-algebra, if there exists some $\omega$ such that $(\boldsymbol{g}, \boldsymbol{k}, j, \omega)$ is a $j$-algebra.

**1.3.** Let $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ be a Kähler algebra. Let $j: \boldsymbol{g} \rightarrow \boldsymbol{g}$ be a linear map satisfying $jx = j'x \pmod{\boldsymbol{k}}$, then $(\boldsymbol{g}, \boldsymbol{k}, j', \rho)$ is also a Kähler algebra (associated with the same manifold as $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$. Replacing $j$ by $j'$ is called an *inessential change* of $j$. We will not distinguish between $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ and $(\boldsymbol{g}, \boldsymbol{k}, j', \rho)$.

**1.4.** For a homogeneous Kähler manifold $M$ we denote by $\mathrm{Aut}(M)$ the group of holomorphic isometries of $M$. We also set $\boldsymbol{g}_M = \mathrm{Lie}\ \mathrm{Aut}(M)$ and denote by $\boldsymbol{k}_M$ the isotropy subalgebra of $\boldsymbol{g}_M$ associated with some fixed base point $O_M$ in $M$.

We recall from [7; **2.1**] that a Kähler algebra $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ is called *quasi-normal* if $\mathrm{ad}\, x$ has only real eigenvalues for all $x \in \mathrm{rad}(\boldsymbol{g})$. It was shown in [7; Thorem 2.1] that to each homogeneous Kähler manifold there exists a quasi-normal Kähler subalgebra. Analysing the proof of [7; Theorem 2.1] as given in [7; **2.4**] one notices that one can even find a quasi-normal subalgebra $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ of $(\boldsymbol{g}_M, \boldsymbol{k}_M, j, \rho)$ such that $\boldsymbol{k}_M = \boldsymbol{k} + \boldsymbol{k}'$, where $\boldsymbol{k}' \subset \mathrm{rad}(\boldsymbol{g}_M)$. Then $\mathrm{rad}(\boldsymbol{g}_M) = \mathrm{rad}(\boldsymbol{g}) + \boldsymbol{k}'$ holds. Now let $\pi: M \rightarrow D$ be the locally trivial fibration of the Fundamental Conjecture. Since $\mathrm{Aut}(M)$ preserves the fibers of $\pi$, the space $\boldsymbol{a} + \boldsymbol{u} + \boldsymbol{k}'$ is a $j$-invariant subalgebra of $\boldsymbol{g}_M = \boldsymbol{g} + \boldsymbol{k}'$ (here $\boldsymbol{a}$ and $\boldsymbol{u}$ are as in [7; Theorem 2.5]). But since $\boldsymbol{a} + \boldsymbol{k}' \subset \mathrm{rad}(\boldsymbol{g}_M)$ we even know that $\boldsymbol{a} + \boldsymbol{k}'$ is a $j$-invariant solvable subalgebra. Now it is easy to show $[\boldsymbol{a}, \boldsymbol{k}'] \subset \boldsymbol{a}$. Therefore the orthogonal complement $\boldsymbol{h} + \boldsymbol{k}'$ of $\boldsymbol{a}$ is a $j$-invariant subalgebra of $\boldsymbol{g}_M$.

Thus we have shown

THEOREM. *Let $M$ be a homogeneous Kähler manifold and $(\boldsymbol{g}_M, \boldsymbol{k}_M, j, \rho)$ the corresponding Kähler algebra. Then $\boldsymbol{g}_M$ is decomposed as*

$$\boldsymbol{g}_M = \boldsymbol{a}_M + \boldsymbol{h}_M, \qquad \text{where } \boldsymbol{a}_M \cap \boldsymbol{h}_M = 0 \text{ and } \rho(\boldsymbol{a}_M, \boldsymbol{h}_M) = 0\,,$$

*where $\boldsymbol{k}_M \subset \boldsymbol{h}_M$ and $\boldsymbol{a}_M$ is an abelian Kähler ideal of $\boldsymbol{g}_M$ and $(\boldsymbol{h}_M, \boldsymbol{k}_M, j)$ is a $j$-algebra. Moreover, there exists some reduclive Kähler subalgebra $\boldsymbol{u}_M$ of $\boldsymbol{h}_M$ satisfying* a) *and* b) *of* [7; Theorem 2.5].

*Finally, if $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ is a quasi-normal subalgebra of $(\boldsymbol{g}_M, \boldsymbol{k}_M, j, \rho)$ and $\boldsymbol{g} = \boldsymbol{a} + \boldsymbol{h}$ the corresponding decomposition of* [7; Theorem 2.5], *then $\boldsymbol{a}_M = \boldsymbol{a}$ and $\boldsymbol{h}_M = \boldsymbol{h} + \boldsymbol{k}_M \cap \mathrm{rad}(\boldsymbol{g}_M)$.*

**1.5.** In view of Theorem 1.4. *we restrict our attention from now on to*

*Kähler algebras* $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ *for which* $(\boldsymbol{g}, \boldsymbol{k}, j)$ *is a* $j$*-algebra* (except where explicitely stated otherwise).

We keep also the assumptions (1.1.8) and (1.1.9) intact.

From [10] we know that the group $G$ associated with $\boldsymbol{g}$ acts transitively on the homogeneous Siegel domain ($\equiv$homogeneous bounded domain) $D$ occuring in the Fundamental Conjecture.

Denoting the isotropy algbra of the origin in $D$ by $\boldsymbol{u}$ we show

LEMMA. a) $\boldsymbol{g}=\boldsymbol{g}'+\boldsymbol{u}'$ *orthogonal sum of* $j$*-invariant ideals.*

b) $\boldsymbol{g}'\subset\mathrm{Lie\ Aut}\,D$, $\boldsymbol{u}'\subset\boldsymbol{u}$.

c) $(\boldsymbol{g}', \boldsymbol{k}\cap\boldsymbol{g}', j, \rho)$ *is a Kähler algebra associated with* $D$ *and satisfies* (1.1.8) *and* (1.1.9).

PROOF. Let $\boldsymbol{u}'$ be the maximal ideal of $\boldsymbol{g}$ contained in $\boldsymbol{u}$. Then $\boldsymbol{g}=\boldsymbol{g}'+\boldsymbol{u}'$ for some ideal $\boldsymbol{g}'$ of $\boldsymbol{g}$. We know from [10; § 3] that the center of $\boldsymbol{u}$ is contained in $\boldsymbol{k}$. Hence the group $U$ corresponding to $\boldsymbol{u}$ is compact, whence also $U'$ corresponding to $\boldsymbol{u}'$ is compact. From [8] and [1] we obtain that $j$ leaves the simple summands of $\boldsymbol{u}$ and the center of $\boldsymbol{u}$ invariant. Thus we can assume that $\boldsymbol{u}'$ is $j$-invariant. Splitting $\boldsymbol{u}'$ into center and semisimple part and using the closedness condition for $\rho$ one obtains $\boldsymbol{g}'+\boldsymbol{k}=\{x\in\boldsymbol{g}\,;\ \rho(x, \boldsymbol{u}')=0\}$. Therefore we can assume that also $\boldsymbol{g}'$ is $j$-invariant. This implies a), b) and all but the last statement of c). But $G/U\cong(G/U')/(U/U')$, whence $\boldsymbol{u}\cap\boldsymbol{g}'=\mathrm{Lie}\,(U/U')$ is the Lie algebra of a compact group.

**1.6.** Since $\boldsymbol{u}'$ is a reductive summand of $\boldsymbol{g}$, it will suffice to study $\boldsymbol{g}'$ in detail. Therefore, where not stated otherwise, we will assume in addition to the assumptions of **1.5** that $\boldsymbol{g}$ can be realized as subalgebra of $\boldsymbol{g}(D)=\mathrm{Lie\ Aut}\,(D)$.

Thus we can apply the results of [3] to the study of $\boldsymbol{g}$.

## § 2. Decompositions of $j$-algebras

**2.1.** In the sections **2.1** to **2.3** we consider Kähler algebras $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ satisfying (1.1.8), (1.1.9) and which can be realized as subalgebras of $\boldsymbol{g}(D)=\mathrm{Lie\ Aut}\,(D)$ for some homogeneous Siegel domain $D$. More precisely, we assume that $\boldsymbol{g}$ generates a connected subgroup $G$ of $\mathrm{Aut}\,(D)$ which acts transitively on $D$. Moreover, the isotropy subgroup $U\subset G$ of the base point $ie\in D$ contains the isotropy subgroup $K$ of $M(\boldsymbol{g})=G/K$. We use as usual $\boldsymbol{u}=\mathrm{Lie}\ U$.

From the proof of Lemma 1.5 we know that $U$ is compact.

**2.2.** It is not hard to see that $\boldsymbol{g}$ contains a solvable subalgebra $\boldsymbol{t}$ which is invariant under $j'$, where $j'x=jx(\bmod \boldsymbol{u})$, and such that $(\boldsymbol{t}, \boldsymbol{t}\cap\boldsymbol{k}, j')$ is a $j$-algebra associated with $D$. We can, [9], —and will— assume that $\boldsymbol{t}$ is associated with affine transformations of $D$. Let $e$ denote the principal idempotent of $\boldsymbol{t}$ and let $R=\mathrm{Re}(\mathrm{ad}\, j'e)$ be defined as in [5; **4.9**]. Then $\boldsymbol{g}$ is invariant under $R$ and $\boldsymbol{g}=\boldsymbol{g}_1+\boldsymbol{g}_{1/2}+\boldsymbol{g}_0+\boldsymbol{g}_{-1/2}+\boldsymbol{g}_{-1}$, where the vector fields induced from $\boldsymbol{g}_\lambda$ are contained in $\boldsymbol{g}(D)_{-\lambda}$ (as defined in [3; §**1.5**]). In particular, $\boldsymbol{g}_a=\boldsymbol{g}_1+\boldsymbol{g}_{1/2}+\boldsymbol{g}_0$ corresponds to the affine automorphisms contained in $G$, and $\boldsymbol{g}_0$ to the linear automorphisms contained in $G$. We have made sure that already the group $G_a$ associated with $\boldsymbol{g}_a$ acts transitively on $D$. Hence $\boldsymbol{g}_1=\boldsymbol{g}_1(D)\subset\boldsymbol{t}$ and $\boldsymbol{g}_{1/2}=\boldsymbol{g}_{1/2}(D)$ $\subset\boldsymbol{t}$. Next we consider the algebraic hull $\tilde{\boldsymbol{g}}_0$ of $\boldsymbol{g}_0$ in $\boldsymbol{g}_0(D)$. We can apply [7; Theorem 6.2] to $\tilde{\boldsymbol{g}}_0$. Since $[\tilde{\boldsymbol{g}}_0, \tilde{\boldsymbol{g}}_0]\subset\tilde{\boldsymbol{g}}_0$, the subspaces $F_{ij}$, $i\neq j$, and the semisimple parts of the $F_{ii}$ are already contained in $\boldsymbol{g}_0$. Thus the difference between $\boldsymbol{g}_0$ and $\tilde{\boldsymbol{g}}_0$ comes at one hand from the fact that the element in $\boldsymbol{g}_0$ corresponding to $f_i$ is in $\boldsymbol{g}_0$ of the form $f_i+f_{i0}$, where $f_{i0}\in\mathrm{center}(F_0)$ and on the other hand that also $\mathrm{center}(F_0)\cap\boldsymbol{g}_0$ is perhaps not the Lie algebra of a compact Lie group.

Since we know that $\boldsymbol{u}$ corresponds to a compact Lie group, the latter cannot happen.

**2.3.** In [3; §7] we have found a semisimple subalgebra of $\boldsymbol{g}(D)=\mathrm{Lie\ Aut}(D)$ that corresponds naturally to the full algebra of infinitesimal automorphisms of a symmetric Siegel domain in a subspace of the original complex vector space.

In [3] we had defined on $\boldsymbol{g}_1=\boldsymbol{g}_1(D)$ the structure of an algebra $\mathcal{A}$ and decomposed the subspaces $\boldsymbol{g}_\lambda(D)$ of $\boldsymbol{g}(D)$ relative to idempotents $e_{ii}$ of $\mathcal{A}$. The description of these decompositions can be found in [3; §§3, 4, 5]. The description of $\tilde{\boldsymbol{g}}_0$ in [7] yields Lie algebras $F_{kk}$ associated with self dual cones. It is not hard to see that $f_k$ corresponds to some idempotent $f_{kk}\in\mathcal{A}$. More precisely, each $e_{ii}$ is a sum of certain $f_{kk}$. Thus the family of $f_{kk}$'s induces a decomposition of the spaces $\boldsymbol{g}_\lambda(D)$ just the same way as the $e_{ii}$'s do. The decomposition relative to $\{f_{kk}\}$ is a refinement of the decomposition relative to $\{e_{ii}\}$. For $\boldsymbol{g}_0(D)$ we obtain $\boldsymbol{g}_0(D)=\bigoplus_{i\leq j} F_{ij}+F_0$, where $F_0\subset F_0(D)$.

For the description of $\boldsymbol{g}_{-1/2}(D)$ and $\boldsymbol{g}_{-1}(D)$ we use the notation introduced in [3; §6]. Thus $\boldsymbol{g}_{-1}(D)$ is parametrized by a subspace $P_1\subset\mathcal{A}_1$. (Note that here we use the Lie algebra which is opposite to the one considered in [3].)

We set

$$\hat{P}_{-1}=\{x\in P_1\,;\, X_{-1}[x]\in\boldsymbol{g}_{-1}\}. \tag{2.3.1}$$

We know [3]:

(2.3.2) $$T^{\sigma}\hat{P}_{-1}\subset\hat{P}_{-1} \quad \text{for all } (T, \hat{T})\in\boldsymbol{g}_0 .$$

Let $g_1, \cdots, g_r$ denote the different $f_k$'s contained in $\boldsymbol{m}_{11}^{\varphi}$ ([3; 3.21]). Then $\mathcal{A}_1=\oplus\mathcal{L}_{ij}$, where $\mathcal{L}_{ij}$ denote the Peirce spaces of $\mathcal{A}_1$ relative to $g_1, \cdots, g_r$. From (2.3.2) we then obtain

(2.3.3) $$\hat{P}_{-1}=\oplus[\hat{P}_{-1}\cap\mathcal{L}_{ij}] .$$

Since the cones corresponding to $f_k$ are irreducible,

(2.3.5) $$\hat{P}_{-1}\cap\mathcal{L}_{kk}=\mathcal{L}_{kk} \quad \text{if } \hat{P}_{-1}\cap\mathcal{L}_{kk}\neq 0 .$$

Moreover,

(2.3.5) If $i<j$ and $\hat{P}_{-1}\cap\mathcal{L}_{ij}\neq 0$, then

$$\hat{P}_{-1}\cap\mathcal{L}_{ij}=\mathcal{L}_{ij} \text{ and } \hat{P}_{-1}\cap\mathcal{L}_{ij}=\mathcal{L}_{ii} .$$

Therefore, if $j_0=\max\{j;\hat{P}_{-1}\cap\mathcal{L}_{jj}\neq 0\}$, then we set $g=\sum_{i=1}^{j_0} g_i$ and obtain (with the usual notation for Peirce decompositions)

(2.3.6) $$\hat{P}_{-1}=(\mathcal{A}_1)_1(g)+\hat{P}_{-1}\cap(\mathcal{A}_1)_{1/2}(g) .$$

From (2.3.3) and (2.3.5) it is clear that the last summand in (2.3.6) is a sum of certain $\mathcal{L}_{ij}$'s.

We claim

PROPOSITION. *$\hat{P}_{-1}=\bigoplus_{k\in I}\mathcal{L}_{kk}$ for some subset $I\subset\{1, \cdots, r\}$.*

PROOF. Let $0\neq b_{ij}\in\hat{P}_{-1}$, $i<j$. Then [3; 6.6] shows $[X_1[g_i], X_{-1}[b_{ij}]]=(T, \hat{T})\in\boldsymbol{g}_0$, where $T=2A_{b_{ij}}(g_i)=2(A_{g_i}(b_{ij}))^{\sigma}$. But since $i<j$, we also know $(A_{g_i}(b_{ij}), \hat{T}^*)\in\boldsymbol{g}_0$ for some $\hat{T}^*$. Hence, by the definition of the $f_k$'s, $i=j$, a contradiction.

In view of (2.3.2) the proposition implies.

COROLLARY 1. *Assume $\mathcal{L}_{kk}$, $\mathcal{L}_{ss}\subset P_{-1}$, then $\mathcal{L}_{kk}=0$.*

As a consequence of this corollary we obtain

COROLLARY 2. *One can assume $g_k=f_k$ for $k=1, \cdots, r$.*

**2.4.** We keep the notation of the last section. We will also use the set $U$ naturally associated with $D$. We set

(2.4.1) $$\mathfrak{g}_{-1k}=X_{-1}[\mathcal{L}_{kk}]$$

(2.4.2) $$\mathfrak{g}_{-(1/2)k}=X_{-1/2}[f_k U]\,.$$

Now we can take over, mutatis mutandis, the proof of [3; Theorem 7.5] and obtain

THEOREM. *There exists some subalgebra $F_{0k}$ of $\mathfrak{g}_0$ such that*

$$\mathfrak{q}_k=X_1[\mathcal{L}_{kk}]+X_{1/2}[f_k U]+(F_{kk}+F_{0k})+X_{-1/2}[f_k U]+X_{-1}[\mathcal{L}_{kk}]$$

*is a semisiple subalgebra of $\mathfrak{g}$.*

Actually it is not hard to see that $\mathfrak{q}_k$ is simple. We set $\mathfrak{q}=\oplus\mathfrak{q}_k$. Then it is easy to obtain

COROLLARY 1. (a) *$\mathfrak{q}$ is a semisimple Lie subalgebra of $\mathfrak{g}$ with simple ideals $\mathfrak{q}_k$*

(b) $\mathfrak{q}_{-1/2}=\mathfrak{g}_{-1/2}$, $\mathfrak{q}_{-1}=\mathfrak{g}_{-1}$.

COROLLARY 2. *If $f_k$ occurs in $\mathfrak{q}$, then $f_k\in\mathfrak{g}$.*

**2.5.** From Theorem 2.4 it is easy to derive that the algebras $\mathfrak{q}_k$ consist of full weight spaces relative to $\{f_k\}$ for nonzero weights. The zero weight in $\mathfrak{q}_k$ is $F_{kk}+F_{0k}$. As in [3; §7] one can show that $F_{0k}$ is an ideal of $F_0\cap\mathfrak{g}_0$. Hence there exists some $F_0'\subset\mathfrak{g}_0$ so that $F_0\cap\mathfrak{g}_0=\oplus F_{0k}+F_0'$ as a sum of ideals.

Now we denote by $\mathfrak{p}$ the "complement of $\mathfrak{q}$ in $\mathfrak{g}$"; i.e. we sum up all weight spaces relative to $\{f_k\}$ for nonzero weights that do not occur in $\mathfrak{q}$, the space $\mathfrak{g}_0\cap(\oplus F_{kk})$ for $F_{kk}\cap\mathfrak{q}=0$ and $F_0'$. Since the latter two types of algebras actually commute with $\mathfrak{q}$, we obtain

THEOREM. *There exists an ideal $\mathfrak{p}$ of $\mathfrak{g}$ such that*

(a) *$\mathfrak{g}=\mathfrak{q}+\mathfrak{p}$ (direct sum of vector spaces)*

(b) *$\mathfrak{p}$ is a sum of weight spaces relative to $\{f_k\}$ for nonzero weights, of the space $\mathfrak{g}_0\cap(\oplus F_{kk})$ for $F_{kk}\cap\mathfrak{q}=0$ and of an ideal $F_0'$ of $F_0\cap\mathfrak{g}_0$.*

(c) *$\mathfrak{p}$ corresponds to affine transformations of $D$.*

REMARK. If $\mathfrak{g}$ is algebraic, then $F_{kk}\cap\mathfrak{q}=0$ implies $F_{kk}\subset\mathfrak{p}$. Moreover, $F_0'$ corresponds to a compact Lie group.

**2.6.** The main purpose of this section is to prove

THEOREM. *Let $(\mathfrak{g}, \mathfrak{k}, j, \rho)$ be a Kähler algebra satisfying* (1.1.8) *and* (1.1.9). *Assume moreover that there exists a homogeneous Siegel domain $D$ such that $\mathfrak{g}\subset$*

Lie Aut$(D)$. *Assume also that there exist a quasi-normal Kähler subalgebra* $(\boldsymbol{g}', \boldsymbol{k}', j, \rho)$ *of* $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ *which is associated with the same manifold as* $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$. *Then* $\boldsymbol{g}$ *is an algebraic Lie algebra.*

PROOF. Clearly, $\boldsymbol{g}=\boldsymbol{g}'+\boldsymbol{k}$. Moreover, since $\boldsymbol{g}'$ is quasi-normal, we can assume $f_i \in \boldsymbol{g}'$ for all $i$. From **2.5** we know that we have $\boldsymbol{g}=\boldsymbol{q}+\boldsymbol{p}$, where $\boldsymbol{q}$ is semisimple and $\boldsymbol{p}$ corresponds to affine transformations. It suffices to show $F_{kk} \subset \boldsymbol{g}_0$ for all $k$ and $F_0 \subset \boldsymbol{g}_0$. But $F_{kk}=\boldsymbol{R}f_k+[F_{kk}, F_{kk}]$ and $f_k \in \boldsymbol{g}'$, whence $F_{kk} \subset \boldsymbol{g}_0$. Moreover, 1.1.9 implies $F_0 \subset \boldsymbol{g}_0$.

COROLLARY. *Let* $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ *be a Kähler algebra satisfying* (1.1.8) *and* (1.1.9). *Assume moreover, that there exists a homogeneous Siegel domain* $D$ *such that* $\boldsymbol{g} \subset$Lie Aut$(D)$. *Assume also that* $\boldsymbol{g}$ *contains a real split solvable subalgebra* $\boldsymbol{s}$ *which generates a transitive subgroup of* $D$. *Then* $\boldsymbol{g}$ *is an algebraic Lie algebra.*

**2.7.** In the last section we considered Kähler algebras which can be realized as subalgebras of the infinitesimal automorphisms of some homogeneous Siegel domain. Using the reductions of § 1 we can derive from this

THEOREM. *Let* $M$ *be a homogeneous Kähler manifold and* $(\boldsymbol{g}_M, \boldsymbol{k}_M, j, \rho)$ *a Kähler algebra associated with* $M$. *Then* ad $\boldsymbol{g}_M$ *is an algebraic Lie algebra.*

PROOF. From Theorem 1.4 we know $\boldsymbol{g}_M=\boldsymbol{a}_M+\boldsymbol{h}_M$, where $\boldsymbol{a}_M$ is an abelian ideal of $\boldsymbol{g}_M$. Since ad $\boldsymbol{a}_M$ consists of nilpotent endomorphisms, it is algebraic by [2; chap V, §3, 4.]. Hence it suffices to show that ad $\boldsymbol{h}_M \subset gl(\boldsymbol{g}_M)$ is algebraic. From **1.5** we know $\boldsymbol{h}_M=\boldsymbol{h}'_M+\boldsymbol{u}'$, where $\boldsymbol{u}'$ is the Lie algebra of a compact group. Hence it suffices to show that ad $\boldsymbol{h}'_M \subset gl(\boldsymbol{g}_M)$ is algebraic by [2; chap II, § 14]. Algebras of type $\boldsymbol{h}'_M$ have been investigated in **2.5** and **2.6.** They are of type $\boldsymbol{h}'_M=\boldsymbol{q}_M+\boldsymbol{p}'+F_0$, where $\boldsymbol{g}_M$ is semisimple, $F_0$ is compact and $\boldsymbol{p}'=$nil$(\boldsymbol{p}')+\boldsymbol{v}+\boldsymbol{e}$, where $\boldsymbol{v}$ is semisimple and $\boldsymbol{c}$ is abelian with only real eigenvalues. Thus by [2; chap II, § 14] it suffices to show that ad $\boldsymbol{c} \subset gl(\boldsymbol{g}_M)$ is algebraic. But ad $\boldsymbol{c}$ is spanned by the ad $f_k$'s and each ad $f_k$ has only the eigenvalues 0, $\pm 1/2$, $\pm 1$ in $\boldsymbol{g}_M$. Hence $\boldsymbol{R}$ ad $f_k$ is algebraic by [2; p. 169]. This finishes the proof of the theorem.

**2.8.** Recall, in **1.4** we have seen that $\boldsymbol{g}_M$ is the orthogonal sum of the $j$-invariant subalgebras $\boldsymbol{a}_M$ and $\boldsymbol{h}_M$ (of course, $\boldsymbol{a}_M$ is even an abelian ideal). In **1.5** we have shown that $\boldsymbol{h}_M$ is the orthogonal and $j$-invariant sum of ideals $\boldsymbol{h}'_M$ and $\boldsymbol{u}'_M$, where $\boldsymbol{h}'_M$ can be realized as subalgebra of $\boldsymbol{g}(D)$=Lie Aut$(D)$ for some homogeneous Siegel domain $D$. From Theorem 2.6 we also know that $\boldsymbol{h}'_M$ is an algebraic subalgebra of $\boldsymbol{g}(D)$. Moreover, in view of Theorem 2.5 we have $\boldsymbol{h}'_M=\boldsymbol{q}+\boldsymbol{p}$, where $\boldsymbol{q}$ is semisimple and described in Theorem 2.5 and $\boldsymbol{p}$ is the sum of the remaining root spaces and part of $F_0$ (where the roots are computed relative to the $f_r$).

In the following two sections we will show that $\boldsymbol{q}$ and $\boldsymbol{p}$ can be assumed to be $j$-invariant.

**2.9.** We want to show that $\boldsymbol{q}$ is (modulo $\boldsymbol{k}$) the orthogonal complement of $\boldsymbol{p}$. To see this we note that $[\boldsymbol{q}, \boldsymbol{p}]\subset \mathrm{nil}(\boldsymbol{h}'_M)\subset \mathrm{nil}(\boldsymbol{p})$ has no component in $\boldsymbol{R}jf_k$ as follows from [3; §6]. Hence, in particular, $\rho(f_k, [\boldsymbol{q}, \boldsymbol{p}])=0$ for all $k$. But then in the formula $(d/dt)\rho(e^{t\,\mathrm{ad}\, f_k}q, e^{t\,\mathrm{ad}\, f_k}p)=\rho(f_k, e^{t\,\mathrm{ad}\, f_k}[q, p])$ the right side vanishes, whence $\rho(q, e^{t\,\mathrm{ad}\, f_k}p)$ is constant in $t$ for all $k$ such that $f_k$ does not occur in $\boldsymbol{q}$. Thus $\boldsymbol{q}$ is perpendicular to all root spaces in $\boldsymbol{p}$ of the abelian family $(\mathrm{ad}\, f_k)$ for which the root is not zero. Therefore we have only to consider $\rho(\boldsymbol{q}, F_{kk})$ and $\rho(\boldsymbol{q}, F''_0)$, where $F''_0=F_0\cap\boldsymbol{p}$ and $F_{kk}\subset\boldsymbol{p}$. Here we can consider $A=\rho([q_1, q_2], f)$ where $q_1, q_2\in\boldsymbol{q}$, $f\in F_{kk}+F_0$. The closedness condition of $\rho$ shows that $A$ vanishes, since $[\boldsymbol{q}, F_{kk}+F''_0]=0$ if $F_{kk}\subset\boldsymbol{p}$. Thus we have shown

PROPOSITION. *The spaces $\boldsymbol{q}$ and $\boldsymbol{p}$ are perpendicular.*

**2.10.** Let $\boldsymbol{v}$ be the radical of $\boldsymbol{h}'_M$. Then, by the Radical Conjecture [6], we can assume that $j\boldsymbol{v}$ is a solvable subalgebra of $\boldsymbol{h}'_M$. Therefore, we can assume that $\boldsymbol{v}+j\boldsymbol{v}$ is a solvable, $j$-invariant subalgebra of $\boldsymbol{h}'_M$. We claim

THEOREM. (a) *The algebras $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular and $j$-invariant*
(b) $\boldsymbol{p}=\boldsymbol{v}+j\boldsymbol{v}+\boldsymbol{u}\cap\boldsymbol{p}$.

PROOF. Since $\boldsymbol{u}$ is $j$-invariant and center $(\boldsymbol{u})\subset\boldsymbol{k}$, we can assume that $\boldsymbol{u}\cap\boldsymbol{q}$ and $\boldsymbol{u}\cap\boldsymbol{p}$ are $j$-invariant. Now let $v\in\boldsymbol{v}$ and $jv=q+p$, where $q\in\boldsymbol{q}$, $p\in\boldsymbol{p}$. We know that $\boldsymbol{h}'_M$ generates a transitive group on $D$. The complex structure induces on $\boldsymbol{h}'_M$ a map $j'$ for which $j'x=jx \bmod (\boldsymbol{u})$ holds. Moreover, from [3; §7] we know that $j'\boldsymbol{q}\subset\boldsymbol{q}$, $j'\boldsymbol{p}\subset\boldsymbol{p}$ holds. Therefore $q\in\boldsymbol{u}\cap\boldsymbol{q}$. But then, in view of Proposition 2.9 we obtain $\rho(jq, q)=\rho(jq, jv-p)=\rho(q, v)-\rho(jq, p)=0$. Hence we can assume $jv\subset\boldsymbol{p}$. Now the remaining claims follow easily.

**2.11.** Before proving one of the main results of this paper we mention one more reduction.

Note that we can assume that $\tilde{F}_0 = F_0 \cap \boldsymbol{p}$ is $j$-invariant. Hence as in the very last part of the proof in **2.9** we see $\boldsymbol{p} = \tilde{\boldsymbol{p}} + \tilde{F}_0$, where $\tilde{\boldsymbol{p}}$ and $\tilde{F}_0$ are $j$-invariant and perpendicular subalgebras.

**2.12.** We have seen in Theorem 1.4 that $(\boldsymbol{h}_M, \boldsymbol{k}_M, j)$ is a $j$-algebra. This means that there exists some linear form $\omega$ on $\boldsymbol{h}_M$ such that $(\boldsymbol{h}_M, \boldsymbol{k}_M, j, d\omega)$ is a Kähler algebra. We would like to know, however, whether we can choose $\omega$ so that actually $\delta | \boldsymbol{h}_M = d\omega$, holds. This will follow from the following:

THEOREM. *Let $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$ be an effective Kähler algebra such that $(\boldsymbol{g}, \boldsymbol{k}, j)$ is a $j$-algebra. Then there exists some linear form $\omega: \boldsymbol{g} \to \boldsymbol{R}$ such that $\rho(x, y) = \omega([x, y])$ for all $x, y \in \boldsymbol{g}$.*

PROOF. Let $M$ denote the simply connected homogeneous Kähler manifold associated with $(\boldsymbol{g}, \boldsymbol{k}, j, \rho)$. It is not hard to verify that it suffices to prove the theorem for $(\boldsymbol{g}_M, \boldsymbol{k}_M, j, \rho)$. Using the orthogonal decompositions listed in **2.5** and **2.8** (note $\boldsymbol{a}_M = 0$ under our assumptions) we see that we have only to consider the case $\boldsymbol{g}_M = \boldsymbol{p}$, $F_0 = 0$. In this case $\boldsymbol{g}_M = \boldsymbol{g}_1 + \boldsymbol{g}_{1/2} + \boldsymbol{g}_0$ where the decomposition is relative to ad $je$, $e$ the maximal idempotent of $\boldsymbol{g}_M$. One shows as usual $\rho(je, \boldsymbol{g}_{1/2}) = 0$, $\rho(F_{ij}, F_{kk}) = 0$ for $i < j$, $\rho(f_i, F_{ii}) = 0$ and $(n+m)\rho(x_n, x_m) = \rho(je, [x_n, x_m])$ for $x_r \in \boldsymbol{g}_r$ and $n, m \in \{0, 1/2, 1\}$. Generalizing slightly a proof of Gindikin and Vinberg we define a linear map $\omega: \boldsymbol{g} \to \boldsymbol{R}$ by

(2.9.1) $\qquad \omega(x) = \rho(je, x) \quad$ for $x \in \boldsymbol{g}_1$

(2.9.1) $\qquad \omega(x) = 0 \quad$ for $x \in \boldsymbol{g}_{1/2}$

(2.9.1) $\qquad \omega(x) = \omega_0(x) \quad$ for $x \in \boldsymbol{g}_0$

where $\omega_0(\sum_{i \leq j} x_{ij}) = \omega_0(\sum x_{ii})$ and $\omega_0 | F_{ii}$ is defined so that $\rho(A, B) = \omega_0([A, B])$ for the reductive algebra $F_{ii}$ (such an $\omega_0$ exists since $\rho$ is closed and $\rho(f_i, F_{ii}) = 0$ holds). Now the claim follows by an easy calculation.

REMARK. The result above has been proved independently in a different way by K. Nakajima [11].

**2.13.** The last Theorem has several immediate algebraic and geometric consequences. The algebraic result is

COROLLARY 1. *For a homogeneous Kähler manifold $M$ and the corresponding Kähler algebra $(\boldsymbol{g}_M, \boldsymbol{k}_M, j, \rho)$ the following are equivalent*

(1) $(\boldsymbol{g}_M, \boldsymbol{k}_M, j, \rho)$ *is a j-algebra,*

(2) $\boldsymbol{a}_M=0$,

(3) $(\boldsymbol{g}_M, \boldsymbol{k}_M, j)$ *is a j-algebra*

(4) $\boldsymbol{g}_M$ *does not contain any abelian j-invariant subalgebra.*

REMARK. In [11] it was shown that this is also equivalent to

(5) *the canonical hermitian form is nondegenerate*

Though in the context of this paper the proof of this last equivalence is fairly straightforward we postpone it to a forthcoming publication where we will investigate a more general situation.

Geometrically one obtains

COROLLARY 2. *For a homogeneous Kähler manifold M the follhwing are equivalent*

(1) *The Kähler form of M is the differential of an* Aut $(M)$*-invariant* 1-*form.*

(2) *M does not contain any locally flat homogeneous complex submanifold,*

(3) *M is analytically the product of a bounded domain and a simply connected compact manifold.*

(4) *There exists some* Aut$(M)$*-invariant Kähler form on M which is the differential of an* Aut$(M)$*-invariant* 1-*form.*

(5) *The Ricci curvature of M is nondegenerate.*

PROOF. With $M$ we associate the Kähler algebra $(\boldsymbol{g}_M, \boldsymbol{k}_M, j, \rho)$.

(1)$\Rightarrow$(2). Since the Kähler form is the differential of an Aut$(U)$-invariant 1-form, we know $\rho=d\omega$, for some linear form $\omega$. Let $N\subset M$ be a locally flat homogeneous complex submanifold. Then there exists $X\in\boldsymbol{g}_M$, $X\notin\boldsymbol{k}_M$, such that $[jx, x]\in\boldsymbol{k}_M$. Therefore $\rho(jx, x)=d\omega([jx, x])=0$, whence $X\in\boldsymbol{k}_M$, a contradiction.

(2)$\Rightarrow$(3). From the Fundamental Conjecture for homogeneous Kähler manifolds we know that $M$ is analytically the product of a flat homogeneous Kähler manifold, a simply connected compact homogeneous Kähler manifold and a homogeneous bounded domain. By the assumption (2) the flat factor does not exist here.

(3)$\Rightarrow$(4). In view of (3), Theorem 1.4 implies $\boldsymbol{a}_M=0$. But then $(\boldsymbol{g}_M, \boldsymbol{k}_M, j)$ is a $j$-algebra, i. e. there exists some $\hat{\rho}=d\hat{\omega}$ such that $(\boldsymbol{g}_M, \boldsymbol{k}_M, j, \hat{\rho})$ is a Kähler algebra. This implies (4).

(4)$\Rightarrow$(1). This follows from Theorem 2.12. The fact that (5) is equivalent with the above four statements follows from (5) in Corollary 1 above.

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