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# ON ALMOST $M$-PROJECTIVES AND ALMOST $M$-INJECTIVES

Dedicated to Professor Tuyosi Oyama on his 60th birthday

By

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We have defined aconcept of almost $M$-projectives and almost $M$-injectives in [4] and [9], respectively. In the first section of this paper we give some relations among lifting modules, mutually almost relative projectivity and locally semi-$T$-nilpotency. After giving a criterion of mutually almost relative projectivity between two hollow modules in the second section, we give a characterization of lifting modules over a right artinian ring. Further we show a difference between $M$-projectives and almost $M$-projectives. Those dual properties are gives in the third and fourth sections with sketch of proofs.

We shall give several characterizations of right Nakayama (resp. right co-Nakayama) rings in terms of almost relative projectives (resp. almost relative injectives) in forthcoming papers (cf. [9]).

## 1. Almost projectives.

Throughout this paper $R$ is an associative ring with identity. Every module $M$ is a unitary right $R$-module. Let $M$ be an $R$-module and $K$ a submodule of $M$. If $M\neq M'+K$ for any proper submodule $M'$ of $M$, then $K$ is called a *small submodule* in $M$. If $K\cap K'\neq 0$ for every non-zero submodule $K'$ of $M$, we say that $K$ is an *essential submodule* of $M$. If every proper submodule of $M$ is always small in $M$, $M$ is called a *hollow module* and we dually call $M$ a *uniform module*, provided every non-zero submodule is essential in $M$. If $\mathrm{End}_R(M)$, the ring of endomorphisms of $M$, is a local ring, $M$ is called an *le module*. By $J(M)$ and $\mathrm{Soc}(M)$ we denote the *Jacobson radical* and the *socle* of $M$, respectively and $|M|$ is the *length* of $M$.

Following K. Oshiro [15] and [16] we define a lifting (resp. extending) module. If for any submodule $N$ of $M$, there exists a direct decomposition $M=M_1\oplus M_2$ such that

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($D_1$) $N\supset M_1$ and $N\cap M_2$ is small in $M_2$ (and hence in $M$)
(resp. ($C_1$) $M_1\supset N$ and $N$ is essential in $M_1$),

then $M$ is called a *lifting* (resp. *extending*) module. If $M$ is a lifting (resp. extending) module with $|M|<\infty$, $M$ is a direct sum of le hollow (resp. uniform) modules from the definition. Hence we shall study, in this paper, a lifting (resp. extending) module which is a direct sum of le and hollow (resp. uniform) modules.

We shall recall notations given in [9]. Let there be given a direct decomposition $M=M_1\oplus M_2$, and let $\pi_1: M\to M_1$ and $\pi_2: M\to M_2$ be the projectives. We shall use the following facts:

(i) Let $f: M_1\to M_2$ be a homomorphism. Define $M_1(f)=\{x+f(x)\,|\,x\in M_1\}$. Then $M_1(f)$ is a submodule of $M$ isomorphic to $M_1$ and $M=M_1(f)\oplus M_2$.

(ii) Let $N_1$, $N^1$, $N_2$ and $N^2$ be submodules of $M$ such that $N_i\subset N^i\subset M_i$ for $i=1, 2$ and let there exist an isomorphism $h: N^1/N_1\to N^2/N_2$. We shall often consider $h$ as a homomorphism $N^1\to N^2/N_2$ in the natural manner, so that $N_1$ is the kernel of $h$. Let $N=\{x+y\,|\,x\in N^1,\ y\in N^2$ and $y+N_2=h(x)\}$. Then, as is easily seen, $N$ is a submodule of $M$ and $\pi_1(N)=N^1$, $\pi_2(N)=N^2$. Further $N\cap M_i=N_i$ for $i=1, 2$. We shall denote this $N$ by

(1) $$N^1(h)N^2\,.$$

(iii) Let $N$ be any submodule of $M$. Put $N_{(i)}=M_i\cap N$ and $\pi_i(N)=N^i$ for $i=1, 2$. Then clearly $N_{(i)}\subset N^i\subset M_i$ for $i=1, 2$. Let $x\in N^1$. Then there is a $y\in N^2$ such that $x+y\in N$. Such a $y$ is not necessarily unique, but is unique modulo $N_2$. By associating $x+N_{(1)}$ with $y+N_{(2)}$, we have an isomorphism $h: N^1/N_{(1)}\to N^2/N_{(2)}$. It is obvious that $N=N^1(h)N^2$ in the sense in (ii).

First we shall decompose a proof of Azumaya's theorem [3] (see [2], Proposition 16.12) for an application to almost projectives, which is the dual observation of [4], Lemma 1.

Let $M_1$, $M_2$ and $N$ be $R$-modules. For a submodule $K$ of $M=M_1\oplus M_2$, take a diagram:

(2) $$\begin{array}{ccccc} M=M_1\oplus M_2 & \xrightarrow{\nu} & (M_1\oplus M_2)/K & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle h} & & \\ & & N & & \end{array}$$

Let $\pi_i: M\to M_i$ be the projection for $i=1, 2$. Put $K^i=\pi_i(K)$, $K_{(i)}=K\cap M_i$ and $K=K^1(f)K^2$ from (1), where $f: K^1/K_{(1)}\to K^2/K_{(2)}$. Since $K\subset K^1\oplus K^2$, there exists the natural epimorphism $\nu': M/K\to M/(K^1\oplus K^2)\approx M_1/K^1\oplus M_2/K^2$. By $\bar{\pi}_i$

we denote the projection onto $M_i/K^i$ in the last decomposition of $M/(K^1 \oplus K^2)$ and we put $\nu_i' = \bar{\pi}_i \nu'$ for $l=1, 2$. We note that $\nu' = \nu_1' + \nu_2'$ and $\nu_i' \nu | M_i$ is nothing but the natural epimorphism $\nu_i$ of $M_i$ onto $M_i/K^i$. Further $\ker \nu' = (K^1 \oplus K^2)/K \approx ((K^1 \oplus K^2)/(K_{(1)} \oplus K_{(2)}))/(K/(K_{(1)} \oplus K_{(2)}))$. While $(K^1 \oplus K^2)/(K_{(1)} \oplus K_{(2)}) \approx K^1/K_{(1)} \oplus K^2/K_{(2)}$ and $K/(K_{(1)} \oplus K_{(2)}) = (K^1(f)K^2)/(K_{(1)} \oplus K_{(2)}) = K^1/K_{(1)}(f) = K^2/K_{(2)}(f^{-1})$, (which is a graph in $(K^1 \oplus K^2)/(K_{(1)} \oplus K_{(2)}) \subset M_1/K_{(1)} \oplus M_2/K_{(2)}$). Hence $\ker \nu' \approx \approx K^1/K_{(1)} \approx K^2/K_{(2)}$. Let $g$ be the canonical monomorphism of $M_1/K_{(1)}$ into $M/K$. Then $g$ gives the above isomorphism: $K^1/K_{(1)} \to \ker \nu'$, and we obtain the commutative diagram:

$$\begin{array}{ccc} K^1/K_{(1)} & \xrightarrow{g|K^1/K_{(1)}} & \ker \nu' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ M_1/K_{(1)} & \xrightarrow{g} & M/K, \end{array}$$

where $i$ and $i'$ are inclusions.

From those observations we obtain two diagrams:

(3)
$$\begin{array}{ccccc} M_1 & \xrightarrow{\nu_1'\nu | M_1} & M_1/K^1 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \nu_1' h} & & \\ & & N, & & \end{array}$$

and

(3')
$$\begin{array}{ccccc} M_2 & \xrightarrow{\nu_2 \nu | M_2} & M_2/K^2 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \nu_2' h} & & \\ & & N. & & \end{array}$$

Here we assume that there exists $\tilde{h}_j: N \to M_j$ such that $(\nu_j' \nu | M_j)\tilde{h}_j = \nu_j' h$ for $j=1, 2$. Put $t = \nu(\tilde{h}_1 + \tilde{h}_2) - h: N \to M/K$. Then $\nu' t = \nu' \nu(\tilde{h}_1 + \tilde{h}_2) - \nu' h = \nu_1 h + \nu_2' h - \nu' h = (\nu' - \nu')h = 0$. Hence $t(N) \subset \ker \nu'$. Put $g' = (g|(K^1/K_{(1)}))^{-1}: \ker \nu' \to K^1/K_{(1)} \subset M_1/K_{(1)}$. Since $\nu(M_1) = g(M_1/K_{(1)})$, $g^{-1}$ exists on $\nu(M_1)$. Thus we obtain a new diagram:

(4)
$$\begin{array}{ccccc} M_1 & \xrightarrow{g^{-1}\nu | M_1} & M_1/K_1 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle g' t} & & \\ & & N. & & \end{array}$$

Finally we assume in (4) that there exists $h_1^*: N \to M_1$ such that $g^{-1}(\nu | M_1)h_1^* = g' t$, i.e. $(\nu | M_1)h_1^* = t$ by operating $g$. Then

$$h=\nu(\tilde{h}_1+\tilde{h}_2)-(\nu|M_1)h_1^*=\nu((\tilde{h}_1-h_1^*)+\tilde{h}_2) \quad \text{and}$$
(5)
$$(\tilde{h}_1-h_1^*)+\tilde{h}_2:\ N\longrightarrow M.$$

We recall the definition of almost $M$-projectives [9]. Let $M$ and $N$ be $R$-modules. For any exact sequence with $K$ a submodule of $M$:

$$\begin{tikzcd} M \arrow[r, "\nu"] & M/K \arrow[r] & 0 \\ & N \arrow[u, "h"'] & \end{tikzcd}$$

if either there exists $\tilde{h}: N\to M$ with $\nu\tilde{h}=h$ or there exist a non-zero direct summand $M_1$ of $M$ and $\tilde{h}: M_1\to N$ with $h\tilde{h}=\nu|M_1$, $N$ is called *almost $M$-projective* (if we always obtain the first half, we say $N$ is *$M$-projective* [3]).

We note the following fact:

(♯) When $N$ is almost $M$-projective and $M$ is indecomposable, if the $h$ in the above diagram is not an epimorphism, then there exists always an $\tilde{h}: N\to M$ with $\nu\tilde{h}=h$.

We frequently use this fact without any reference.

The following lemma is useful on almost projectives.

LEMMA 1. *Let $M_1, M_2, \cdots, M_n$ be hollow modules and $N$ an $R$-moddle. Assume that $N$ is almost $M_i$-projective for all $i$. Take a diagram with $K$ a submodule of $\sum\oplus M_i$:*

$$\begin{tikzcd} \sum_{i=1}^n\oplus M_i \arrow[r, "\nu"] & (\sum\oplus M_i)/K \arrow[r] & 0 \\ & N. \arrow[u, "h"'] & \end{tikzcd}$$

*If $h(N)$ is small in $(\sum\oplus M_i)/K$, $h$ is liftable to $\tilde{h}: N\to\sum\oplus M_i$, i.e. $h=\nu\tilde{h}$.*

PROOF. We shall prove the lemma by induction on $n$. If $n=1$, it is clear from the definition. We assume that the lemma holds true for $M^*=\sum_{j=2}^n\oplus M_j$ and put $M=M_1\oplus M^*$. Let $\pi_i$ be the projection of $M=\sum_{j=1}^n\oplus M_j$ onto $M_i$. Assume first that $\pi_1(K)$ $(=K^1)=M_1$. Put $\pi^*=\sum_{j\geq 2}\pi_j: M\to M^*$, $K^*=\pi^*(K)$, $K_{(1)}=K\cap M_1$ and $K_{(*)}=K\cap M^*$. Further set $\overline{M}=M/(K_{(1)}\oplus K_{(*)})\supset\overline{K}=K/(K_{(1)}\oplus K_{(*)})$. Since $K=K^1(h)K^*$ with $h: K^1/K_{(1)}\approx K^*/K_{(*)}$ from (1), we obtain $\overline{K}\subset M_1/K_{(1)}\oplus M^*/K_{(*)}=(M_1/K_{(1)})(h)\oplus M^*/K_{(*)}=\overline{M}$ and $\overline{K}=(M_1/K_{(1)})(h)$. Hence $M^*/K_{(*)}\approx\overline{M}/\overline{K}\approx M/K$, and by $\varphi$ we denote this isomorphism of $M^*/K_{(*)}$ onto $M/K$. Accordingly we have a commutative diagram:

$$\begin{tikzcd} M^* \arrow[r, "\nu^*"] \arrow[d, "i"] & M^*/K_{(*)} \arrow[r] \arrow[d, "\varphi"] & 0 \\ M \arrow[r] & M/K \arrow[r] & 0 \\ & N. \arrow[u, "h"] & \end{tikzcd}$$

Since $\varphi$ is an isomorphism, by assumptions there exists $\tilde{h}^*: N \to M^*$ such that $\nu^*\tilde{h}^* = \varphi^{-1}h$, and so $\nu(i\tilde{h}^*) = \varphi\nu^*h^* = h$. Hence $i\tilde{h}^*: N \to M$ is the desired map. Thus we can assume that $K^1 \neq M_1$. Since $h(N)$ is small in $M/K$, for $\nu_i'h$ in the diagrams (3) and (3′), $\nu_1'h(N)$ and $\nu_2'h(N)$ are small in $M_1/K^1$ and $M^*/K^*$, respectively. Hence by assumption and induction hypothesis, there exist $\tilde{h}_1: N \to M_1$ and $\tilde{h}^*: N \to M^*$, which make the diagrams (3) and (3′) commutative. Let $t$ and $g'$ be the mappings defined after (3′). Since $M_1$ is indecomposable, $g't(N) \subset K^1/K_{(1)}$ and $K^1 \neq M_1$, there exists $\tilde{h}_1': N \to M_1$ which makes the diagram (4) commutative. Therefore $h$ is liftable to $\tilde{h}: N \to \sum \oplus M_i$ as is shown in (5).

By definition we have

LEMMA 2. *Let $\{M_\alpha\}_I$ be a set of almost $M$-projectives for a fixed $R$-module $M$. Then $\sum_I \oplus M_\alpha$ is almost $M$-projective.*

We have given some relationships between lifting modules and almost projectives in [9]. We give here a simpler relation for a finite direct sum. This is dual to [14], Theorem 12, however the proof is not, because we used injective hulls in [14], but we can not take here projective covers.

THEOREM 1. *Let $\{M_i\}_{i=1}^n$ be a set of* le *and hollow modules. Then the following are equivalent:*

1) *$M = \sum_{i=1}^n \oplus M_i$ is lifting.*

2) *$M_i$ is almost $M_j$-projective for any $i \neq j$.*

3) *For any subset $J$ in $I = \{1, 2, \cdots, n\}$ $\sum_J \oplus M_j$ is almost $\sum_{I-J} \oplus M_i$-projective.*

PROOF. 1)→3)→2). This is clear from the definition of almost projectives, Lemma 2 and [9], Theorem 1″.

2)→1). If we can show that every non small submodule $N$ in $M$ contains a non-zero direct summand of $M$ (i.e., $M$ satisfies $(1-D_1)$ in [9]), then $M$ is lifting by [9], Theorem 1″. In order to get the above fact, we shall show

(6) every non small submodule in $M$ contained in $M_1'\oplus M_2'\oplus\cdots M_k'\oplus T_{k+1}\oplus\cdots\oplus T_n$ contains a non-zero direct summand of $M$, where $M=\sum_{i=1}^n\oplus M_i'$ is any direct decomposition into indecomposable modules $M_i'$ $(\approx M_i)$, and the $T_i$ are small in $M_i'$ for $i\geqq k+1$.

We may assume $M_i'=M_i$ in (6). If (6) is true for all $k$, taking $k=n+1$ $(M_{n+1}'=T_{n+1}=0)$, we are done. Consider (6) with $k=1$. Let $N$ be a non-small submodule contained in $M_1\oplus\sum_{i=2}^n\oplus T_i$, and put $M^*=M_2\oplus M_3\oplus\cdots\oplus M_n$. Let $\pi_1: M\to M_1$ and $\pi^*: M\to M^*$ be the projections. Since $N$ is not small in $M$ and the $T_i$ is small in $M_i$ for all $i\geqq 2$, $\pi_1(N)=N^1=M_1$. Then from (1) $N=M_1(h)N^*$, where $N^*=\pi^*(N)$, $N_{(1)}=N\cap M_1$, $N_{(*)}=N\cap M^*$, and $h: M_1/N_{(1)}\approx N^*/N_{(*)}$. Since $N^*\subset\sum_{i=2}^n\oplus T_i$, $N^*$ is small in $M^*$ and hence $N^*/N_{(*)}$ is small in $M_*/N_{(*)}$. From those datas we obtain the diagram:

$$\begin{array}{ccccc}
M^*=M_2\oplus M_3\oplus\cdots\oplus M_n & \xrightarrow{\nu} & M^*/N_{(*)} & \longrightarrow & 0\\
 & & \uparrow h & & \\
 & & M_1/N_{(1)} & & \\
 & & \uparrow \nu_1 & & \\
 & & M_1 & &
\end{array}$$

Since $M_1$ is almost $M_j$-projective for all $j\geqq 2$ by assumption and $h(M_1/N_{(1)})=N^*/N_{(*)}$ is small in $M^*/N_{(*)}$, there exists $\tilde{h}: N\to M^*$ with $\nu\tilde{h}=h\nu_1$ by Lemma 1. Hence $N$ contains $M_1(\tilde{h})$ a direct summand of $M$ (consider $M/(N_{(1)}\oplus N_{(*)})\supset N/(N_{(1)}\oplus N_{(*)})$, cf. the proof of [9], Theorem 1). Assume that (6) is true for all $k'\leqq k$ and let $N\subset M_1\oplus\cdots\oplus M_{k+1}\oplus T_{k+2}\oplus\cdots\oplus T_n$ $(k\geqq 1)$. We may assume $\pi_1(N)=M_1$. Let $\rho$ be the projection of $M$ onto $M^{**}=M_1\oplus M_2$. Since $\pi_1(N)=M_1$, $\rho(N)$ is not small in $M^{**}$. Then $M^{**}$ being lifting by [9], Theorem 1″, $M^{**}=L_1\oplus L_2$ and $\rho(N)=L_1\oplus(L_2\cap\rho(N))$ with $L_2\cap\rho(N)$ small in $M^{**}$. Since $L_i$ is a direct sum of at most two direct summands, we put $L_1=M_1''\oplus M_2''$ $(M_1''\neq 0)$, $L_2=M_3''$, where $M_k''\approx$one of $\{M_1, M_2, (0)\}$. Then $M=M^{**}\oplus M_3\oplus\cdots\oplus M_n\supset M_1''\oplus M_2''\oplus M_3''\oplus M_3\oplus\cdots\oplus M_{k+1}\oplus T_{k+2}\oplus\cdots\oplus T_n\supset N$. If $M_2''=0$, i.e., $L_1=M_1''$ and $L_2=M_3''$, $N$ satisfies (6) by induction, since $\rho(N)=M_1''\oplus(M_3''\cap\rho(N))$ and $M_3''\cap\rho(N)$ is small in $M_3''$. Assume $M_2''\neq 0$ (and hence $M_3''=0$) i.e., $\rho(N)=M_1''\oplus M_2''=M^{**}$. Let $\pi_2''$ be the projection of $M$ onto $M_2''$. Since $\rho(N)=M^{**}$, $N\cap\pi_2''^{-1}(0)$ is not small in $M$ and $N\cap\pi_2''^{-1}(0)\subset M_1''\oplus 0\oplus M_3\oplus\cdots\oplus M_{k+1}\oplus T_{k+2}\oplus\cdots\oplus T_n$. Hence $(N\supset)$ $N\cap\pi_2''^{-1}(0)$ contains a non zero direct summand of $M$ by assumption of induction. Therefore (6) is true for any $k$, and so $N$ always contains a non-zero direct summand of $M$.

THEOREM 1 is not true if $\{M_a\}_I$ is an infinite set, even though $\{M_a\}_I$ is locally semi $T$-nilpotent, which is given in [7], p. 174, and briefly lsTn (see example before Theorem 2 below). In [9], Theorem 1$''$ the locally semi-$T$-nilpotency is important. Concerning this fact we have the following lemma. In the proof we make use of certain factor categories given in [7]. We do not know a module theoretical proof.

LEMMA 3. *Let* $\{M_a\}_I$ *be a set of* le *modules. If* $M=\sum_I \oplus M_a$ *is lifting, then* $\{M_a\}_I$ *is* lsTn.

PROOF. From the definition of lsTn, we way assume that $I$ is an infinite set. Let $M_0=\sum_{i=1}\oplus M_i$ and $\{f_i : M_i \to M_{i+1}\}$ a set of non-isomorphisms and $M_i'=M_i(f_i)\subset M_i\oplus M_{i+1}$. Since $M_0$ is lifting, for $M_*=\sum_{i=1}^{\infty}\oplus M_i'$, $M_0=T_1\oplus T_2$; $M_*=T_1\oplus M_*\cap T_2$ and $M_*\cap T_2$ is small in $M_*$. Here we shall apply some theorems on factor categories $\boldsymbol{A}/\boldsymbol{J}'$ induced from le modules (see [7], Chapters 6 and 7), and use the same terminologies given there. First we note that $M_*$ is also a direct sum of le modules, i.e., $M_*\in \boldsymbol{A}$. Let $T_i^*$ and $(M_*\cap T_2)^*$ be full submodules in $T_i$ and $(M_*\cap T_2)$, respectively ([7], p. 169). Let $i_{M_*}$, $i_{T_i}$ and $i_{M_*\cap T_2}$ be inclusions in $M$, Since $M_*\cap T_2$ is small in $M_0$, $\bar{i}_{M_*\cap T_2}=0$ by the definition of $\boldsymbol{J}'$ in [7], p. 148. Further $\bar{i}_{M_*}$ is an isomorphism by [7], Theorem 7.3.13, and $\bar{i}_{M_*}=\bar{i}_{T_1}+\bar{i}_{M_*\cap T_2}=\bar{i}_{T_1}$. On the other hand, $\bar{i}_{M_0}=\bar{i}_{T_1}+\bar{i}_{T_2}$. Hence $\bar{i}_{T_2}=0$, since $\bar{i}_{T_1}=\bar{i}_{M_*}$ is an isomorphism and $\bar{i}_{T_1}$, $\bar{i}_{T_2}$ are mutually orthogonal idempotents, and so $T_2=0$ by [7], Theorem 7.1.2. According $M_0=M_*$. Therefore $\{M_a\}_I$ is lsTn by [7]. Theorem 7.2.7.

THEOREM 2. *Let* $\{M_a\}_I$ *be a set of* le *hollow and cyclic modules. Then the following are equivalent*:

1) $M=\sum_I\oplus M_a$ *is lifting.*

2) $M_a$ *is almost* $M_b$*-projective for any* $a\neq b$ *and* $\{M_a\}_I$ *is* lsTn.

3) $\sum_J\oplus M_a$, *is almost* $\sum_{I-J}\oplus M_b$, *-projective for any subset* $J$ *in* $I$ *and* $\{M\}_I$ *is* lsTn. (*cf. Theorem* 4 *below.*)

PROOF. This is clear from Theorem 1, Lemma 2 and 3 and [9], Theorem 1$''$.

We prepare the following lemma for an example below.

LEMMA 4. *Let* $M$ *be an* le *and hollow module. If any infinite direct sum of copies of* $M$ *is always lifting,* $M$ *is cyclic.*

PROOF. Assume that $M$ is not cyclic. Then $xR$ is a small submodule in $M$ for any $x$ in $M$. Put $D=\sum_{x\in M}\oplus M_x$ $(M_x=M)$ and $S=\sum_x\oplus xR$, Taking an epimorphism $f: D\to M$ such that $f|M_x=1_M$, we know that $S$ is not small in $M$. Hence $M$ is not lifting from [9], Corollary 2.

Let $Z$ be the ring of integers. Then $E(Z/p)$, injective hull of $Z/p$ ($p$ is prime) is almost $E(Z/p)$-projective (see [12]). However $\sum_{i=1}^{\infty}\oplus E_i$ $(E_i=E(Z/p))$ is not lifting by Lemma 4, even though $\{M_i=E(Z/p)\}$ is lsTn. On the other hand $\sum_p\oplus E(Z/p)$ is lifting.

## 2. Lifting property.

First we shall give a relationship between lifting module and lifting property.

Let $X\supset Y$ be $R$-modules and $\nu: X\to X/Y$ the natural epimorphism. If, for a direct summand $T$ of $X/Y$, there exists a direct summand $T_0$ of $X$ such that $T=\nu(T_0)$, we say that $T$ is lifted to $T_0$. If every direct summand of any factor module $X/Y'$ is lifted, we say that $X$ has the *lifting property of direct summands modulo submodules.* If, for any submodule $Y$ of $X$ and for any direct decomposition $X/Y=\sum\oplus T_i$, there exists a direct decomposition $X=\sum\oplus T_i'$ with $\nu(T_i')=T_i$ for all $i$, we say that $X$ has the *lifting property of direct sums modulo submodules.*

We take a direct decomposition $M=\sum\oplus M_i$. For a submodule $N_i$ of $M_i$ we call $\sum\oplus N_i$ a *standard submodule* of $M$ with respect to this decomposition $\sum\oplus M_i$. If we say a standard submodule in the following, that is a standard submodule with respect to decomposition into indecomposable modules. We note that $J(X)$ and $\mathrm{Soc}(X)$ are always standard submodules with respect to any decompositions.

PROPOSITION 1. *Let $\{M_\alpha\}_I$ be a set of hollow and* le *modules and $M=\sum_I\oplus M_\alpha$. Assume that $\{M_\alpha\}_I$ is* lsTn. *Then the following are equivalent:*

1) *$M$ is lifting.*

2) *$M$ has the lifting property of direct summands modulo submodules (cf.* [15], §4).

PROOF. 1)→2) (The argument below is valid for any lifting module). Let $N$ be a submodule of $M$ and $T$ a direct summand of $M/N$. Let $\nu: M\to M/N$ be the natural epimorphism of $M$. We apply $(D_1)$ to the inverse image $T_0$ of $T$. Then there exists a decomposition $M=M'\oplus M''$ such that $T_0=M'\oplus T_0\cap M''$

and $T_0\cap M''$ is small in $M$. Then $T=\nu(T_0)=\nu(M')+\nu(T_0\cap M'')$. Since $T_0\cap M''$ is small in $M$ and $T$ is a direct summand of $M/N$, $\nu(T_0\cap M'')$ is small in $T$. Hence $T=\nu(M')$.

2)→1). Let $T_0$ be a non-small submodule in $M$. Then there exists a submodule $X$ ($\neq M$) of $M$ such that $M=T_0+X$. Now $M/(T_0\cap X)=T_0/(T_0\cap X)\oplus X/(T_0\cap X)$ and $T_0/(T_0\cap X)\neq 0$. Since $M$ has the lifting property, $M=M'\oplus M''$ and $(M'+T_0\cap X)/(T_0\cap X)=T_0/(T_0\cap X)$, and so $0\neq M'\subset T_0$. Therefore $M$ is lifting by [9], Theorem 1$''$.

The following corollary shows us a difference between $M$-projectives and almost $M$-projectives.

COROLLARY. *Assume* $|I|=n<\infty$ *and* $|M_i|<\infty$ *in the above. Then the following two conditions are equivalent*:

1) *$M_i$ is almost $M_j$-projective for all $i\neq j$.*

2) *$M$ has the lifting property of any indecomposable direct summands modulo standard submodules.*

*Similarly the following two conditions are equivalent*:

3) *$M_i$ is $M_j$-projective for all $i\neq j$.*

4) *$M$ has the lifting property of direct sums modulo standard submodules,* (cf. [15], §4).

PROOF. 1)→2). This is clear from Theorem 1 and Proposition 1.

2)→1). Put $M^*=M_1\oplus M_2$. We can show by routine work that $M^*$ has the lifting property of indecomposable direct summands modulo standard submodules, since so does $M$. Let $X$ be a non-small submodule of $M^*$. Then $\pi_1|X$ or $\pi_2|X$ is an epimorphism, where $\pi_i\colon M^*\to M_i$ is the projection, say $\pi_1|X$. Then $X/(X_{(1)}\oplus X_{(2)})$ is a graph of $M_1/X_{(1)}$ in $M^*/(X_{(1)}\oplus X_{(2)})$ provided $X_{(1)}\neq M_1$, where $X_{(i)}=X\cap M_i$, and hence a direct summand of $M^*/(X_{(1)}\oplus X_{(2)})$. Further $X/(X_{(1)}\oplus X_{(2)})$ is indecomposable, and $X/(X_{(1)}\oplus X_{(2)})$ is lifted to a direct summand $X'$ of $M^*$ by assumption. Hence $X'\subset X$. If $X_{(1)}=M_1$, $M_1\subset X$. Accordingly $M^*$ is lifting, and hence $M_1$ and $M_2$ are mutually almost relative projective by Theorem 1.

3)→4) First assume that $M_1$, $M_2$ are mutually relative projective and $M=M_1\oplus M_2$. Put $\tilde{M}=M/(N_1\oplus N_2)$. Let $C$ be any submodule in $M$. We denote $(C+(N_1\oplus N_2))/(N_1\oplus N_2)$ by $\tilde{C}$ ($\subset\tilde{M}$). It is clear that $\tilde{M}=\tilde{M}_1\oplus\tilde{M}_2$ and $\tilde{M}_i\approx M_i/N_i$. Let $\tilde{M}=A\oplus B$. We note that if an $R$-module $L$ is a finite direct sum of le modules $L_i$, every non-zero indecomposable direct summand of $L$ is given by a graph of some $L_i$ (see [7], Proposition 6.3.3). Since $M_i/N_i$ is an le module by assumption, we can assume $A=\tilde{M}_1(\tilde{f}_1)$; $\tilde{f}_1:\tilde{M}_1\to\tilde{M}_2$. Then there exists a decomposition $M=M_1(f_1)\oplus M_2$, where $f_1$ is a lifted one of $\tilde{f}_1$. Clearly $\widetilde{M_1(f_1)}=A$. Since $\tilde{M}$ $(=\tilde{M}_1(\tilde{f}_1)\oplus\tilde{M}_2)=A\oplus\tilde{M}_2=A\oplus B$, $B=\tilde{M}_2(\tilde{f}_2)$; $\tilde{f}_2:\tilde{M}_2\to A=\widetilde{M_1(f_1)}\approx M_1(f_1)/(M_1(f_1)\cap(N_1\oplus N_2))$, (take the projection of $\tilde{M}$ onto $\tilde{M}_2$). Hence there exists $f_2:M_2\to M_1(f_1)$ and $\widetilde{M_2(f_2)}=B$. Therefore $M=M_1(f_1)\oplus M_2(f_2)$ is the desired decomposition. Finally we study in a general case. Let $\tilde{M}=\sum_{i=1}^n\oplus M_i/N_i=\sum_{i=1}^m\oplus A_i$. Since $M_i/N_i$ is an le module, the $A_i$ is a direct sum of hollow modules by Krull-Schmidt's theorem. Hence we may assume that all $A_i$ are hollow. Without loss of generality we can put $A_1=\tilde{M}_1(\tilde{f}_1)$; $\tilde{f}_1:\tilde{M}_1\to\sum_{i\geq 2}\oplus\tilde{M}_i$, and $\tilde{M}=A_1\oplus\sum_{i\geq 2}\oplus\tilde{M}_i=A_1\oplus A_2\oplus\cdots\oplus A_n$. Let $\rho$ be the projection of $\tilde{M}$ onto $\sum_{i\geq 2}\oplus\tilde{M}_i$ on the first decomposition of the above. Since $\rho|(A_2\oplus\cdots\oplus A_n)$ is an isomorphism onto $\sum_{i\geq 2}\oplus\tilde{M}_i$, there exists, from the above remark, a projection $\theta_j:\sum_{i\geq 2}\oplus\tilde{M}_i\to\tilde{M}_j$ such that $\theta_j\rho|A_2$ is an isomorphism, say $j=2$, whence $A_2=\tilde{M}_2(\tilde{f}_2)$; $\tilde{f}_2:\tilde{M}_2\to\tilde{M}_1(\tilde{f}_1)\oplus\tilde{M}_3\oplus\cdots\oplus\tilde{M}_n$. Similarly $A_i=\tilde{M}_i(\tilde{f}_i)$ with $\tilde{f}_i:\tilde{M}_i\to\tilde{M}_1(\tilde{f}_1)\oplus\cdots\oplus\tilde{M}_{i-1}(\tilde{f}_{i-1})\oplus\tilde{M}_i\oplus\cdots\oplus\tilde{M}_n$. By virtue of Azumaya's theorem [3] we can apply the initial argument to those decompositions and obtain finally a lifted direct decomposition $M=\sum\oplus M_i(f_i)$.

4)→3) It is clear that if $M=\sum_{i=1}^n\oplus M_i$ satisfies 4), then so does $M_1\oplus M_2$. Let $\tilde{f}:M_1\to M_2/N_2$ be a homomorphism $(N_2\subset M_2)$. Then $\tilde{M}=M_1\oplus M_2/N_2=M_1(\tilde{f})\oplus M_2/N_2$ is lifted to $M=T_1\oplus T_2$ such that $\tilde{T}_1=M_1(\tilde{f})$ and $\tilde{T}_2=M_2/N_2$. Let $\rho:M\to T_1$ and $\pi_2:M\to M_2$ be the projections. Then $\pi_2\rho|M_1$ is a lifted one of $\tilde{f}$ (see the proof of [8], Theorem 2). Hence $M_1$ is $M_2$-projective.

Next we shall give some criterion of almost relative projectivity for two hollow (local) modules. Let $e$ be a local idempotent, i.e., $eR$ is hollow. Let $A$ and $B$ be $R$-submodules in $eR$. We note that any element in $\mathrm{Hom}_R(eR/A, eR/B)$ is given by $x_l$ $(x\in eRe)$, the left-sided multiplication of $x$.

From the definition and a fact: $(eR/A)/J(eR/eA))\approx eR/eJ$ we have

LEMMA 5. *Assume that $eR/A$ is almost $eR/B$-projective. Then for any unit $u$ in $eRe$ there exists a unit $x$ such that $xA\subset B$ and $x\equiv u$* (mod $eJe$) *or $xB\subset A$ and $u^{-1}\equiv x$* (mod $eJe$).

LEMMA 6. *Let $M$ be an indecomposable $R$-module and assume that $eR/A$ is almost $M$-projective, and take a non-epic homomorphism $f$ of $eR$ to $M$. Then $f(A)$ $=0$* ([11]; [7], Theorem 5.4.11).

PROOF. Consider a derived diagram from $f$:

$$\begin{array}{ccc} M \longrightarrow & M/f(A) & \longrightarrow 0 \\ & \uparrow \bar{f} & \\ & eR/A\,. & \end{array}$$

Since $\bar{f}$ is not epic, $\tilde{h}$ is same. Hence there exists $h: eR/A \to M$ with $\nu\tilde{h}=\bar{f}$ by assumption. Let $\rho: eR \to eR/A$ be the natural epimorphism and put $h=\tilde{h}\rho: eR \to M$. Since $\nu\tilde{h}=\bar{f}$,

$$\nu f(e)=\bar{f}(e+A)=\nu\tilde{h}(e+A)=\nu\tilde{h}\rho(e)=\nu h(e)\,,$$

Hence

(7) $$f(e)-h(e)=f(a) \quad \text{for some } a \text{ in } A\,.$$

Now $0=h(a)=h(e)a=f(a)-f(a)a=f(a)(1-a)$ from (7). Hence, $f(a)=0$ for $a \in A \subset eJ$, and so $f(A)=f(e)A=h(e)A=h(A)=0$ from (7).

PROPOSITION 2. *Let $e$ and $e'$ be local idempotents. Then*

1) *$eR/A$ is $e'R/B$-projective if and only if $e'ReA \subset B$. If $e \not\approx e'$, $eR/A$ is $e'R/B$-projective if and only if $eR/A$ is almost $e'R/B$-projective.*

2) *If $eR/A$ is almost $eR/B$-projective, $eJeA \subset B$.*

3) *$eR/A$ and $eR/B$ are mutually almost relative projective if and only if $eJeA \subset B$, $eJeB \subset A$ and for any unit element $u$ in $eRe$, $uA \subset B$ or $B \subset uA$. In particular $A \subset B$ or $B \subset A$.*

PROOF. 1) is clear from [1], p. 22, Exercise 4 and 2) is clear from Lemma 6.

3) (This is the same argument given in [10]). Assume that $eR/A$ and $eR/B$ are mutually almost relative projective. Then $eJeA \subset B$ and $eJB \subset A$ from 2). First assume that $eR/B$ is almost $eR/A$-projective. Let $u$ be any unit in $eRe$. Then by Lemma 5 there exists $j$ in $eJe$ (resp. $j'$) such that

a) $(u+j)A \subset B$ or b) $(u^{-1}+j')B \subset A$.

a): $uA=((u+j)-j)A \subset (u+j)A+jA \subset B$ since $eJeA \subset B$. We obtain similarly $u^{-1}B \subset A$ in case b).

The converse is clear from definition and the initial remark before Lemma 5.

Let $R$ be a right artinian (basic) ring and $\{e_i\}_{i=1}^n$ a complete set of mutually orthogonal primitive idempotents. Then every hollow module is of a form $e_iR/A$. Take an $R$-module $M$ which is a direct sum of hollow modules:

(8) $$M=\sum_i \sum_{n(ij) \in I_i} \oplus (e_iR/A_{ij})^{(n(ij))}\,;\ e_iR/A_{ij} \not\approx e_{i'}R/A_{i'j'} \text{ if } (i,j) \neq (i'j') \text{ (and } n(ij)\neq 0, \text{ which may be infinite, for all } i \text{ and } j),$$

where $K^{(n(ij))}$ is the direct sum of $n(ij)$-copies of $K$.

If $M$ is lifting, then from Theorem 2 and Proposition 2, we obtain,

i) $|I_i|=n_i<\infty$ for all $i$.

After changing induces

ii) If $n_i\geqq 2$

$$e_iR\supset A_{i1}\supset R_iA_{i2}\supset A_{i2}\supset\cdots\supset R_iA_{in_i}\supset A_{in_i}\supset \tag{9}$$
$$\textstyle\sum_{k=1}^{n} e_iJe_kA_{k1}, \qquad \text{where } R_i=e_iRe_i\,.$$

If $n_i=1$, $e_iR\supset A_{i1}\supset\sum_{k\neq i}e_iJe_kA_{k1}$.

iii) If $n(ij)\geqq 2$, $A_{ij}$ is characteristic.

Thus we obtain from Theorem 2 and [8], Corollary to Theorem 4

THEOREM 3. *Let $R$ be a right artinian ring and $M$ an $R$-module. Then the following are equivalent*:

1) *$M$ is lifting.*

2) *$M$ is a direct sum of hollow modules as in* (8), *which satisfy* (9).

### 3. Almost injectives.

Following [4] we recall the definition of almost $V$-injectives and study some properties of them.

Let $V$ and $U$ be $R$-modules and $V\supset V'$. Consider the following diagram with $i$ the inclusion and two conditions 1) and 2):

$$\begin{array}{ccccc} 0 & \longrightarrow & V' & \xrightarrow{i} & V \\ & & \big\downarrow h & & \\ & & U & & \end{array}$$

1) There exists $\tilde{h}: V\to U$ such that $\tilde{h}i=h$ or

2) There exist a non-zero direct summand $V_0$ of $V$ and $\tilde{h}: U\to V_0$ such that $\tilde{h}h=\pi i$, where $\pi: V\to V_0$ is the projection of $V$ onto $V_0$. $U$ is called *almost $V$-injective* if the above 1) or 2) holds for any submodule $V'$ of $V$ and any $h: V'\to U$ ($U$ is called *$M$-injective* if we have only 1) [3]).

The following lemma is dual to a special case of Theorem 1.

LEMMA 8. *Let $U_1$ and $U_2$ be* le *and uniform modules and $U=U_1\oplus U_2$. Then the following are equivalent*:

1) *$U$ is extending.*

2) *$U_1$ and $U_2$ are mutually almost relative injective.*

PROOF. 1)→2). Let $V$ be a submodule in $U=U_1\oplus U_2$. We may assume that $V$ is uniform. Let $\pi_i$ be the projection of $U$ onto $U_i$. Since $V$ is uniform, $V=U_i'(f_i)$ ($i=1$ or 2), where $U_i'=\pi_i(V)$ and $f_i: U_i'\to U_j'$ ($j\neq i$). Assume $V=U_1'(f_1)$ and take a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & U_1' & \xrightarrow{i} & U_1 \\ & & \downarrow{\scriptstyle f_1} & & \\ & & U_2 & & \end{array}$$

Then since the $U_i$ are indecomposable, there exists $\tilde{f}_1: U_1\to U_2$ or $U_2\to U_1$ with $\tilde{f}_1 f_1=i$ or $\tilde{f}_1 i=f_1$ by 2). Hence $V=U_1'(f_1)\subset U_1(\tilde{f}_1)$ or $V\subset U_2(\tilde{f}_1)$, which is a direct summand of $U$.

1)→2). Consider the above diagram and define $U'=U_1'(f_1)$ in $U_1\oplus U_2$. Since $U'$ is uniform, there exists a decomposition $U=V_1\oplus V_2$ and $V_1\supset U'$. Since $V_1$ has the exchange property, $U=V_1\oplus U_1$ or $=V_1\oplus U_2$. If the latter case occurs, $\tilde{h}=\pi_2'|U_1$ is a desired homomorphism, where $\pi_2': U\to U_2$. We obtain a similar result for the former (note, in this case, that $f_1$ is a monomorphism).

The following theorem is the dual to Theorem 1, which is essentially given in [14].

THEOREM 4. *Let $\{U_a\}_I$ be a set of* le *uniform modules and $U=\sum_I\oplus U_a$. Assume that $\{U_a\}_I$ is* lsTn. *Then the following are equivalent:*

1) *$U$ is extending.*

2) *$U_a$ is almost $U_b$-injective for all $a\neq b$.*

PROOF. 1)→2). It is clear from Lemma 8.

2)→1). (Essentially due to [14]) $U=\sum_I\oplus U_a$ satisfies $(1-C_1)$ (i.e., $N$ is uniform in $C_1$) by Lemma 8 and [14], Lemma 11, and so every closed submodule $A$ in $U$ contains a non-zero indecomposable direct summand $X$ of $U$ by [14], Proposition 6. Hence we can define a non-empty set $\boldsymbol{F}$ of direct sums of uniform modules in $U$ as follows: $\boldsymbol{F}=\{\sum_{c'}\oplus X_{c'}|\subset A,\ X_{c'}$ is uniform and $\sum_{c'}\oplus X_{c'}$ is a locally direct summand of $U\}$. We can find a maximal member $\sum_c\oplus X_c$ in $\boldsymbol{F}$ by Zorn's lemma. Since $\{U_a\}$ is lsTn, $\sum_c\oplus X_c$ is a direct summand of $U$ by [7], Theorem 7.3.15, say $U=(\sum_c\oplus X_c)\oplus U'$ and $A=(\sum_c\oplus X_c)\oplus U'\cap A$. It is clear that $U'\cap A$ is also closed in $U$. Hence $U'\cap A=0$ by the maximality of $\sum_c\oplus X_c$. Therefore $U$ is extending.

We consider a result similar to Lemma 3 for extending modules.

PROPOSITION 3. *Let $U=\sum\oplus U_\alpha$ be as above. Assume that $U$ is extending. Then there do not exist any infinite sets* $\{U_1 \xrightarrow{f_1} U_2 \xrightarrow{f_2} U_n \xrightarrow{f_n} \cdots$; *the $f_i$ are monomorphisms but not isomorphisms*$\}$.

PROOF. Let $\{f_i : U_i \to U_{i+1}\}$ be a set of non-isomorphisms and put $U^*= \sum\oplus U_i(f_i) \subset \sum\oplus U_i$. Then we obtain a decomposition $U'$ $(=\sum\oplus U_i)=X\oplus Y$ and $U^* \subset' X$, i.e. $U^*$ is essential in $X$. Since $\bar{i}_* : U^* \to U'$ is an isomorphism in $\boldsymbol{A}/\boldsymbol{J}'$, $Y=0$ (see the proof of Lemma 3). Hence $U^* \subset' U'$, and so $U_1 \cap U^* \neq 0$. If we use this argument for the case where all $f_i$ are monomorphisms, we know that $\{f_i\}$ must be finite.

EXAMPLE. $R_1$ (resp. $R_2$) is the ring of upper (lower) triangular matrices over a field $K$ with infinite degree. Let $e_i=e_{ii}$ be matrix units. Then $e_k R_i$ is almost $e_s R_i$-projective and almost $e_s R_i$-injective for any $k$, $s$ and a fixed $i=1$ or 2, and further $\sum_k \oplus e_k R_1$ is lifting and extending by Theorems 2 and 4. On the other hand $e_k R_2$ is almost $\sum_{j\neq k}\oplus e_j R_2$-projective and almost $\sum_{j\neq k}\oplus e_j R_2$-injective (cf. [4], Theorem) for all $k$, however $\sum_i \oplus e_i R_2$ is neither lifting nor extending by Lemma 3 and Proposition 3, since we have an infinite chain of submodules; $e_1R_2 \subset e_2R_2 \subset \cdots \subset e_nR_2 \subset \cdots$. Further $e_1R_2$ is always almost $\sum_{i\geq 2}\oplus e_iR_2$-injective for any $n$, but $e_1R_2$ is not almost $\sum_{i\geq 2}\oplus e_iR_2$-injective. Because, we assume that $e_1R$ were almost $\sum_{i\geq 2}\oplus e_iR$-injective, where $R=R_2$. Put $U= \sum_{i\geq 2}\oplus e_iR$. Then $\mathrm{Soc}(U)=\sum_{i\geq 2}\oplus e_iRe_1$ and $e_iRe_1 \approx e_1R_1=e_1R$ as $R$-modules. Take a diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \sum\oplus e_iRe_1 & \xrightarrow{i} & U \\ & & \downarrow{\scriptstyle f} & & \\ & & e_1R\,, & & \end{array}$$

where $f$ is given by the above isomorphisms. Since $\mathrm{Hom}_R(e_iR, e_1R)=0$ for $i\geq 2$, we should have a decomposition $U=A\oplus B$ and $\tilde{h} : e_1R \to A$ such that $\tilde{h}f=\pi_A i$ with $\pi_A : U \to A$. Further $\mathrm{Soc}(U)=\mathrm{Soc}(A)\oplus\mathrm{Soc}(B)$ and $\pi_A|\mathrm{Soc}(A)=1_{\mathrm{Soc}(A)}$. Hence $\tilde{h}f=\pi_A i$ implies that $\mathrm{Soc}(A)$ is simple, and so $A$ is indecomposable and $B$ is a direct sum of indecomposable modules $B_j$ $(j\geq 2)$ by [7], Theorem 8.3.3. Accordingly we may assume that $A=e_{n_1}R(f_1)$; $f_1 : e_{n_1}R \to \sum_{k\neq n_1}\oplus e_kR$ and $B_j= e_{n_j}R(f_j)$; $f_j : e_{n_j}R \to \sum_{k\neq n_j}\oplus e_kR$. Since $e_iR \not\approx e_jR$ if $i\neq j$, $n_i \neq n_j$ by Krull-Remark-Schmidt-Azumaya's theorem. Hence we can assume that $A=e_nR(f_n)$ for some $n$ and $B_j=e_jR(f_j)$ $(j\neq n$ and $B_n=e_2R(f_2))$; $n$ may be 2. Since $\mathrm{Hom}_R(e_iR, e_jR)$

$=0$ for $i>j$, we know $e_{n+1}R\subset\sum_{j\geq n+1}\oplus B_j\subset B$ from the structure of $B_j$. $\tilde{h}f(e_{n+1}Re_1)=\tilde{h}(e_1Re_1)\neq 0$ since $e_1R=e_1Re_1$ is simple and $\mathrm{Soc}(A)\subset\mathrm{Soc}(U)=\sum_{i\geq 2}\oplus e_iRe_1$, while $\tilde{h}f(e_{n+1}Re_1)=\pi_A(e_{n+1}Re_1)\subset\pi_A(B)=0$, a contradiction.

## 4. Extending property.

We shall consider a dual concept to § 2 (cf. [6] and [16]). Let $U\supset V$ be $R$-modules. Take a direct summand $V_1$ of $V$, i.e., $V=V_1\oplus V_2$. If $U$ has a decomposition $U=U_1\oplus U_2$ such that $U_1\cap V=V_1$, we say that $V_1$ is extendible to $U_1$. If, for any submodule $V$, every direct summand of $V$ is extendible to a direct summand of $U$, we say that $U$ has the *extending property of direct summands*. If $U$ has a decomposition $U=U_1\oplus U_2$ such that $V_i=V\cap U_i$ $(i=1, 2)$ for all $V$ and $V_i$, we say that $U$ has the *extending property of direct sums.*

The following results are dual to ones in § 2. Hence we shall skip proofs except Lemma 9 below.

In order to show a difference between $U$-injectives and almost $U$-injectives, we shall give the dual to corollary to Proposition 1.

PROPOSITION 4. *Let* $\{U_i\}_{i\in I}$ *be a set of* le *and uniform modules and* $U=\sum_{i=1}^{n}\oplus U_i$. *Then the following are equivalent*:

1) $U_i$ *is almost* $U_j$*-injective for all* $i\neq j$.

2) $U$ *has the extending property of direct summands.*

*Further the following are equivalent*:

3) $U_i$ *is* $U_j$*-injective for all* $i\neq j$.

4) $U$ *has the extending property of direct sums.*

Let $E$ be an indecomposable and injective module and $T=\mathrm{End}_R(E)$. Then $T$ is a local ring with radical $=\{f\mid\in T, \ker f\subset' E\}$ (see, [12] and [7], Proposition 5.4.9). Let $U_1$ and $U_2$ be uniform modules and $E_i=E(U_i)$. It is clear from the definition that if $E_1\not\approx E_2$, $U_1$ is almost $U_2$-injective if and only if $U_1$ is $U_2$-injective.

Dually to Lemma 6 we have

LEMMA 9 ([12]; [7], Theorem 5.4.2). *Let* $U_1$ *and* $U_2$ *be uniform modules and* $E_i$ *an injective hull of* $U_i$ *for* $i=1, 2$. *Assume that* $U_1$ *is almost* $U_2$*-injective. Let* $f$ *be not a monomorphism of* $E_2$ *to* $E_1$. *Then* $f(U_2)\subset U_1$.

PROOF. Put $U=f^{-1}(U_1)\cap U_2$, and take a diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & U & \longrightarrow & U_2 \\ & & \Big\downarrow f|U & & \\ & & U_1, & & \end{array}$$

Since $f^{-1}(0)\cap U\neq 0$, there exists $g: U_2\to U_1$ such that $g|U=f|U$ by assumption. We may assume that $g$ is an element in $\mathrm{Hom}_R(E_2, E_1)$. If $(f-g)(U_2)\neq 0$, then since $E_1'\supset U$, there exist $u_1\neq 0\in U_1$, $u_2\in U_2$ such that $(f-g)(u_2)=u_1$. However $g(u_2)\in U_1$, and so $u_2\in U_2\cap f^{-1}(U_1)=U$. Therefore $(f-g)(u_2)=0$, a contradiction. Hence $f(U_2)=g(U_2)\subset U_1$.

Finally we exhibit the following proposition dual to Proposition 2.

PROPOSITION 5. *Let $E$ be an indecomposable and injective module and $U_1$, $U_2$ submodules of $E$. Then*

1) *If $U_1$ is almost $U_2$-injective, $J(T)U_2\subset U_1$.*

2) *$U_1$ and $U_2$ are mutually almost injective if and only if $J(T)U_1\subset U_2$, $J(T)U_2\subset U_1$ and for any unit $f$ in $T$, $f(U_1)\subset U_2$ or $U_2\subset f(U_1)$, where $T=\mathrm{End}_R(E)$.*

PROOF. We can prove the proposition by virtue of Lemma 9 and its proof.

If either $U_1$ or $U_2$ has finite length, for every unit $f$ we have only a fixed side of $f(U_1)\subset U_2$ and $U_2\subset f(U_1)$ in 2). While let $Z_p$ be a local ring over the ring of integers $Z$, where $p$ is prime. Then $(p^n)$ and $Z_p$ are mutually almost injective. For units 1 and $p^{-(n+1)}$ in $Q=\mathrm{End}_{Z_p}(Q)$, $Z_p\subset p^{-(n+1)}(p^n)$ and $(p^n)\subset 1\cdot Z_p$.

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