

ON REAL HYPERSURFACES OF A COMPLEX SPACE FORM WITH η -PARALLEL RICCI TENSOR

By

Young Jin SUH*

Introduction.

Let $M_n(c)$ denote an n -dimensional complex space form with constant holomorphic sectional curvature c . It is well known that a complete and simply connected complex space form consists of a complex projective space CP^n , a complex Euclidean space C^n or a complex hyperbolic space CH^n , according as $c > 0$, $c = 0$ or $c < 0$. In this paper we consider a real hypersurface M of CP^n or CH^n .

The study of real hypersurfaces of CP^n was initiated by Takagi [10], who proved that all homogeneous hypersurfaces of CP^n could be divided into six types which are said to be of type A_1 , A_2 , B , C , D and E . Moreover, he showed that if a real hypersurface M of CP^n has two or three distinct constant principal curvatures, then M is locally congruent to one of the homogeneous ones of type A_1 , A_2 and B ([11]). Recently, to give another characterization of homogeneous hypersurfaces of type A_1 , A_2 and B in CP^n Kimura and Maeda [6] introduced the notion of an η -parallel second fundamental form, which was defined by $g((\nabla_X A)Y, Z) = 0$ for any vector fields X, Y and Z orthogonal to the structure vector field ξ , where A means the second fundamental form of M in CP^n , and g and ∇ denote the induced Riemannian metric and the induced Riemannian connection, respectively.

On the other hand, real hypersurfaces of CH^n have also been investigated by many authors (Berndt [1], Montiel [8], Montiel and Romero [9]).

Using some results about focal sets, Berndt [1] proved the following.

THEOREM A. *Let M be a connected real hypersurface of CH^n ($n \geq 2$). Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following.*

(A_0) *a horosphere in CH^n .*

Received September 16, 1988. Revised January 24, 1989.

*) Partially supported by TGRC-KOSEF.

- (A₁) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane CH^{n-1} .
- (A₂) a tube over a totally geodesic submanifold CH^k for $k=1, \dots, n-2$.
- (B) a tube over a totally real hyperbolic space RH^n .

It is necessary to remark that real hypersurfaces of type A_0 or A_1 appearing in Theorem A, are *totally η -umblical* hypersurfaces with two distinct constant principal curvatures. In the paper of Montiel [7] the real hypersurface of type A_0 in Theorem A is said to be self-tube.

In §3 we also consider the η -parallel second fundamental form in CH^n and give a further characterization of type A_0, A_1, A_2 , and B in CH^n . Now we introduce the notion of an η -parallel Ricci-tensor of M in $M_n(c)$, $c \neq 0$, which is defined by $g((\nabla_X S)Y, Z) = 0$ for any X, Y , and Z orthogonal to ξ , where S is the Ricci-tensor of M in $M_n(c)$, $c \neq 0$. It is easily seen that if the second fundamental form is η -parallel, then so is the Ricci-tensor, under the condition such that ξ is principal. Thus the purpose of this paper is to investigate this converse problem. By using the classification theorem due to Takagi [10] and Kimura and Maeda [6], we get the following.

THEOREM B. *Let M be a real hypersurface of CP^n . Then the Ricci-tensor of M is η -parallel and ξ is principal if and only if M is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 and B .*

By applying the Theorem A we can also prove the following.

THEOREM C. *Let M be a real hypersurface of CH^n ($n \geq 2$). Then the Ricci-tensor of M is η -parallel and ξ is principal if and only if M is locally congruent to one of type A_0, A_1, A_2 and B .*

§ 1. Preliminaries.

Let M be a real hypersurface of a complex n -dimensional complex space form $M_n(c)$, and let C be its unit normal vector field. Since $M_n(c)$ admits an almost complex structure, let us denote by F its almost complex structure. For any tangent vector field X and normal vector field C on M , the transformations of X and C under F can be given by

$$FX = \phi X + \eta(X)C, \quad FC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of

M , while η and ξ denote a 1-form and a vector field on a neighborhood of x in M , respectively. In which it is seen that $g(\xi, X)=\eta(X)$, where g denotes the induced Riemannian metric on M . By the properties of the almost complex structure F , they satisfy the following

$$(1.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. The set of tensors (ϕ, ξ, η, g) is called an almost contact structure on M .

Furthermore, the covariant derivatives of the structure tensors are given by

$$(1.2) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,$$

where ∇ is the induced Riemannian connection of g and A denotes the shape operator with respect to C on M .

Since the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c , the equation of Gauss and Codazzi are respectively given as follows:

$$(1.3) \quad R(X, Y)Z \\ = c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}/4 \\ + g(AY, Z)AX - g(AX, Z)AY,$$

$$(1.4) \quad (\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X .

The Ricci-tensor S' of M is the tensor of type $(0, 2)$ given by $S'(X, Y) = \text{tr}\{Z \rightarrow R(Z, X)Y\}$. Also it may be regarded as the tensor of type $(1, 1)$ and denoted by $S: TM \rightarrow TM$; it satisfies $S'(X, Y) = g(SX, Y)$. From (1.3) we see that the Ricci tensor S of M is given by

$$(1.5) \quad S = c\{(2n+1)I - 3\eta \otimes \xi\}/4 + hA - A^2,$$

where we have put $h = \text{Tr} A$. The covariant derivative of (1.5) are given as follows

$$(1.6) \quad (\nabla_X S)Y = \frac{c}{4}\{-3(\nabla_X \eta)(Y)\xi - 3\eta(Y)\nabla_X \xi\} + (Xh)AY + h(\nabla_X A)Y - (\nabla_X A^2)Y.$$

The Ricci-tensor on the real hypersurface of $M_n(c)$, $c \neq 0$, is said to be η -parallel if it satisfies $g((\nabla_X S)Y, Z) = 0$ for any tangent vector fields X, Y , and Z in ξ^\perp . In the sequel, assume that the hypersurface M is with η -parallel Ricci-tensor. Thus for any X, Y , and Z in ξ^\perp , (1.6) gives

$$(1.7) \quad g((\nabla_X S)Y, Z) = (Xh)g(AY, Z) + hg((\nabla_X A)Y, Z) - g((\nabla_X A^2)Y, Z) = 0.$$

It follows from (1.7) that if ξ is principal and if the second fundamental form is η -parallel, then the Ricci-tensor is η -parallel.

§ 2. Certain lemmas.

Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. The shape operator A of M can be considered as a symmetric $(2n-1, 2n-1)$ -matrix. Now we suppose that the structure vector ξ is a principal curvature vector of A , that is, $A\xi = \alpha\xi$, where α is the principal curvature corresponding to ξ .

Then the covariant derivative gives

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX,$$

where we have used the second formular of (1.2). Thus it follows that

$$(2.1) \quad g((\nabla_X A)Y, \xi) = (X\alpha)\eta(Y) + \alpha g(Y, \phi AX) - g(Y, A\phi AX),$$

for any tangent vector fields X , and Y on M . By using the equation of Codazzi to (2.1) and using the fact $X\alpha = (\xi\alpha)\eta(X)$, we have

$$(2.2) \quad 2A\phi AX - c\phi X/2 = \alpha(\phi A + A\phi)X.$$

We now introduce the following fact without proof.

LEMMA 2.1. ([3]) *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If ξ is a principal curvature vector of A , then its principal curvature α is locally constant.*

REMARK. Maeda [7] proved that α is constant for the real hypersurface of CP^n .

Since CP^n has constant holomorphic sectional curvature $c=4$, (2.2) gives the following.

LEMMA 2.2. ([7]) *Let M be a real hypersurface of CP^n . Assume that ξ is a principal curvature vector and the corresponding principal curvature is α . If $AX = \lambda X$ for any X in ξ^\perp , then $A\phi X = ((\alpha\lambda + 2)/(2\lambda - \alpha))\phi X$.*

§ 3. Real hypersurfaces of CH^n with η -parallel second fundamental form.

It is well known that the complex hyperbolic space CH^n admits the Bergmann metric normalized so that the constant holomorphic sectional curvature c is -4 .

Thus (2.2) gives the following equation for the real hypersurface of CH^n whose structure vector field ξ is principal.

$$(3.1) \quad 2A\phi AX + 2\phi X = \alpha(\phi A + A\phi)X$$

for any tangent vector field X in M . It follows that if $AX = \lambda X$ for any X in ξ^\perp , then

$$(3.2) \quad (2\lambda - \alpha)A\phi X = (\alpha\lambda - 2)\phi X.$$

Now we need the following lemmas which will be used in the later.

LEMMA 3.1. (Montiel and Romero [9]) *Let M be a real hypersurface of CH^n . Then*

$$(3.3) \quad A\phi = \phi A \text{ holds on } M \text{ if and only if } M \text{ is of type } A_0, A_1 \text{ or } A_2.$$

LEMMA 3.2. *Let M be a real hypersurface of CH^n . Then*

$$(3.4) \quad A\phi + \phi A = k\phi \text{ (} k \neq 0 \text{: constant) holds on } M \text{ if and only if } M \text{ is of type } A_0, A_1 \text{ or } B.$$

PROOF. From (3.4) we have that $A\xi = \alpha\xi$, that is, ξ is the principal curvature vector. If $AX = \lambda X$ for any X in ξ^\perp , then $A\phi X = (k - \lambda)\phi X$.

By Lemma 2.1 α is constant. Thus we can consider the following two cases: $\alpha^2 - 4 \neq 0$ and $\alpha^2 - 4 = 0$.

For $\alpha^2 - 4 \neq 0$ we then have $2\lambda - \alpha \neq 0$ by (3.2). Thus also from (3.2) it follows that $k - \lambda = (\alpha\lambda - 2)/(2\lambda - \alpha)$. Hence it follows that $2\lambda^2 - 2k\lambda + \alpha k - 2 = 0$. Since λ satisfies the above quadratic equation with constant coefficients, all principal curvatures are constant on M . Thus due to Theorem A, M is of type A_1 , A_2 or B . Suppose that M is of type A_2 . By Lemma 3.1 $A\phi = \phi A$ holds on M . This fact and (3.4) imply $2A\phi = k\phi$. Thus from the almost contact structure it follows that $A = aI + b\eta \otimes \xi$, that is, M is totally η -umbilical. Then it is seen by Montiel and Romero [9] that M is of type A_0 or A_1 , a contradicts. Thus the type of A_2 can not occur.

Now we consider for the case $\alpha^2 - 4 = 0$. Let $M_0 = \{x \in M \mid (2\lambda - \alpha)_x \neq 0\}$. Then λ also satisfies $2\lambda^2 - 2k\lambda + \alpha k - 2 = 0$. Thus λ is constant on M_0 . On the other hand, we have $2\lambda - \alpha = 0$ on $M - M_0$. Then (3.2) gives $\alpha\lambda = 2$. Thus $\lambda = \pm 1$ on $M - M_0$.

The continuity of principal curvatures implies that if the set $M - M_0$ is not empty, then $\lambda = \pm 1$ on M . Hence M is of type A_0 .

For the case where M_0 coincides with the whole M , it is of type A_1 , A_2 or B and therefore it must be of type A_1 or B by the same argument as that of the above half, a contradiction.

Conversely, suppose that M is of type A_0 , A_1 or B . It is seen by Montiel

and Romero [9] that the type of A_0 and A_1 are the only totally η -umblical real hypersurfaces of \mathbf{CH}^n . Thus it naturally satisfies $A\phi + \phi A = k\phi$.

For the type of B we can take an orthonormal basis $\{X_1, \dots, X_{n-1}, \phi X_1, \dots, \phi X_{n-1}, \xi\}$ of $T_x(M)$ such that $AX_i = \coth \theta X_i$, $A\phi X_i = \tanh \theta \phi X_i$, $i=1, \dots, n-1$, and $A\xi = 2 \tanh 2\theta \xi$. Then we have $A\phi X + \phi AX = (\tanh \theta + \coth \theta)\phi X$ for any X in $T_x(M)$. Thus we complete the above lemma.

LEMMA 3.3. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If the structure vector field ξ is principal and if the second fundamental form A satisfies the following quadratic formula:*

$$(3.5) \quad A^2 + aA + cI = 0 \quad (a^2 - 4b \neq 0, \quad a, b : \text{constant}) \quad \text{on } \xi^\perp,$$

then the second fundamental form A is η -parallel.

PROOF. By taking covariant derivative of (3.5), we get

$$(3.6) \quad g((\nabla_X A)AY, Z) + g(A(\nabla_X A)Y, Z) + ag((\nabla_X A)Y, Z) = 0$$

for any X, Y , and Z in ξ^\perp .

Taking the skew-symmetric part of (3.6) and using the equation of Codazzi, we have

$$g((\nabla_X A)AY, Z) = g((\nabla_Y A)AX, Z),$$

from which together with $g(AX, (\nabla_Z A)Y) = g((\nabla_Z A)AX, Y) = g((\nabla_X A)AZ, Y)$, we get

$$(3.7) \quad g((\nabla_X A)AY, Z) = g(A(\nabla_X A)Y, Z)$$

for any X, Y , and Z in ξ^\perp , where we have used the fact that ξ^\perp is invariant under the transformation of A because ξ is the principal curvature vector.

Combining (3.6) and (3.7), we obtain for any X, Y , and Z in ξ^\perp

$$(3.8) \quad 2g(A(\nabla_X A)Y, Z) + ag((\nabla_X A)Y, Z) = 0.$$

Transforming (3.8) with A and using (3.5) again, we get

$$(3.9) \quad 2bg((\nabla_X A)Y, Z) = -ag(A(\nabla_X A)Y, Z).$$

From which, substituting into (3.8), we have

$$(3.10) \quad g(A(\nabla_X A)Y, Z) = 0,$$

where we have used the fact $a^2 - 4b \neq 0$. Thus (3.9) gives $g((\nabla_X A)Y, Z) = 0$ for $b \neq 0$.

For the case where $b = 0$, $a^2 - 4b \neq 0$ implies $a \neq 0$. From which together

with (3.8) and (3.10) it follows that $g((\nabla_X A)Y, Z)=0$ for any X, Y and Z in ξ^\perp . Hence we get the above lemma.

These Lemmas 3.1, 3.2 and 3.3 and Theorem A enable us to prove the following.

THEOREM 3.4. *Let M be a real hypersurface of CH^n . Then the second fundamental form of M is η -parallel and the structure vector field ξ is principal if and only if M is locally congruent to one of type A_0, A_1, A_2 or B .*

PROOF. First we shall show that the second fundamental form of type A_0, A_1, A_2 or B is η -parallel.

Now let M be a of type A_0, A_1 or A_2 . Then by Lemma 3.1 $A\phi=\phi A$ holds on M . Thus $\phi A\xi=0$ implies that ξ is principal, that is, $A\xi=\alpha\xi$. From which and (3.1) it follows that

$$A^2-\alpha A+I=0 \quad \text{on } \xi^\perp.$$

Thus Lemma 3.3 gives that the second fundamental form is η -parallel for the case $\alpha^2-4\neq 0$. For the case where $\alpha^2=4$ all the principal curvatures λ are ± 1 . Thus M is of type A_0 and totally η -umbilical. Hence the second fundamental form is also η -parallel in this case.

Now we consider that M is of type B . Then by Lemma 3.2 $A\phi+\phi A=k\phi$ ($k\neq 0$: constant) holds on M . From which we also get $A\xi=\alpha\xi$. Thus from (3.1) it follows that

$$A^2-kA-(1-\alpha k/2)I=0 \quad \text{on } \xi^\perp.$$

On the other hand, due to Berndt's classification [1] all the principal curvatures of type B are given as follows: $\lambda=\coth \theta, \mu=\tanh \theta, \alpha=2 \tanh 2\theta$. Since $\lambda+\mu=2 \coth 2\theta=4/\alpha, A\phi+\phi A=k\phi$ implies $k=4/\alpha$. Hence we conclude that $k^2+4(1-\alpha k/2)\neq 0$. Hence by Lemma 3.3 we also get our result.

Conversely, it suffices to show that all the principal curvatures are constant on M . If $AX=\lambda X$ for any X in ξ^\perp , then $g((\nabla_Y A)X, X)=(Y\lambda)g(X, X)$. Thus from the assumption we have that $Y\lambda=0$ for any Y in ξ^\perp .

On the other hand, using equation of Codazzi and making use of (2.1) and Lemma 2.1, we get the following.

$$\xi\lambda=g((\nabla_\xi A)X, X)=g((\nabla_X A)\xi, X)=0.$$

From these facts and Theorem A, we conclude that M is of type A_0, A_1, A_2 , and B . This completes the above Theorem.

REMARK. Kimura and Maeda [6] showed that a real hypersurface of CP^n with η -parallel second fundamental form and principal structure vector field ξ is locally congruent to one of homogeneous real hypersurfaces of type A_1 , A_2 and B .

§ 4. Real hypersurfaces of $M_n(c)$, $c \neq 0$, with η -parallel Ricci-tensor.

Let M be a real hypersurface of $M_n(c)$ with η -parallel Ricci-tensor, that is, $g((\nabla_X S)Y, Z) = 0$ for any X, Y and Z in ξ^\perp . It is easily seen that if ξ is principal, then the second fundamental form A of M in $M_n(c)$ is η -parallel implies that the Ricci-tensor S is η -parallel. In this section we are investigated to study this converse problem by using Kimura and Maeda's [6] result and Theorem 3.4. Then we can state another characterization as the following.

THEOREM 4.1. *Let M be a real hypersurface of CP^n . Then the Ricci-tensor is η -parallel and the structure vector field ξ is principal if and only if M is of type A_1 , A_2 and B .*

PROOF. For any X, Y in ξ^\perp , the fact that the Ricci-tensor is η -parallel implies

$$(\nabla_X S)Y = -3(\nabla_X \eta)(Y)\xi + (Xh)AY + h(\nabla_X A)Y - (\nabla_X A^2)Y$$

belongs to $[\xi]$, where $[\xi]$ means 1-dimensional vector space spanned by ξ . Thus $g((\nabla_X S)Y, Y) = 0$ for any Y in ξ^\perp . Hence if we put $AY = \lambda Y$, then

$$(4.1) \quad \lambda(Xh) + h(X\lambda) - (X\lambda^2) = 0 \quad \text{for any } X \text{ in } \xi^\perp.$$

Also for any Y in ξ^\perp such that $AY = \lambda Y$ we have $(\nabla_\xi A)Y = (\xi\lambda)Y + (\lambda I - A)\nabla_\xi Y$. Thus $\xi\lambda = g((\nabla_\xi A)Y, Y) = g((\nabla_Y A)\xi, Y) = 0$. Hence the mean curvature h is also constant on ξ -direction. Together with this fact and (4.1), we conclude that $\lambda h - \lambda^2$ is constant on M . Thus we can put as the following.

$$(4.2) \quad \lambda h - \lambda^2 = a, \quad (4.3) \quad \mu h - \mu^2 = b.$$

By Lemma 2.2, (4.2) and (4.3) can be rewritten as follows

$$(4.4) \quad \lambda^2 - h\lambda + a = 0,$$

$$(4.5) \quad (2h\alpha - \alpha^2 - 4b)\lambda^2 - \{(\alpha^2 - 4)h + 4\alpha - 4b\alpha\}\lambda - (2\alpha h + b\alpha^2 + 4) = 0.$$

Substituting $h\lambda = \lambda^2 + a$ into (4.5), we then have

$$(4.6) \quad 2\alpha\lambda^4 - (2\alpha^2 + 4b - 4)\lambda^3 + 2(a\alpha + 2b\alpha - 3\alpha)\lambda^2 - (a\alpha^2 - 4a + b\alpha^2 + 4)\lambda - 2a\alpha = 0.$$

From which we see that λ satisfies an algebraic equation with constant coefficient-

ents. Thus M has at most five constant principal curvatures. According to Kimura's theorem [4], M is homogeneous.

On the other hand, due to Takagi's classification of homogeneous real hypersurface of CP^n , we conclude that M is of type A_1, A_2, B, C, D and E .

In order to prove this theorem we shall show that the shape operator is η -parallel.

Let $A(\lambda)$ be an eigenspace of A with eigenvalue λ . Then the subspace ξ_x^\perp of the tangent space $T_x(M)$ at x can be decomposed as $\xi_x^\perp = A(\lambda_1) \oplus A(\lambda_2) \oplus \dots \oplus A(\lambda_s)$. Now in what follows we consider the following eigenvector such that $X \in A(\lambda)$, $Y \in A(\mu)$ and $Z \in A(\sigma)$, where λ, μ and σ are corresponding constant principal curvatures. Then we have that

$$(4.7) \quad g((\nabla_X A)Y, Z) = (\mu - \sigma)g(\nabla_X Y, Z).$$

On the other hand, from the η -parallel Ricci-tensor it follows that

$$(4.8) \quad (h - \mu - \sigma)g((\nabla_X A)Y, Z) = 0.$$

For the case where $\mu = \sigma$, (4.7) implies that $g((\nabla_X A)Y, Z) = 0$. Thus it suffices to show that the shape operator is η -parallel for the case where $\mu \neq \sigma$.

In the case where $h - \mu - \sigma \neq 0$, (4.8) gives our result. Thus it remains to consider for the case where $h - \mu - \sigma = 0$. Thus the η -parallel Ricci-tensor gives

$$(4.9) \quad g((\nabla_Y S)X, Z) = (h - \lambda - \sigma)g((\nabla_Y A)X, Z) = 0.$$

If $\lambda \neq \mu$, then $h - \mu - \sigma = 0$ implies $h - \lambda - \sigma \neq 0$. From which together with (4.9) it follows $g((\nabla_X A)Y, Z) = g((\nabla_Y A)X, Z) = 0$. If $\lambda = \mu$, (4.7) gives $g((\nabla_X A)Y, Z) = g((\nabla_Z A)X, Y) = 0$. Summing up, we conclude that the shape operator is η -parallel. Thus, due to Kimura and Maeda's Theorem [6], M is of type A_1, A_2 and B .

Conversely, if M is of type A_1, A_2 or B , then by Kimura and Maeda's Theorem [6] the second fundamental form is η -parallel and its structure vector field ξ is principal. Since η -parallel second fundamental form with the principal structure vector ξ implies η -parallel Ricci-tensor, we get the above Theorem.

REMARK. Kimura [5] showed that a real hypersurface of CP^n with the condition $(\nabla_X S)Y = c\{g(\phi AX, Y) + \eta(Y)\phi AX\}$, where c is constant, is locally congruent to homogeneous hypersurfaces with 2 or 3 distinct principal curvatures. Thus this condition implies that the Ricci-tensor S is η -parallel and structure vector field ξ is principal.

On the other hand, for a real hypersurface of CH^n we get the following.

THEOREM 4.2. *Let M be a real hypersurface of CH^n , $n \geq 2$. Then the Ricci-tensor is η -parallel and the structure vector field ξ is principal if and only if M is of type A_0, A_1, A_2 or B .*

PROOF. The converse is trivial by Theorem 3.4.

Let M be a real hypersurface of CH^n with η -parallel Ricci-tensor and principal structure vector field ξ . Then similarly as in Theorem 4.1 we can put

$$(4.13) \quad \lambda h - \lambda^2 = a,$$

$$(4.14) \quad \mu h - \mu^2 = b.$$

By Lemma 2.1 we can consider the following two cases.

CASE I. $\alpha^2 - 4 \neq 0$.

Then $2\lambda - \alpha \neq 0$. In fact, suppose that $2\lambda - \alpha = 0$. Then (3.2) gives $\alpha\lambda = 2$. Together with this fact we have $\alpha^2 - 4 = 0$, a contradiction. Thus from (3.2) it follows $A\phi X = \mu\phi X$, $\mu = (\alpha\lambda - 2)/(2\lambda - \alpha)$. From which, substituting (4.14), then we get

$$(4.15) \quad (2\alpha h - \alpha^2 - 4b)\lambda^2 + \{4\alpha + 4b\alpha - (\alpha^2 + 4)h\}\lambda + (2\alpha h - 4 - b\alpha^2) = 0.$$

Substituting (4.13) into (4.15), then λ satisfies the following equation with constant coefficients

$$2\alpha\lambda^4 - 2(\alpha^2 + 2b + 2)\lambda^3 + 2\alpha(a + 2b + 3)\lambda^2 - (a\alpha^2 + b\alpha^2 + 4a + 4)\lambda + 2a\alpha = 0.$$

In the case where $\alpha = 0$, $a = -1$ and $b = -1$, coefficients of the above equation are all vanishing. Thus it suffices to prove that principal curvatures are also constant on M in this case.

For the case where $a = -1$, and $b = -1$ it follows from (4.13) and (4.14) that $\lambda = \mu$ or $h = \lambda + \mu$. Since $\mu = -1/\lambda$ for $\alpha = 0$, $\lambda = \mu$ implies $\lambda^2 + 1 = 0$. This contradicts. Thus we have $h = \lambda + \mu$. From which together with $h = m_1\lambda + m_2(-1/\lambda)$ for $\alpha = 0$, it follows that $(m_1 - 1)\lambda^2 - (m_2 - 1) = 0$. Since $m_1 \neq 1$, principal curvatures are constant on M in this case. Hence all principal curvatures are constant on M . Thus due to Theorem A we conclude that M is of type A_1, A_2 or B .

CASE II. $\alpha^2 = 4$.

Now we consider for the case $\alpha = 2$. Then (3.2) gives

$$(\lambda-1)A\phi X=(\lambda-1)\phi X.$$

Let us take an open set $M_0=\{x\in M|\lambda\neq 1\}$. Then $A\phi X=\phi X$. Thus $\mu=1$. From which and (4.14) it follows $h=b+1$ on M_0 . Since $\lambda=1$ on $M-M_0$, also from (4.13) it follows $h=a+1$. Hence h is constant and $a=b$ on M . Thus λ satisfies a quadratic equation with constant coefficients: $\lambda^2-h\lambda+a=0$. Hence all principal curvatures are constant on M .

Similarly, for the case $\alpha=-2$ we also get the same conclusion. By virtue of Theorem A, M is of type A_0, A_1, A_2 or B . Since $\alpha=\pm 2$, then M is of type A_0 .

References

- [1] Berndt, J., Real hypersurfaces with constant principal curvatures in complex hyperbolic space, *J. reine angew. Math.* **395** (1989), 132-141.
- [2] Cecil T.E. and Ryan., P. J., Focal sets and real hypersurfaces in complex projective space, *Trans. Amer. Math. Soc.* **269** (1982), 481-499.
- [3] Ki, U-H. and Suh., Y. J., On real hypersurfaces of a complex space form, to appear in *Okayama Math. J.*
- [4] Kimura, M., Real hypersurfaces and complex submanifolds in a complex projective space, *Trans. Amer. Math. Soc.* **296** (1986), 137-149.
- [5] Kimura, M., Real hypersurfaces of a complex projective space, *Bull. Austral. Math. Soc.* Vol. **33** (1986), 383-387.
- [6] Kimura, M. and Maeda, S., On real hypersurfaces of a complex projective space, *Math. Z.* **202** (1989), 299-311.
- [7] Maeda, Y., On real hypersurfaces of a complex projective space, *J. Math. Soc. Japan* **28** (1976), 529-540.
- [8] Montiel, S., Real hypersurfaces of a complex hyperbolic spaces, *J. Math. Soc. Japan* **37** (1985), 515-435.
- [9] Montiel, S. and Romero, A., On some real hypersurfaces of a complex hyperbolic space, *Geometriae Dedicata*, **20** (1986), 245-261.
- [10] Takagi, R., On homogeneous real hypersurfaces of a complex projective space, *Osaka J. Math.* **10** (1973), 495-506.
- [11] Takagi, R., Real hypersurfaces in a complex projective space with constant principal curvatures. I, II. *J. Math. Soc. Japan* **27** (1975), 43-53, 507-516.
- [12] Yano, K. and Kon, M., *CR*-submanifolds of Kaehlerian and Sasakian manifold, Birkhauser 1983.

University of Tsukuba
Institute of Mathematics
Ibaraki, 305
Japan

and
Andong University
Department of Mathematics
Andong, 760-749
Korea